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ON THE FAST GROWTH OF ANALYTIC FUNCTIONS BY MEANS OF LAGRANGE POLYNOMIAL APPROXIMATION AND INTERPOLATION IN C^{N*}

Abstract. The present paper is concerned with the fast growth of analytic functions in the sets of the form $\{z\epsilon C^N : \phi_K(z) < R\}$ (where $\phi_K(z)$ is the Siciak extremal function of a compact set K) by means of the Lagrange polynomial approximation and interpolation on K having rapidly increasing maximum modulus. To study the precise rates of growth of such functions the concept of index has been used.

Key words: index$-q$, approximation and interpolation, fast growth and analytic functions.

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1. Introduction

Let K be a compact set in C^N and let $\|\|$ denote the supremum norm on K. For $n\epsilon N$ denote by P_n the space of all polynomials from C^N to C, of degree at most n. Throughout the paper we assume that the set K is $L-$regular, i.e., the Siciak extremal function of K[20], [21],

$$\phi_K(z) = \sup\left\{|p(z)|^{1/n} : p\epsilon P_n, \|p\| \le 1, \quad n \ge 1\right\}, \quad z\epsilon C^N,$$

is continuous in C^N.

Let f be a function defined and bounded on K and let t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K. It is known that [20], [21] if K is $L-$regular and $\limsup_{n\to\infty} \|f-t_n\|^{1/n} = \frac{1}{R}$ with $1 < R < \infty$, then there exists a function g analytic in $K_R = \left\{z\epsilon C^N : \phi_K(z) < R\right\}$ such that $g|_K = f$.

Reddy [14], [15] connected classical order and type with polynomial approximation error of the (entire) function which is an extension of a continuous function defined on $[-1, 1]$. Contemporarily, Rice [16], Massa [11],

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Rizvi and Juneja [18], Nautiyal and Rizvi [12], Rizvi [17], Kapoor and Nautiyal [6] and Winiarski [23] studied these results for different approximation errors of a continuous function on the arbitrary domains, Shah [19], Kapoor and Nautiyal [7] and Nautiyal, Rizvi and Kapoor [13] have studied in this direction for continuous functions on the domain $[-1, 1]$.

They studied the results for $(\alpha, \beta)-$orders. Later on, Kasana and Kumar [8], [9] extended these results to the $(p, q)-$scale introduced by Juneja et.al [2], [3]. Jurgen Muller [5] studied accelerated polynomial approximation problem of finite order entire functions by growth reduction. All these results have been studied for $R = \infty$ and $N = 1$ i.e., in single complex variables. Kumar [10] and Winiarski [24] obtained various results regarding the entire function of given growth. It has been noticed that the case $R < +\infty$ in several complex variables has not been studied so extensively. The aim of this paper is to study the case $R < +\infty$ in C^N. Our results apply satisfactorily for functions of fast growth or simply speaking if maximum modulus is increasing so rapidly that the order of function is infinite. It is significant to mention here that our results generalize various results contained in Juneja and Kapoor[4].

For $R > 1$ let $D(R) = D_K(R)$ denote the set of all functions analytic in K_R and not continuable to any $K_{R'}$ with $R' > R$. Given a function $g \epsilon D(R)$, we put

$$M(r, g) = \sup \{|g(z)| : \phi_K(z) = r\}, \quad r < R.$$

For a function $g \epsilon D(R)$, set

$$\rho_R(q) = \limsup_{r \to R} \frac{\log^{[q]} M(r, g)}{-\log(1 - r/R)} \tag{1}$$

where $\log^{[0]} M(r, g) = M(r, g)$ and $\log^{[q]} M(r, g) = \log(\log^{[q-1]} M(r, g))$, $q = 1, 2, \cdots$. To avoid the trivial cases we shall assume throughout that $M(r, g) \to \infty$ as $r \to R$.

Definition 1. *A function $g \epsilon D(R)$ is said to have the index$-q$ if $\rho_R(q) < \infty$ and $\rho_R(q - 1) = \infty$, $q = 1, 2, \cdots$. If q is the index of $g(z)$, then $\rho_R(q)$ is called the $q-$order of g.*

Definition 2. *A function $g \epsilon D(R)$ having $q-$order $\rho_R(q)$, ρ_R $(q > 0$, $q = 2, 3, \cdots)$, is said to have $q-$type $\sigma_R(q)$ if*

$$\sigma_R(q) = \limsup_{r \to R} \frac{\log^{[q-1]} M(r, g)}{(1 - r/R)^{-\rho_R(q)}}. \tag{2}$$

(For the definition of index-q etc. see[4]).

