F A S C I C U L I M A T H E M A T I C I

Nr 43 2010

Memudu O. Olatinwo

SOME STABILITY RESULTS FOR NONEXPANSIVE AND QUASI-NONEXPANSIVE OPERATORS IN UNIFORMLY CONVEX BANACH SPACE USING TWO NEW ITERATIVE PROCESSES OF KIRK-TYPE

Abstract. In this paper, we examine the stability of Kirk-Ishikawa and Kirk-Mann iteration processes for nonexpansive and quasi-nonexpansive operators in uniformly convex Banach space. To the best of our knowledge, apart from the results of Olatinwo [19], stability of fixed point iteration processes has not been investigated in uniformly convex Banach space. Our results generalize, extend and improve some of the results of Harder and Hicks [11], Rhoades [26, 27], Osilike [23], Berinde [2, 3] as well as Imoru and Olatinwo [12].

Key words: uniformly convex Banach space; Ishikawa iteration process.

AMS Mathematics Subject Classification: 47H06, 54H25.

1. Introduction

Let (E, d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{ p \in E \mid Tp = p \}$ is the set of fixed points of T.

There are several iteration processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iteration process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, \cdots , \tag{1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx, Ty) \le \alpha d(x, y), \ \ \forall \ x, y \in E \ \text{ and } \ \alpha \in [0, 1). \tag{2}$$

Condition (2) is called the *Banach's contraction condition.* Any operator satisfying (2) is called *strict contraction.* Also, condition (2) is significant in the proof of celebrated Banach's fixed point theorem [1].

In the Banach space setting, we shall state some of the iteration processes generalizing (1) as follows:

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0, 1, \cdots, \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iteration process (see Mann [18]).

For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T z_n \\ z_n = (1-\beta_n)x_n + \beta_n T x_n \end{cases} \quad n = 0, 1, \cdots, \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$, is called the Ishikawa iteration process (see Ishikawa [13]). See Berinde [3] for detail on various iteration processes.

Kannan [15] established an extension of the Banach's fixed point theorem by using the following contractive definition: For a selfmap T, there exists $\beta \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right], \quad \forall\, x, y \in E. \tag{5}$$

Chatterjea [6] used the following contractive condition: For a selfmap T, there exists $\gamma \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right], \quad \forall\, x, y \in E. \tag{6}$$

Zamfirescu [35] established a nice generalization of the Banach's fixed point theorem by combining (2), (5) and (6). That is, for a mapping $T : E \to E$, there exist real numbers α, β, γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$, $0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x, y \in E$, at least one of the following is true:

$$\begin{cases} (z_1)\ d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2)\ d(Tx, Ty) \leq \beta \left[d(x, Tx) + d(y, Ty)\right] \\ (z_3)\ d(Tx, Ty) \leq \gamma \left[d(x, Ty) + d(y, Tx)\right]. \end{cases} \tag{7}$$

The mapping $T : E \to E$ satisfying (7) is called the *Zamfirescu contraction.* Any mapping satisfying condition (z_2) of (7) is called a *Kannan mapping*, while the mapping satisfying condition (z_3) is called *Chatterjea operator.* The contractive condition (7) implies

$$||Tx - Ty|| \leq 2\delta ||x - Tx|| + \delta ||x - y||, \quad \forall\, x, y \in E, \tag{8}$$

where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $0 \leq \delta < 1$.

Rhoades [29, 30] used condition (7) to obtain some convergence results for Mann and Ishikawa iteration processes in a uniformly convex Banach space, while Berinde [4] extended the results of [29, 30] to arbitrary Banach space for the same iteration processes.

The following definition of stability of iteration process is due to Harder and Hicks [11].

Definition 1. *Let (E, d) be a complete metric space, $T : E \to E$ a selfmap of E. Suppose that $F_T = \{p \in E \mid Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving T which is defined by*

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, \cdots, \tag{9}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n))$, $n = 0, 1, \cdots$. Then, the iteration procedure (1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$. Since the metric is induced by the norm, we have

$$\epsilon_n = ||y_{n+1} - f(T, y_n)||, \quad n = 0, 1, \cdots,$$

in place of

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, \cdots,$$

in the definition of stability whenever we are working in normed linear space or Banach space.