Remark 1. Definition 1 is the generalization of the Beuermann definition of the order of an analytic function in the unit disc [1] (compare [22]).

2. Auxilliary results

Let K be a fixed compact, $L-$regular set in C^N and let $(p_n)_{n\epsilon N}$ be a sequence of polynomials such that

(i) $p_n \epsilon P_n, n\epsilon N$

(ii) $\sum\limits_{n=0}^{\infty} p_n \epsilon D(R)$ with $1 < R < +\infty$,

(iii) for every positive $r < R$ the set $\{\|p_n\| r^n : n\epsilon N\}$ is bounded.

Set

$$\begin{aligned} M^*(r &= \max\left\{\|p_n\|r^n : n\epsilon N\right\}, 1 < r < R, \\ \rho_R^*(q) &= \limsup_{r\to R} \frac{\log^{[q]} M^*(r)}{-\log(1-r/R)}; \end{aligned}$$

if $\rho_R(q) = \rho_R\left(q, \sum\limits_{n=0}^{\infty} p_n\right)$ is positive, we put

$$\sigma_R^*(q) = \limsup_{r\to R} \frac{\log^{[q-1]} M^*(r)}{(1-r/R)^{-\rho_R(q)}}.$$

Now we shall prove some auxiliary results which will be used in the sequel.

Lemma 1. *Under the assumptions of section 2*

(i) $\rho_R(q) \le \rho_R^*(q)$,

(ii) if $\rho_R(q)\epsilon(0,+\infty)$, then $\sigma_R(q) \le \sigma_R^*(q)$,

where $\rho_R(q)$ and $\sigma_R(q)$ are the $q-$order and $q-$type of the function $\sum\limits_{n=0}^{\infty} p_n$ respectively.

Proof. We have

$$M(r) \le \sum_{n=0}^{\infty} \sup\{|p_n(z)|\} \ : \ z\epsilon K_r.$$

Using the property of [20] we get

$$\begin{aligned} |p_n(z)| &\le \|p_n\|r^n, \quad z\epsilon K_r, \ \ n\epsilon N. \\ M(r) &\le \sum_{n=0}^{\infty} \|p_n\|r^n. \end{aligned}$$

Let us write $r = r^\delta R^{1-\delta}(r/R)^{1-\delta}$ in above, we obtain

$$qM(r) \leq \sum_{n=0}^{\infty} M^*(r^\delta R^{1-\delta})(r/R)^{(1-\delta)n}$$

or

$$M(r) \leq \frac{M^*(r^\delta R^{1-\delta})}{1-(r/R)^{1-\delta}}.$$

For every $\delta < 1$, we get

$$\log^+ \log^+ M(r) \leq \log^+ \left[\log^+ M^*(r^\delta R^{1-\delta}) + \log \frac{1}{1-(r/R)^{1-\delta}} \right], \tag{3}$$

where $\log^+ x = \max(\log x, 0), 0 \leq x \leq \infty$. Here we shall assume that $M^*(r^\delta R^{1-\delta}) \to +\infty$ as $r \to R$ because if the function $r \to M^*(r^\delta R^{1-\delta})$ is bounded, then $\rho_R(q) = \rho^*_R(q) = 0$. Then for r sufficiently close to R, (3) gives after a simple calculation

$$\frac{\log^{[q]} M(r)}{-\log(1-r/R)} \leq \frac{\log^{[q]} M^*(r^\delta R^{1-\delta})}{-\log(1-r/R)}(1+0(1))$$

and

$$\frac{\log^{[q-1]} M(r)}{(1-r/R)^{-\rho_R(q)}} \leq \frac{\log^{[q-1]} M^*(r^\delta R^{1-\delta})}{(1-r/R)^{-\rho_R(q)}}(1+0(1)).$$

Proceeding to the upper limits in above inequalities the proof of (i) and (ii) completed. ■

Remark 2. *If the set K is balanced and $(p_n)_{n\epsilon N}$ is a sequence of homogeneous polynomials, then using the Cauchy inequalities $M^*(r) \leq M(r)$, it gives $\rho^*_R(q) \leq \rho_R(q)$ and $\sigma^*_R(q) \leq \sigma_R(q)$ provided $0 < \rho_R(q) < +\infty$.*