If in (9),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, \cdots,$$

then we have the Picard iteration process defined in (1), while we obtain the Ishikawa iteration process (4) from (9) if

$$\begin{aligned} f(T, x_n) &= (1 - \alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n = 0, 1, \cdots, \ \alpha_n, \beta_n \in [0, 1]. \end{aligned}$$

Several stability results established in metric space and normed linear space are available in the literature. Some of the various authors whose contributions are of colossal value in the study of stability of the fixed point iteration procedures are Ostrowski [25], Harder and Hicks [11], Rhoades [26, 28], Osilike [23], Osilike and Udomene [24], Jachymski [14], Berinde [2, 3] and Singh et al [34]. Harder and Hicks [11], Rhoades [26, 28], Osilike [23] and Singh et al [34] used the method of the summability theory of infinite matrices to prove various stability results for certain contractive definitions. The method has also been adopted to establish various stability results for certain contractive definitions in Olatinwo et al [20, 21]. Osilike and Udomene [24] introduced a shorter method of proof of stability results and this has also been employed by Berinde [2], Imoru and Olatinwo [12], Olatinwo et al [22] and some others. In Harder and Hicks [11], the contractive definition stated in (2) was used to prove a stability result for the Kirk's iteration process. The first stability result on $T-$ stable mappings was proved by Ostrowski [25] for the Picard iteration using condition (2). In addition to (2), the contractive condition in (9) was also employed by Harder and Hicks [11] to establish some stabilty results for both Picard and Mann iteration processes. Rhoades [26, 28] extended the stability results of [11] to more general classes of contractive mappings. Rhoades [26] extended the results of [11] to the following independent contractive condition: there exists $c \in [0,1)$ such that

$$d(Tx,Ty) \leq c \max\{d(x,y), d(x,Ty), d(y,Tx)\}, \quad \forall\, x,y \in E. \tag{10}$$

Rhoades [28] used the following contractive definition: there exists $c \in [0,1)$ such that

$$d(Tx,Ty) \leq c \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, d(x,Ty), d(y,Tx)\right\}, \tag{11}$$

$\forall\, x,y \in E$.

Moreover, Osilike [23] generalized and extended some of the results of Rhoades [28] by using a more general contractive definition than those of Rhoades [26, 28]. Indeed, he employed the following contractive definition: there exist $a \in [0,1]$, $L \geq 0$ such that

$$d(Tx,Ty) \leq Ld(x,Tx) + ad(x,y), \quad \forall\, x,y \in E. \tag{12}$$

Osilike and Udomene [24] introduced a shorter method to prove stability results for the various iteration processes using the condition (12). Berinde [2] established the same stability results for the same iteration processes using the same set of contractive definitions as in Harder and Hicks [11] but the same method of shorter proof as in Osilike and Udomene [24].

More recently, Imoru and Olatinwo [12] established some stability results which are generalizations of some of the results of [2, 11, 23, 24, 26, 28]. In Imoru and Olatinwo [12], the following contractive definition was employed: there exist $a \in [0,1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that

$$d(Tx,Ty) \leq \varphi(d(x,Tx)) + ad(x,y), \quad \forall\, x,\ y \in E. \tag{13}$$

Condition (13) was also employed in Olatinwo et al [20] to establish some stability results in normed linear space setting with additional condition of continuity imposed on φ. In the next section, we shall state our new iteration processes, some remarks, definition and lemmas which are required in the sequel.

2. Preliminaries

We shall employ the following iteration processes: Let E be a Banach space, $T : E \to E$ a selfmap of E and $x_0 \in E$. Then, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$(14) \qquad \begin{cases} x_{n+1} = \alpha_{n,0}x_n + \sum\limits_{i=1}^{k} \alpha_{n,i}T^i z_n, & \sum\limits_{i=0}^{k} \alpha_{n,i} = 1, \quad n = 0, 1, 2, \cdots, \\ z_n = \sum\limits_{j=0}^{s} \beta_{n,j}T^j x_n, & \sum\limits_{j=0}^{s} \beta_{n,j} = 1, \end{cases}$$

$k \geq s$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i} \geq 0$, $\beta_{n,0} \neq 0$, $\alpha_{n,i}$, $\beta_{n,j} \in [0, 1]$, where k and s are fixed integers.

If $s = 0$ in (14), we also obtain the following interesting iteration process in a Banach space:

$$(15) \qquad x_{n+1} = \sum_{i=0}^{k} \alpha_{n,i}T^i x_n, \quad \sum_{i=0}^{k} \alpha_{n,\ i} = 1, \quad n = 0, 1, 2, \cdots,$$

$\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0, 1]$, where k is a fixed integer.

The iteration process defined in (14) will be called the *Kirk-Ishikawa* iteration process, while that of (15) will be called the *Kirk-Mann* iteration process.

(i) If $s = 0$, $k = 1$ in (14), then we have $z_n = \beta_{n,0}x_n = x_n$, $\beta_{n,0} = 1$ and

$$x_{n+1} = (1 - \alpha_{n,1})x_n + \alpha_{n,1}Tx_n,$$

which is the usual Mann iteration process with $\sum\limits_{i=0}^{1} \alpha_{n,i} = 1$, $\alpha_{n,1} = \alpha_n$.

(ii) Also, if $s = k = 1$, in (14), we get

$$\begin{cases} x_{n+1} = (1 - \alpha_{n,1})x_n + \alpha_{n,1}Tz_n \\ z_n = (1 - \beta_{n,1})x_n + \beta_{n,1}Tx_n, \end{cases}$$

which is the usual Ishikawa iteration process with $\sum\limits_{i=0}^{1} \alpha_{n,i} = \sum\limits_{i=0}^{1} \beta_{n,i} = 1$, $\alpha_{n,1} = \alpha_n$, $\beta_{n,1} = \beta_n$.