Lemma 2. *Let $(p_n)_{n\epsilon N}$ be a sequence of polynomials such that $p_n \epsilon P_n$ for $n \epsilon N$. If $\sum_{n=0}^{\infty} p_n \epsilon D(R)$ and if there exist positive constants β, n_0 and $\alpha > 1$ such that*

$$\|p_n\| \leq R^{-n} \exp\left[\frac{\beta n}{(\log^{[q-2]} n)^{1/\alpha}}\right], \quad n \geq n_0,$$

then

(i) $\rho_R(q) \leq \alpha - A(q)$,

(ii) if $\rho_R(q) = \alpha - A(q)$, then $\sigma_R(q) \leq \beta^{\rho_R(q)+A(q)}/B_R(q)$,

where

$$A(q) = \begin{cases} 1 & if \quad q=2 \\ 0 & otherwise \end{cases} \tag{4}$$

$$B_R(q) = \begin{cases} \frac{(\rho_R(q)+1)^{\rho_R(q)+1}}{(\rho_R(q))^{\rho_R(q)}} & if \quad q=2 \\ 1 & otherwise \end{cases}.$$

Proof. We have

$$\|p_n\| r^n \le (r/R)^n \exp\left[\frac{\beta n}{\left(\log^{[q-2]} n\right)^{1/\alpha}}\right], \quad n \ge n_0, \;\; r < R. \tag{5}$$

First we consider the case for $q = 2$,

$$\|p_n\| r^n \le (r/R)^n \exp\left\{\beta n^{(1-1/\alpha)}\right\}.$$

Let $A^* = (r/R)^x \exp\{\beta x^{(1-1/\alpha)}\}$, $0 < x < +\infty$. The suprimum of A^* is attained at $x = \frac{\beta^\alpha(\alpha-1)^\alpha}{\alpha^\alpha}\frac{1}{(\log R/r)^\alpha}$ and we obtain

$$\begin{aligned} \|p_n\| r^n &\le \sup\left\{(r/R)^x \exp\left\{\beta x^{(1-1/\alpha)}\right\}\right\} \\ &= \exp\left[\frac{\beta^\alpha(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(\log R/r)^{1-\alpha}\right], \quad n \ge n_0, \;\; r < R. \end{aligned} \tag{6}$$

It can be seen that for every $r < R$ there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \|p_{\nu(r)}\| r^{\nu(r)}$$

and

$$M^*(r) > \|p_n\| r^n, \quad n > \nu(r).$$

If $\nu(r)$ is bounded for $r < R$ then $M^*(r)$ also is bounded, hence $\rho_R^*(q) = 0$ and consequently $\rho_R(q) = 0$. So we may take $\nu(r) \ge n_0$ for r sufficiently close to R. Putting $n = \nu(r)$ in (6) we get

$$M^*(r) \le \exp\left[\frac{\beta^\alpha(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(\log R/r)^{1-\alpha}\right], \quad r_0 < r < R,$$

or

$$\frac{\log^+\log^+ M^*(r)}{-\log(1-r/R)} \le \frac{\log\left[\frac{\beta^\alpha(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\right]}{-\log(1-r/R)} + (\alpha-1)\frac{\log\log R/r}{\log(1-r/R)}$$

and

$$\frac{\log^+ M^*(r)}{(1-r/R)^{-\rho_R(2)}} \le \beta^{\rho_R(\dot{2})+1}\frac{(\rho_R(2))^{\rho_2(2)}}{(\rho_R(2)+1)^{\rho_R(2)+1}}\,\frac{(\log R/r)^{-\rho_R(2)}}{(1-r/R)^{-\rho_R(2)}}$$

provided $\rho_R(2) = \alpha - 1$. Proceeding to limits as $r \to R$ and using the Lemma 1 we obtain the inequalities (i) and (ii) for $q = 2$.