(iii) If $s = 0$ and $\alpha_{n,i} = \alpha_i$ in (14), then we obtain the usual Kirk's iteration process

$$(16) \qquad x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n, \quad \sum_{i=0}^{k} \alpha_i = 1, \quad n = 0, 1, 2, \cdots,$$

with $z_n = \beta_{n,0}x_n = x_n$, since $\beta_{n,0} = 1$.

Eq. (15) is also a generalization of Picard, Schaefer, Mann and the Kirk's iteration processes. See Berinde [3] for detail on the various already existing fixed point iteration processes.

Remark 1. If $\alpha = 1$ in (2), then the mapping $T : E \to E$ is called *nonexpansive*, that is,

$$||Tx - Ty|| \leq ||x - y||, \quad \forall\, x, y \in E. \tag{17}$$

We shall use both condition (17) and the following contractive definition to obtain our results: there exist $a \in [0, 1)$ and a subadditive monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = 0$ and $\forall\, x, y \in X$, we have

$$||Tx - Ty|| \leq \varphi(||x - Tx||) + a||x - y||. \tag{18}$$

Remark 2. Since metric is induced by norm, then the contractive condition (18) is a reformulation of (13) in terms of norm.

Definition 2. *A Banach space $(E, ||\cdot||)$ is called uniformly convex if, given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x,\ y \in E$ with $||x|| \leq 1$, $||y|| \leq 1$, and $||x - y|| \geq \epsilon$, we have $\frac{1}{2}||x + y|| \leq 1 - \delta$.*

Lemma 1 (Berinde [2, 3]). *If δ is a real number such that $0 \leq \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \quad n = 0, 1, \cdots,$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Lemma 2 (Groetsch [10]). *Let X be a uniformly convex Banach space and let $x, y \in X$. If $||x|| \leq 1$, $||y|| \leq 1$ and $||x - y|| \geq \epsilon > 0$, then $||\lambda x + (1 - \lambda)y|| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$ for $0 \leq \lambda < 1$.*

Lemma 3 (Olatinwo et al [22]). *Let $(E, ||\cdot||)$ be a normed linear space and let $T : E \to E$ be a selfmap of E satisfying (18). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a subadditive, monotone increasing function such that $\varphi(0) = 0$, $\varphi(Lu) \leq L\varphi(u)$, $L \geq 0$, $u \in \mathbb{R}_+$. Then, $\forall\, i \in \mathbb{N}$ and $\forall\, x, y \in E$,*

$$||T^i x - T^i y|| \leq \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(||x - Tx||) + a^i ||x - y||. \tag{19}$$

The contractive condition (18) is more general than a number of other contractive conditions in the following sense:

If in (18), $\varphi(u) = Lu$, $L \geq 0$, $\forall\ u \in I\!R_+$, then we obtain the condition (12) employed in [23, 24].

If $\varphi(u) = 0$, $\forall\ u \in I\!R_+$, then (18) reduces to the Banach's contraction condition (2). In a similar manner, condition (18) reduces to the Zamfirescu contractive condition. The Banach's contraction condition and the Zamfirescu contractive condition were used in [2, 11] to obtain stability results for both the Picard and Mann iteration processes.

For a nonexpansive operator $T, F_T \neq \phi$ is not generally true. A generalization of a nonexpansive operator, with at least one fixed point, is that of the quasi-nonexpansive operators.

Definition 3. *An operator $T : E \to E$ is said to be quasi-nonexpansive if T has at least one fixed point in E and, for each fixed point $p \in F_T$, we have $||Tx - p|| \leq ||x - p||$, $\forall\ x \in E$.*

Remark 3. (i) If T is an operator satisfying (18), then T is a (strict) quasi-nonexpansive operator. That is, from (18) with $p \in F_T$, we have

$$||Tx - p|| = ||Tp - Tx|| \leq a||p - x|| + \varphi(||p - Tp||) < ||x - p||.$$

(ii) The Zamfirescu and Kannan operators are also nice examples of quasi-nonexpansive operators. See [3, 5, 15, 35] for detail on the Zamfirescu and Kannan operators.

(iii) It follows from the Remark 3 (i) that if T is a quasi-nonexpansive operator satisfying (18), then

$$||T^i x - p|| < ||x - p||, \quad \forall\ x, y \in E \ \text{ and } \ \mathrm{p} \in \mathrm{F_T}. \tag{20}$$

That is, from condition (18) and Lemma 3, we have that

$$\begin{aligned} ||T^i x - p|| &= ||T^i x - T^i p|| = ||T^i p - T^i x|| \\ &\leq \sum_{j=1}^{i} \binom{i}{j} a^{i-j} \varphi^j(||p - Tp||) + a^i ||p - x|| \\ &= a^i ||x - p|| < ||x - p||. \end{aligned}$$

See Berinde [3, 5] and Olatinwo [19] for detail on nonexpansive and quasi-nonexpansive operators.