Now for $q = 3, 4, \cdots$. The suprimum of right hand side of (5) is attained at

$$n = x = \exp^{[q-2]}\left[\frac{\beta^{\alpha}\left(1-\frac{x}{\log^{[q-2]} x \prod_{i=3}^{q}\log^{[q-i]} x}\right)^{\alpha}}{(\log R/r)^{\alpha}}\right],$$

and we get

$$\log^{+} M^{*}(r) \leq \exp^{[q-2]}\left[\frac{\beta^{\alpha}}{(\log R/r)^{\alpha}}\left(1-\frac{n}{\log^{[q-2]} n \prod_{i=3}^{q}\log^{[q-i]} n}\right)^{\alpha}\right]\log(R/r)$$
$$\times\left[\frac{1}{1-\frac{n}{\log^{[q-2]} n \prod_{i=3}^{q}\log^{[q-i]} n}}-1\right]$$

or

$$\log^{[q]} M^{*}(r) \leq \log\beta^{\alpha}\left(1-\frac{n}{\log^{[q-2]} n \prod_{i=3}^{q}\log^{[q-i]} n}\right)^{\alpha} + \log(\log R/r)^{-\alpha} + 0(1)$$

or

$$\frac{\log^{[q]} M^{*}(r)}{-\log(1-r/R)} \leq \frac{\alpha\log\log(R/r)}{-\log(1-r/R)} + 0(1).$$

Proceeding to limits as $r \to R$ and using the Lemma 1, we get

$$\rho_R(q) \leq \alpha.$$

Similarly, we can obtain easily from above inequality that

$$\sigma_R(q) \leq \beta^{\rho_R(q)}\alpha.$$

Hence the proof is completed. ∎

Let K be a compact, $L-$regular set in C^N. Given a function f defined and bounded on K we put for $n\epsilon N$ [19]

$$\begin{aligned} E_n^{(1)} &= E_n^{(1)}(f,K) = \|f-t_n\|, \\ E_n^{(2)} &= E_n^{(2)}(f,K) = \|f-l_n\|, \\ E_{n+1}^{(3)} &= E_{n+1}^{(3)}(f,K) = \|l_{n+1}-l_n\|, \end{aligned}$$

where t_n denotes the n^{th} chebysev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation polynomial for f with nodes at extremal points of K.

We have the following inequalities [23, Lemma 3.3]

$$E_n^{(1)} \leq E_n^{(2)} \leq (n_* + 2)E_n^{(1)}, \tag{7}$$

$$E_n^{(3)} \leq 2(n_* + 2)E_{n-1}^{(1)}, \quad n \geq 1, \tag{8}$$

where $n_* = \left(\begin{array}{c} n+N \\ n \end{array} \right)$.

Theorem A [**20**]. *The function f is the restriction to K of a function from $D(R)$ if and only if*

$$\limsup_{n\to\infty} \left(E_n^{(s)}\right)^{1/n} = \frac{1}{R}, \quad s = 1, 2 \text{ or } 3.$$

Lemma 3. *Let K be a compact, $L-$regular, balanced set in C^N. Then for every $g\epsilon D(R)$*

$$E_n^{(1)}(g|_K, K) \leq \frac{M(r,g)}{r^n(r-1)}, \quad 1 < r < R, \ \ n \in N. \tag{9}$$

Proof. The proof of this lemma follows immediately from a result of Siciak [20, p. 344, inequality (7)] and from the Cauchy inequality. ■

3. Main results

Theorem 1. *f is the restriction to K of a function $g\epsilon D(R)$ having the index$-q$ and $q-$order $\rho_R(q)(0 < \rho_R(q) < \infty)$ if and only if*

$$\rho_R(q) + A(q) = \limsup_{n\to\infty} \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+ \left(E_n^{(s)} R^n\right)} = \gamma_s(q), \tag{10}$$

$q = 2, 3, \cdots,\ s = 1, 2, 3.$

Proof. By inequalities (7) and (8), $\gamma_3(q) \leq \gamma_2(q) = \gamma_1(q)$, so it is sufficient to show that $\gamma_1(q) \leq \rho_R(q) + A(q) \leq \gamma_3(q)$.