It is our purpose in this paper to examine the stability of two new iteration processes for nonexpansive and quasi-nonexpansive operators in uniformly convex Banach space. Our results are improvements, generalizations and extensions of some of the results of Harder and Hicks [11], Rhoades [26, 27], Osilike [23], Osilike and Udomene [24], Berinde [2, 3] as well as Imoru and Olatinwo [12].

3. Main results

The following are stability results for the Kirk-Ishikawa and Kirk-Mann type iteration processes in uniformly convex Banach space.

Theorem 1. *Let E be a closed convex subset of a uniformly convex Banach space X and $T : E \to E$ a nonexpansive operator. Suppose that T has at least a fixed point p. Let $x_0 \in E$ and let $\{x_n\}_{n=0}^{\infty}$ be the Kirk-Ishikawa iteration process defined by (14), where $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,j}\}_{n=0}^{\infty} \subset [0,1]$ such that $\alpha = \min(\alpha_{n,0},\ 1-\alpha_{n,0})$, $n = 0,1,\cdots,$ and $\alpha^2\delta(\epsilon) \geq \frac{k}{2}$, $0 < \alpha < 1$, $0 < k < 1$. Then, the Kirk-Ishikawa iteration process is T-stable.*

Proof. We proceed as follows using by using the nonexpansiveness condition (17): Suppose that $||y_n - p|| \neq 0$, $s_n = \sum_{j=0}^{s} \beta_{n,j} T^j y_n$, $\sum_{j=0}^{s} \beta_{n,j} = 1$, $u_n = \frac{y_n - p}{||y_n - p||}$ and $v_{n,i} = \frac{\sum_{i=1}^{k} \alpha_{n,i}(T^i s_n - T^i p)}{(1-\alpha_{n,0})||y_n - p||}$.

Then, we have

$$||u_n|| = ||\left(\frac{y_n - p}{||y_n - p||}\right)|| \leq \frac{||y_n - p||}{||y_n - p||} = 1$$

and

$$\begin{aligned} ||v_{n,i}|| &= ||\left(\frac{\sum_{i=1}^{k} \alpha_{n,i}(T^i s_n - T^i p)}{(1-\alpha_{n,0})||y_n - p||}\right)|| \leq \frac{\sum_{i=1}^{k} \alpha_{n,i}||s_n - p||}{(1-\alpha_{n,0})||y_n - p||} \\ &= \frac{\sum_{i=1}^{k} \alpha_{n,i}||\sum_{j=0}^{s} \beta_{n,j}(T^j y_n - T^j p)||}{(1-\alpha_{n,0})||y_n - p||} \\ &\leq \frac{\sum_{i=1}^{k} \alpha_{n,i}\sum_{j=0}^{s} \beta_{n,j}||y_n - p||}{(1-\alpha_{n,0})||y_n - p||} = 1, \end{aligned}$$

since $\sum_{i=1}^{k} \alpha_{n,i} = 1 - \alpha_{n,0}$.

Now let $\epsilon_n = ||y_{n+1} - \alpha_{n,0} y_n - \sum_{i=1}^{k} \alpha_{n,i} T^i s_n||$, $n = 0,1,\cdots,$ and suppose $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$, using the triangle inequality as follows: Therefore, we have

$$\begin{aligned} ||y_{n+1} - p|| &\leq ||y_{n+1} - \alpha_{n,0} y_n - \sum_{i=1}^{k} \alpha_{n,i} T^i s_n|| \\ &\quad + ||\alpha_{n,0} y_n + \sum_{i=1}^{k} \alpha_{n,i} T^i s_n - \sum_{i=0}^{k} \alpha_{n,i} T^i p|| \\ &= ||\alpha_{n,0}(y_n - p) + \sum_{i=1}^{k} \alpha_{n,i}(T^i s_n - T^i p)|| + \epsilon_n \\ &= ||(||y_n - p||)[\alpha_{n,0} u_n + (1-\alpha_{n,0}) v_{n,i}]|| + \epsilon_n \\ &\leq ||\alpha_{n,0} u_n + (1-\alpha_{n,0}) v_{n,i}||\ ||y_n - p|| + \epsilon_n. \end{aligned} \tag{21}$$

By using Lemma 2 in (21) as well as applying the condition on $\alpha_{n,0}$ and $1-\alpha_{n,0}$, we obtain that

$$\begin{aligned} ||y_{n+1}-p|| &\leq [1-2\alpha_{n,0}(1-\alpha_{n,0})\delta(\epsilon)]||y_n-p||+\epsilon_n \\ &\leq [1-2\alpha^2\delta(\epsilon)]||y_n-p||+\epsilon_n \\ &\leq (1-k)||y_n-p||+\epsilon_n. \end{aligned}$$

Since $0\leq 1-k<1$, $0<k<1$, using Lemma 1 in (22) yields $\lim\limits_{n\to\infty}||y_n-p||=0$, that is, $\lim\limits_{n\to\infty} y_n=p$.

Conversely, let $\lim\limits_{n\to\infty} y_n = p$. Then, we have by the triangle inequality and Lemma 2 that

$$\begin{aligned} \epsilon_n &= ||y_{n+1}-\alpha_{n,0}y_n-\sum_{i=1}^{k}\alpha_{n,i}T^i s_n|| \\ &\leq ||y_{n+1}-p||+||\sum_{i=0}^{k}\alpha_{n,i}T^i p-\alpha_{n,0}y_n-\sum_{i=1}^{k}\alpha_{n,i}T^i s_n|| \\ &= ||y_{n+1}-p||+||(||y_n-p||)[\alpha_{n,0}u_n+(1-\alpha_{n,0})v_{n,i}]|| \\ &\leq ||y_{n+1}-p||+||\alpha_{n,0}u_n+(1-\alpha_{n,0})v_{n,i}||\;||y_n-p|| \\ &\leq ||y_{n+1}-p||+[1-2\alpha_{n,0}(1-\alpha_{n,0})\delta(\epsilon)]||y_n-p|| \\ &\leq ||y_{n+1}-p||+[1-2\alpha^2\delta(\epsilon)]||y_n-p|| \\ &\leq ||y_{n+1}-p||+(1-k)||y_n-p||\to 0 \text{ as } \mathrm{n}\to\infty. \end{aligned}$$

■

Theorem 2. *Let E be a closed convex subset of a uniformly convex Banach space X and $T: E\to E$ a quasi-nonexpansive operator satisfying (18). Suppose that T has at least a fixed point p. Let $x_0\in E$ and let $\{x_n\}_{n=0}^{\infty}$ be the Kirk-Ishikawa iteration process defined by (14), where $\{\alpha_{n,i}\}_{n=0}^{\infty}$, $\{\beta_{n,j}\}_{n=0}^{\infty}\subset[0,1]$ such that $\alpha=\min(\alpha_{\mathrm{n},0},\ 1-\alpha_{n,0})$, $n=0,1,2,\cdots$, and $\alpha^2\delta(\epsilon)\geq\frac{k}{2}$, $0<\alpha<1$, $0<k<1$. Suppose also that $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a monotone increasing function such that $\varphi(0)=0$. Then, the Kirk-Ishikawa iteration process is T-stable.*

Proof. We proceed as follows using the Remark 3 (*iii*): Suppose that $||y_n-p||\neq 0$, $s_n=\sum_{j=0}^{s}\beta_{n,j}T^j y_n$, $\sum_{j=0}^{s}\beta_{n,j}=1$, $u_n=\frac{y_n-p}{||y_n-p||}$ and $v_{n,i}=\frac{\sum_{i=1}^{k}\alpha_{n,i}(T^i s_n-T^i p)}{(1-\alpha_{n,0})||y_n-p||}$.

Then, we have $||u_n||\leq 1$ and

$$||v_{n,i}|| = ||\left(\frac{\sum_{i=1}^{k}\alpha_{n,i}(T^i s_n-T^i p)}{(1-\alpha_{n,0})||y_n-p||}\right)|| \tag{22}$$

$$
\begin{aligned}
&\leq \frac{\sum_{i=1}^{k}\alpha_{n,i}||T^i s_n - T^i p||}{(1-\alpha_{n,0})||y_n - p||} < \frac{\sum_{i=1}^{k}\alpha_{n,i}||s_n - p||}{(1-\alpha_{n,0})||y_n - p||} \\
&\leq \frac{\sum_{i=1}^{k}\alpha_{n,i}\sum_{j=0}^{s}\beta_{n,j}||T^j y_n - T^j p||}{(1-\alpha_{n,0})||y_n - p||} \\
&= \frac{\sum_{i=1}^{k}\alpha_{n,i}\left[\sum_{j=1}^{s}\beta_{n,j}||T^j y_n - T^j p|| + \beta_{n,0}||y_n - p||\right]}{(1-\alpha_{n,0})||y_n - p||} \\
&< \frac{\sum_{i=1}^{k}\alpha_{n,i}\left[\sum_{j=1}^{s}\beta_{n,j}||y_n - p|| + \beta_{n,0}||y_n - p||\right]}{(1-\alpha_{n,0})||y_n - p||} \\
&= \frac{\sum_{i=1}^{k}\alpha_{n,i}\sum_{j=0}^{s}\beta_{n,j}||y_n - p||}{(1-\alpha_{n,0})||y_n - p||} = 1,
\end{aligned}
$$

since $\sum_{i=1}^{k}\alpha_{n,i} = 1-\alpha_{n,0}$ and $\sum_{j=0}^{s}\beta_{n,j} = 1$. ■

The second part of the proof of this theorem is the same as that of Theorem 3.1.