1. $\gamma_1(q) \leq \rho_R(q) + A(q)$. From (1) and Lemma 3 for $\mu(q) > \rho_R(q)$ there exists $r_{\mu(q)} > 1$ such that

$$\begin{aligned} \log^+ \left(E_n^{(1)} R^n\right) &\leq \exp^{[q-2]}(1 - r/R)^{-\mu(q)} \\ &\quad + \log(R/r_\mu)^n + \log\frac{1}{r_\mu - 1}, \quad r_{\mu(q)} < r < R, \ \ n \in N. \end{aligned} \tag{11}$$

Substituting

$$r = R\left[1 - \left(\log^{[q-2]} \frac{\mu(q)}{n}\right)^{\frac{1}{\mu(q)+A(q)}}\right], \tag{12}$$

we get for $q = 2$.

$$\log^+ \left(E_n^{(1)} R^n\right) \leq \left(\frac{n}{\mu(2)}\right)^{\frac{\mu(2)}{\mu(2)+1}} - n \log \left[1 - \left(\frac{\mu(2)}{n}\right)^{\frac{1}{\mu(2)+1}}\right]$$
$$- \log(r_{\mu(2)} - 1), \quad n \geq n(\mu(2)).$$

For every $\varepsilon > 0$ and sufficiently large n

$$-\log \left[1 - \left(\frac{\mu(2)}{n}\right)^{\frac{1}{\mu(2)+1}}\right] \leq (1+\varepsilon) \left(\frac{\mu(2)}{n}\right)^{\frac{1}{\mu(2)+1}}$$
$$-\log(r_{\mu(2)} - 1) \leq \varepsilon(\mu(2))^{\frac{1}{\mu(2)+1}} n^{\frac{\mu(2)}{\mu(2)+1}}.$$

Hence

$$\frac{\log^+ \log^+ \left(E_n^{(1)} R^n\right)}{\log n} \leq \frac{\log \left[\mu(2)^{-\frac{\mu(2)}{\mu(2)+1}} + (1+2\varepsilon)\mu(2)^{\frac{1}{\mu(2)+1}}\right]}{\log n} + \frac{\mu(2)}{\mu(2)+1}.$$

Proceeding the limit as $n \to \infty$ we get

$$\frac{\gamma_1(2) - 1}{\gamma_1(2)} \leq \frac{\mu(2)}{\mu(2)+1}.$$

Since $\rho_R(2) < \mu(2)$ is arbitrary, it gives

$$\frac{\gamma_1(2) - 1}{\gamma_1(2)} \leq \frac{\rho_R(2)}{\rho_R(2)+1}.$$

or

$$\gamma_1(2) \leq \rho_R(2) + 1.$$

For $q = 3, 4, \cdots$, (11) and (12) together give for $n \geq n(\mu(q))$

$$\log^+(E_n^{(1)} R^n) < \exp^{[q-2]} \left(\log^{[q-2]} \left(\frac{n}{\mu(q)}\right)\right) [1 + 0(1)].$$

After a simple calculation, we get

$$\mu(q) \geq \limsup_{n\to\infty} \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+(E_n^{(1)} R^n)} = \gamma_1(q).$$

By the arbitrariness of $\mu(q) > \rho_R(q)$, we obtain $\gamma_1(q) \leq \rho_R(q)$.

2. $\rho_R(q) + A(q) \leq \gamma_3(q)$. We shall prove this inequality by contradiction. Suppose that $\rho_R(q) + A(q) > \gamma_3(q)$ then there exists $\rho_R(q) + A(q) > \alpha(q)$ such that $\alpha(q) > \gamma_3(q)$, so

$$\frac{\log^{[q-1]} n}{\log n - \log^+ \log^+(E_n^{(3)} R^n)} \leq \alpha(q)$$

for sufficiently large n. Thus

$$\log^+(E_n^{(3)}R^n) \leq \frac{n}{(\log^{[q-2]} n)^{1/\alpha(q)}}. \tag{13}$$

Using Lemma 2 for polynomials $p_n = l_n - l_{n-1}$, $n \geq 1$, $p_0 = l_0$, we get

$$\rho_R(q) + A(q) \leq \alpha(q).$$

Since $\alpha(q) > \gamma_3(q)$ is arbitrary it follows that $\rho_R(q) + A(q) \leq \gamma_3(q)$. Hence we get a contradiction. Hence the proof of the theorem is completed. ■

Remark 3. For $q = 2$ and $N = 1$ the above Theorem 1 gives a theorem of Juneja and Kapoor ([4], Thm. 4.5.5, pp. 238) as a particular case. It has been noticed that if any two functions $f \epsilon D(R)$ have the same $q-$order then the above theorem does not give a precise information about their comparative rates of growth. For this purpose we have the following theorem.