Remark 4. *Theorem 1 is a generalization and extension of Theorem 3.1 of Olatinwo [19] while Theorem 2 is a generalization and extension of Theorem 3.2 of Olatinwo [19], results of the author in [12, 20, 21, 22], Theorem 2 and Theorem 3 of Harder and Hicks [11], some results of Rhoades [26, 27, 28], Theorem 1, Theorem 2 and Theorem 3 of Osilike [23], Theorem 4 and Theorem 5 of Osilike and Udomene [24] as well as the results of Berinde [2].*

Remark 5. *If $s = 0$ and $k = 1$ in iteration process of (14), then Theorem 1 and Theorem 2 reduce to stability results for the Mann iteration process in uniformly convex Banach space for nonexpansive and quasi-nonexpansive operators. In a similar manner, we can obtain stability results in uniformly convex Banach space for some other celebrated fixed point iteration processes using more general contractive definitions.*

We now establish the following stability results in uniformly convex Banach space for the Kirk-Mann iteration process defined in (15).

Theorem 3. *Let E be a closed convex subset of a uniformly convex Banach space X and $T : E \to E$ a nonexpansive operator. Suppose that T has at least a fixed point p. Let $x_0 \in E$ and let $\{x_n\}_{n=0}^{\infty}$ be the Kirk-Mann iteration process defined by (15), where $\{\alpha_{n,i}\}_{n=0}^{\infty} \subset [0,1]$ such that $\alpha = \min(\alpha_{n,0},\ 1-\alpha_{n,0})$, $n = 0,1,2,\cdots$, and $\alpha^2\delta(\epsilon) \geq \frac{k}{2}$, $0 < \alpha < 1$, $0 < k < 1$. Then, the Kirk-Mann iteration process is T-stable.*

Proof. We proceed as follows by using the nonexpansiveness condition (17): Suppose that $||y_n - p|| \neq 0$, $u_n = \frac{y_n - p}{||y_n - p||}$ and $v_{n,i} = \frac{\sum_{i=1}^k \alpha_{n,i}(T^i y_n - T^i p)}{(1-\alpha_{n,0})||y_n - p||}$. Then, we have $||u_n|| \leq 1$ and $||v_{n,i}|| \leq 1$, since $\sum_{i=1}^k \alpha_{n,i} = 1 - \alpha_{n,0}$.

Now let $\epsilon_n = ||y_{n+1} - \sum_{i=0}^k \alpha_{n,i} T^i y_n||$, $n = 0, 1, \cdots$, and suppose $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$, using the triangle inequality as follows: Therefore, we have

$$
\begin{aligned}
(23) \qquad ||y_{n+1} - p|| &\leq ||y_{n+1} - \sum_{i=0}^k \alpha_{n,i} T^i y_n|| \\
&\quad + ||\sum_{i=0}^k \alpha_{n,i} T^i y_n - \sum_{i=0}^k \alpha_{n,i} T^i p|| \\
&= ||\alpha_{n,0}(y_n - p) + \sum_{i=1}^k \alpha_{n,i}(T^i y_n - T^i p)|| + \epsilon_n \\
&= ||(||y_n - p||)[\alpha_{n,0} u_n + (1 - \alpha_{n,0}) v_{n,i}]|| + \epsilon_n \\
&\leq ||\alpha_{n,0} u_n + (1 - \alpha_{n,0}) v_{n,i}|| \; ||y_n - p|| + \epsilon_n.
\end{aligned}
$$

By using Lemma 2 in (23) as well as applying the condition on $\alpha_{n,0}$ and $1 - \alpha_{n,0}$, we obtain that

$$
\begin{aligned}
(24) \quad ||y_{n+1} - p|| &\leq [1 - 2\alpha_{n,0}(1 - \alpha_{n,0})\delta(\epsilon)]||y_n - p|| + \epsilon_n \\
&\leq [1 - 2\alpha^2 \delta(\epsilon)]||y_n - p|| + \epsilon_n \leq (1 - k)||y_n - p|| + \epsilon_n.
\end{aligned}
$$

Since $0 \leq 1 - k < 1$, $0 < k < 1$, using Lemma 1 in (24) yields $\lim_{n\to\infty} ||y_n - p|| = 0$, that is, $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, we have by the triangle inequality and Lemma 2 that

$$
\begin{aligned}
\epsilon_n &= ||y_{n+1} - \sum_{i=0}^k \alpha_{n,i} T^i y_n|| \\
&\leq ||y_{n+1} - p|| + ||\sum_{i=0}^k \alpha_{n,i} T^i p - \sum_{i=0}^k \alpha_{n,i} T^i y_n|| \\
&\leq ||y_{n+1} - p|| + ||\alpha_{n,0} u_n + (1 - \alpha_{n,0}) v_{n,i}|| \; ||y_n - p|| \\
&\leq ||y_{n+1} - p|| + [1 - 2\alpha^2 \delta(\epsilon)] \; ||y_n - p|| \\
&\leq ||y_{n+1} - p|| + (1 - k)||y_n - p|| \to 0 \quad \text{as n} \to \infty.
\end{aligned}
$$

■

Theorem 4. *Let E be a closed convex subset of a uniformly convex Banach space X and $T : E \to E$ a quasi-nonexpansive operator satisfying (18). Suppose that T has at least a fixed point p. Let $x_0 \in E$ and let $\{x_n\}_{n=0}^{\infty}$ be the Kirk-Mann iteration process defined by (15), where $\{\alpha_{n,i}\}_{n=0}^{\infty} \subset [0,1]$ such that $\alpha = \min(\alpha_{n,0}, 1-\alpha_{n,0})$, $n = 0, 1, \cdots$, and $\alpha^2\delta(\epsilon) \geq \frac{k}{2}$, $0 < \alpha < 1$, $0 < k < 1$. Suppose also that $\varphi : I\!R_+ \to I\!R_+$ is a monotone increasing function such that $\varphi(0) = 0$. Then, the Kirk-Mann iteration process is T-stable.*

Proof. We proceed as follows using the Remark 3 (iii): Suppose that $||y_n - p|| \neq 0$, $u_n = \frac{y_n - p}{||y_n - p||}$ and $v_{n,i} = \frac{\sum_{i=1}^{k} \alpha_{n,i}(T^i y_n - T^i p)}{(1-\alpha_{n,0})||y_n - p||}$. Then, we have $||u_n|| \leq 1$ and $||v_{n,i}|| \leq 1$, since $\sum_{i=1}^{k} \alpha_{n,i} = 1 - \alpha_{n,0}$. ■

The second part of the proof of this theorem is the same as that of Theorem 3.

Remark 6. To the best of our knowledge, Theorem 3 is new, while Theorem 4 is a generalization and extension of Theorem 3.2 of Olatinwo [19], the results of [12, 20, 21, 22], Theorem 2 and Theorem 3 of Harder and Hicks [11], some results of Rhoades [26, 27, 28], Theorem 1 of Osilike [23], Theorem 1 of Osilike and Udomene [24] as well as the results of Berinde [2].

Remark 7. *To the best of our knowledge, apart from the results of Olatinwo [19], stability of fixed point iteration processes has not been investigated in uniformly convex Banach space. We therefore, claim that all the stability results established in this paper and Olatinwo [19] are new and original.*

Example 1. The iteration process defined in (15) has been employed to approximate the fixed point for nonexpansive mapping. For instance, the fixed point of the nonexpansive mapping $T : [0,1] \to [0,1]$ defined by

$$Tx = 1 - x, \quad x \in [0,1],$$

has been obtained by both Kirk's iteration process and the iteration process defined in (15) for the same choice of initial point $x_0 \in [0,1]$. The fixed point of T is given by $F_T = \{\frac{1}{2}\}$. Specifically, Kirk's iteration process converges to the fixed point of T after seven iterations when $x_0 = 0.6$, $k = 2$ and $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{3}$.

However, our new iteration process defined in (15) converges to the same fixed point of T after the second iteration when $x_0 = 0.6$, $k = 2$ and $\alpha_{n,0} = 1 - \frac{1}{n+1} - \frac{1}{n+1}$, $\alpha_{n,1} = \alpha_{n,2} = \frac{1}{n+1}$. Hence, our iteration process converges faster to the fixed point of T than the Kirk's iteration process.

The operator T in (15) may not even be nonexpansive. This is an advantage over the Kirk iteration process.

Also, our new scheme converges faster than Mann iterative process too.

References

[1] Banach S., Sur les Operations dans les Ensembles Abstraits et leur Applications aux Equations Integrales, *Fund. Math.,* 3(1922), 133-181.

[2] Berinde V., On the stability of some fixed point procedures, *Bul. Stiint. Univ. Baia Mare, Ser. B, Matematica-Informatica,* Vol. XVIII (1)(2002), 7-14.

[3] Berinde V., *Iterative Approximation of Fixed Points,* Editura Efemeride, 2002.

[4] Berinde V., On the convergence of the Ishikawa iteration in the class of quasi-contractive operators, *Acta Math. Univ. Comenianae,* Vol. LXXIII (1)(2004), 119-126.

[5] Berinde V., *Iterative Approximation of Fixed Points,* Springer-Verlag Berlin Heidelberg, 2007.

[6] Chatterjea S.K., Fixed-point theorems, *C. R. Acad. Bulgare Sci.,* 10(1972), 727-730.

[7] Chidume C.E., *Geometric Properties of Banach Spaces and Nonlinear iterations,* The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, October 2000.