Theorem 2. *If a function $g \epsilon D(R)$ has a positive finite $q-$order $\rho_R(q)$ and a finite $q-$type $\sigma_R(q)$, then*

$$\lambda_s(q) = \limsup_{n\to\infty} (\log^{[q-2]} n)\left(\frac{\log^+(E_n^{(s)}R^n)}{n}\right)^{\rho_R(q)+A(q)} = \sigma_R(q)\beta_R(q).$$

Proof. By inequalities (7) and (8), $\lambda_3(q) \leq \lambda_2(q) = \lambda_1(q)$, so it suffices to show that $\lambda_1(q) \leq B_R(q)\sigma_R(q) \leq \lambda_3(q)$ for $q = 2$ and $\lambda_1(q) \leq \sigma_R(q) \leq \lambda_3(q)$ for $q = 3, 4, \cdots$.

By Definition 2, for every $w(q) > \sigma_R(q)$ there exists $r_{w(q)} > 1$ such that

$$M(r) \leq \exp^{[q-1]}\left[w(q)(1 - r/R)^{-\rho_R(q)}\right], \qquad r \epsilon (r_{w(q)}, R).$$

Using Lemma 3, we get

$$E_n^{(1)}R^n \leq \left(\frac{R}{r}\right)^n \exp^{[q-1]}\left[w(q)(1 - r/R)^{-\rho_R(q)}\right]\frac{1}{r_{w(q)} - 1}, \qquad n \in N. \tag{14}$$

For $q = 2$, let r be given by

$$r = R\left[1 - \left(\frac{\rho_R(2)w(2)}{n}\right)^{1/\rho_R(2)+1}\right]$$

then (14) gives for $n > n(w(2))$ that

$$\log^+(E_n^{(1)}R^n) \leq w(2)(w(2)\rho_R(2))^{-\frac{\rho_R(2)}{\rho_R(2)+1}} n^{\frac{\rho_R(2)}{\rho_R(2)+1}} - n\log\left[1 - \left(\frac{w(2)\rho_R(2)}{n}\right)^{\frac{1}{\rho_R(2)+1}}\right] + \log\frac{1}{r_{w(2)-1}}.$$

For every $\varepsilon > 0$ and sufficiently large value of n, we get

$$\begin{aligned}\log^+(E_n^{(1)}R^n) \leq w(2)(w(2)\rho_R(2))^{-\frac{\rho_R(2)}{\rho_R(2)+1}} n^{\frac{\rho_R(2)}{\rho_R(2)+1}} \\ + n(1+\varepsilon)\left(\frac{w(2)\rho_R(2)}{n}\right)^{\frac{1}{\rho_R(2)+1}} \\ + \varepsilon(w(2)\rho_R(2))^{\frac{1}{\rho_R(2)+1}}\ n^{\frac{\rho_R(2)}{\rho_R(2)+1}}.\end{aligned}$$

Proceeding $n \to \infty, \varepsilon \to 0$ and $w(2) \to \sigma_R(2)$ we get

$$\limsup_{n\to\infty}(n)\left(\frac{\log^+(E_n^{(1)}R^n)}{n}\right)^{\rho_R(2)+1} \leq \frac{(1+\rho_R(2))^{1+\rho_R(2)}}{(\rho_R(2))^{\rho_R(2)}}\sigma_R(2)$$

or

$$\lambda_1(2) \leq B_R(2)\sigma_R(2).$$

For $q = 3, 4, \cdots$. Choosing r such that

$$r = R\left[1 - \left(\frac{w(q)}{\log^{[q-2]}(n/\rho_R(q))}\right)^{1/\rho_R(q)}\right].$$

Substituting r in (14), we obtain

$$\begin{aligned}\log^+(E_n^{(1)}R^n) &\leq -n\log\left[1 - \left(\frac{w(q)}{\log^{[q-2]}(n/\rho_R(q))}\right)^{1/\rho_R(q)}\right] \\ &\quad + \frac{n}{\rho_R(2)} + \log\frac{1}{r_{w(q)} - 1} \\ &\leq n(1+\varepsilon)\left(\frac{w(q)}{\log^{[q-2]}(n/\rho_R(q))}\right)^{\rho_R(q)} \\ &\quad + \frac{n}{\rho_R(q)} + \varepsilon(w(q)\rho_R(q))^{\frac{1}{\rho_R(q)}}n.\end{aligned}$$

or

$$\log^{[q-2]} n^{(1+0(1))}\left(\frac{\log^+(E_n^{(1)}R^n)}{n}\right)^{\rho_R(q)} \leq w(q)(1+0(1)).$$

Proceeding to limits as $n \to \infty, \varepsilon \to 0$ and $w(q) \to \sigma_R(q)$ we get

$$\lambda_1(q) \leq \sigma_R(q).$$

Now we have to prove that $B_R(q)\sigma_R(q) \leq \lambda_3(q)$ for $q = 2$ and $\sigma_R(q) \leq \lambda_3(q)$ for $q = 3, 4, \cdots$.

Suppose that $\lambda_3(q) < B_R(q)\sigma_R(q)$ for $q = 2$. Then there exists $T_R(2) < \sigma_R(2)$ such that $\lambda_3(2) < B_R(2)T_R(2)$, so

$$n\left(\frac{\log^+(E_n^{(3)}R^n)}{n}\right)^{\rho_R(2)+1} \le B_R(2)T_R(2), \quad \text{for sufficiently large } n.$$

Thus

$$E_n^{(3)} \le R^{-n}\exp\left\{(B_R(2)T_R(2))^{1/\rho_R(2)+1}.n^{\frac{\rho_R(2)}{\rho_R(2)+1}}\right\}.$$

By Lemma 2, $\sigma_R(2) \le B_R(2)T_R(2)$ which is a contradiction. Hence

$$B_R(2)\sigma_R(2) \le \lambda_3(2).$$

For $q = 3, 4, \cdots$, suppose that $\lambda_3(q) < \sigma_R(q)$. Then there exists $T_R(q) < \sigma_R(q)$ such that $\lambda_3(q) < T_R(q)$, so

$$(\log^{[q-2]} n)\left(\frac{\log^+(E_n^{(3)}R^n)}{n}\right)^{\rho_R(q)} \le T_R(q)$$

provided n is sufficiently large. Thus

$$E_n^{(3)} \le R^{-n}\exp\left\{n\left(\frac{T_R(q)}{\log^{[q-2]} n}\right)^{1/\rho_R(q)}\right\}.$$

Therefore by Lemma 2, $T_R(q) \ge \sigma_R(q)$ and we et a contradiction, because $T_R(q)$ has been chosen less than $\sigma_R(q)$. Hence $\sigma_R(q) \le \lambda_3(q)$.
Hence the proof is completed. ■

Remark 4. For $q = 2$ and $N = 1$ the Theorem 2 gives a Theorem 4.5.6. of Juneja and Kapoor ([4], pp. 238) as a particular case.

4. Convergence of sequence of errors

Now we shall show how the speed of convergence to 0 of the sequence $(E_n^{(s)}(f,K))_{n\epsilon N}$ estimates the set on which the function f can be extended analytically and determines the growth of this extension.

Theorem 3. *Given a function f, defined and bounded on K, set*

$$\alpha^*(q) = \limsup_{n\to\infty}\frac{\log^{[q-1]} n}{\log n - \log^+\log^+(E_n^{(1)}R^n)}.$$

If $\alpha^*(q)\epsilon(0,\infty)$, then the function $\widetilde{f} = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$ belongs to $D(R)$, $\widetilde{f}|_K = f$ and $\widetilde{f}$ has the $q-$order $\rho_R(q) = \alpha^*(q) - A(q)$.

Proof. For every $\mu^*(q)\epsilon(\alpha^*(q),\infty)$

$$\frac{\log^{[q-1]} n}{\log n - \log^+ \log^+(E_n^{(1)} R^n)} \leq \mu^*(q)$$

provided n is sufficiently large. Hence

$$E_n^{(1)} R^n \leq \exp\left\{\frac{n}{(\log^{[q-2]} n)^{1/\mu^*(q)}}\right\} \tag{15}$$

or

$$\limsup_{n\to\infty} (E_n^{(1)} R^n)^{1/n} \leq 1.$$

Since $\alpha^*(q) > 0$, the sequence $(E_n^{(1)} R^n)_{n\epsilon N}$ is unbounded, which gives that

$$\limsup_{n\to\infty} (E_n^{(1)} R^n)^{1/n} \geq 1.$$

Thus

$$\limsup_{n\to\infty} (E_n^{(1)})^{1/n} = \frac{1}{R}.$$

Hence in view of Theorem A, $\widetilde{f}\epsilon D(R)$. Moreover inequalities (15), (7) and (8) give

$$\|l_n - l_{n-1}\| \leq R^{-n} \exp\left\{\frac{2n}{(\log^{[q-2]} n)^{\frac{1}{\mu^*(q)}}}\right\}.$$