[8] Ciric Lj.B., Generalized contractions and fixed point theorems, *Publ. Inst. Math. (Beograd) (N. S.),* 12(26)(1971), 19-26.

[9] Ciric Lj.B., A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.,* 45(1974), 267-273.

[10] Groetsch C.W., A note on segmenting Mann iterates, *J. Math. Anal. Appl.,* 40(1972), 369-372.

[11] Harder A.M., Hicks T.L., Stability results for fixed point iteration procedures, *Math. Japonica,* 33(5)(1988), 693-706.

[12] Imoru C.O., Olatinwo M.O., On the stability of Picard and Mann iteration processes, *Carp. J. Math.,* 19(2)(2003), 155-160.

[13] Ishikawa S., Fixed point by a new iteration method, *Proc. Amer. Math. Soc.,* 44(1)(1974), 147-150.

[14] Jachymski J.R., An extension of A. Ostrowski's theorem on the round-off stability of iterations, *Aequ. Math.,* 53(1997), 242-253.

[15] Kannan R., Some results on fixed points, *Bull. Calcutta Math. Soc.,* 10(1968), 71-76.

[16] Liu L., Fixed points of local strictly pseudo-contractive mappings using Mann and Ishikawa iteration with errors, *Indian J. Pure Appl. Math.,* 26(7)(1995), 649-659.

[17] Liu L., Ishikawa and Mann iteration processes with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.,* 194 (1995), 114-125.

[18] Mann W.R., Mean value methods in iteration, *Proc. Amer. Math. Soc.,* 44(1953), 506-510.

[19] Olatinwo M.O., Some Stability Results for Nonexpansive and Quasi-Nonexpansive Operators in Uniformly Convex Banach Space Using the Ishikawa Iteration Process, *Carpathian J. Math.,* 24(1)(2008), 82 - 87.

[20] Olatinwo M.O., Owojori O., Imoru C.O., On Some Stability Results for Fixed Point Iteration Procedure, *J. Math. Stat.,* 2(1)(2006), 339-342.

[21] OLATINWO M.O., OWOJORI O., IMORU C.O., Some stability results on Krasnoselskij and Ishikawa fixed point iteration procedures, *J. Math. Stat.,* 2(1)(2006), 360-362.

[22] OLATINWO M.O., OWOJORI O., IMORU C.O., Some stability results for fixed point iteration processes, *Aus. J. Math. Anal. Appl.,* 3(2)(2006), Article 8, 1-7.

[23] OSILIKE M.O., Some stability results for fixed point iteration procedures, *J. Nigerian Math.Soc.,* 14/15(1995), 17-29.

[24] OSILIKE M.O., UDOMENE A., Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings, *Indian J. Pure Appl. Math.,* 30(12)(1999), 1229-1234.

[25] OSTROWSKI A.M., The round-off stability of iterations, *Z. Angew. Math. Mech.,* 47(1967), 77-81.

[26] RHOADES B.E., Fixed point theorems and stability results for fixed point iteration procedures, *Indian J.Pure Appl. Math.,* 21(1)(1990), 1-9.

[27] RHOADES B.E., Some fixed point iteration procedures, *Int. J. Math. Math. Sci.,* 14(1)(1991), 1-16.

[28] RHOADES B.E., Fixed point theorems and stability results for fixed point iteration procedures II, *Indian J. Pure Appl. Math.,* 24(11)(1993), 691-703.

[29] RHOADES B.E., Fixed point iteration using infinite matrices, *Trans. Amer. Math. Soc.,* 196(1974), 161-176.

[30] RHOADES B.E., Comments on two fixed point iteration methods, *J. Math. Anal. Appl.,* 56(2)(1976), 741-750.

[31] RHOADES B.E., A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.,* 226 (1977), 257-290.

[32] RUS I.A., *Generalized Contractions and Applications,* Cluj Univ. Press, Cluj Napoca, 2001.

[33] RUS I.A., PETRUSEL A., PETRUSEL G., *Fixed Point Theory,* 1950-2000, Romanian Contributions, House of the Book of Science, Cluj Napoca, 2002.

[34] SINGH S.L., BHATNAGAR C., MISHRA S.N., Stability of Jungck-type iterative procedures, *International J. Math. & Math. Sc.,* 19(2005), 3035-3043.

[35] ZAMFIRESCU T., Fix point theorems in metric spaces, *Arch. Math.,* 23(1972), 292-298.

[36] ZEIDLER Z., *Nonlinear Functional Analysis and its Applications, Fixed-Point Theorems I.,* Springer-Verlag New York, Inc. (1986).

MEMUDU OLAPOSI OLATINWO
DEPARTMENT OF MATHEMATICS
OBAFEMI AWOLOWO UNIVERSITY
ILE-IFE, NIGERIA
e-mail: polatinwo@oauife.edu.ng *or* molaposi@yahoo.com

Received on 13.01.2009 and, in revised form, on 04.03.2009.