Thus by Lemma 2 and Theorem 1 we get $\rho_R(q) = \alpha^*(q) - A(q)$. ■

Theorem 4. *Let f be a function defined and bounded on K. If for some positive and finite $q-$order $\rho_R(q)$*

$$\beta(q) = \limsup_{n\to\infty} (\log^{[q-2]} n)^{\frac{1}{\rho_R(q)+A(q)}} \left(\frac{\log^(E_n^{(1)} R^n)}{n}\right)$$

is finite, then the function $\widetilde{f} = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1})$ belongs to $D(R)$, $\widetilde{f}|_K = f, \rho_R(q)$ is the $q-$order of $\widetilde{f}$ and

$$\sigma_R(q) = \frac{(\beta(q))^{\rho_R(q)+A(q)}}{B_R(q)}$$

is the $q-$type of $\widetilde{f}$.

Proof. Following on the lines of Theorem 2 one may easily prove that $\widetilde{f}\epsilon D_R$. In order to estimate the growth of $\widetilde{f}$ take any $\widehat{\iota}(q) > \beta(q)$. Then

$$(\log^{[q-2]} n)^{\frac{1}{\rho_R(q)+A(q)}} \left(\frac{\log^+(E_n^{(1)} R^n)}{n} \right) \leq \widehat{\iota}(q), \quad n > n(\widehat{\iota}(q))$$

or

$$E_n^{(1)} \leq R^{-n} \exp\left\{ \frac{\widehat{\iota}(q) n}{(\log^{[q-2]} n)^{\frac{1}{\rho_R(q)+A(q)}}} \right\}.$$

Using Inequalities (7) and (8) we get

$$\|l_n - l_{n-1}\| \leq R^{-n} \exp\left\{ \frac{2\widehat{\iota}(q) n}{(\log^{[q-2]} n)^{\frac{1}{\rho_R(q)+A(q)}}} \right\}$$

for n sufficiently large. Hence by Lemma 2, $\widetilde{\rho}_R(q)$ the $q-$order of $\widetilde{f}$, satisfies $\widetilde{\rho}_R(q) \leq \rho_R(q)$. Suppose that $\widetilde{\rho}_R(q) < \rho_R(q)$. Then, in view of Theorem 1 for every $\rho_R^*(q)\epsilon(\widetilde{\rho}_R(q), \rho_R(q))$

$$\rho_R^*(q) + A(q) \geq \frac{\log^{[q-1]} n}{\log n - \log^+ \log^+(E_n^{(1)} R^n)}, \quad n > n'(\rho_R^*(q)).$$

Thus

$$\limsup_{n\to\infty} (\log^{[q-2]} n)^{\frac{1}{\rho_R^*(q)+A(q)}} \left(\frac{\log^+(E_n^{(1)} R^n)}{n} \right) \leq 1.$$

or

$$\begin{aligned} &\limsup_{n\to\infty} (\log^{[q-2]} n)^{\frac{1}{\rho_R^*(q)+A(q)}} \left(\frac{\log^+(E_n^{(1)} R^n)}{n} \right) \\ &\qquad = \limsup_{n\to\infty} (\log^{[q-2]} n)^{\frac{1}{\rho_R^*(q)+A(q)}} \left(\frac{\log^+(E_n^{(1)} R^n)}{n} \right) \\ &\qquad\quad \times n(\log^{[q-2]} n)^{-\frac{1}{\rho_R(q)+A(q)}} \frac{(\log^{[q-2]} n)^{\frac{1}{\rho_R^*(q)+A(q)}}}{n} = \infty. \end{aligned}$$

Which is a contradiction, whence $\widetilde{\rho}_R(q) = \rho_R(q)$. Moreover, by Lemma 2 and Theorem 2

$$\sigma_R(q) = \frac{(\beta(q))^{\rho_R(q)+A(q)}}{B_R(q)}.$$

Hence the proof is completed. ∎

Remark 5. In view of inequalities (7) and (8), one may replace $E_n^{(1)}$ by $E_n^{(2)}$ or $E_n^{(3)}$ in Theorems 3 and 4.

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