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INTEGRAL EQUATION ARISING IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS

Abstract. The aim of the present paper is to study some basic qualitative properties of solutions of a certain integral equation arising in the theory of partial differential equations. The well known Banach fixed point theorem and the new integral inequality with explicit estimate obtained in the present paper are used to establish the results.

Key words: integral equation, partial differential equations, Banach fixed point theorem, integral inequality, explicit estimate, existence and uniqueness, Bielecki's norm, discrete analogues.

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1. Introduction

The theory of partial differential equations is an important source to give rise to integral equations (see [1, 3-8, 13-16]). For instance, the partial differential equation of the form

$$u_{tt}(x,t) - au_{xxt}(x,t) = F(x,t,u(x,t)), \quad x \in [0,L], \; t \in [0,T], \tag{1}$$

with the initial conditions

$$u(x,0) = \phi(x), u_t(x,0) = \psi(x), \quad x \in [0,L], \tag{2}$$

and boundary conditions

$$u(0,t) = u(L,T) = 0, \quad t \in [0,T], \tag{3}$$

can be reduced to an integral equation of the form

$$u(x,t) = f(x,t) + \int_0^t \int_0^s \int_0^L G(x,y,s-\tau) F(y,\tau,u(y,\tau))\, dy d\tau ds, \tag{4}$$

if appropriate conditions are satisfied by the functions involved in (1)-(3), where $G(x,y,t)$ is the Green's function for the heat equation $w_t(x,t) =$

$aw_{xx}(x,t)$, with zero Dirichlet boundary data, a is a positive constant and $L > 0$, $T > 0$ are finite but can be arbitrarly large constants. For more details, see [1,13]. Indeed, in the study of certain basic results, the equation (4) can be dealt with in a more satisfactory manner than dealing directly with the equations (1)-(3).

In this paper we consider the more general integral equation of the form

$$u(x,t) = f(x,t) + \int_0^t \int_0^s \int_0^L K(x,t,s,y,\tau,u(y,\tau))\, dy d\tau ds, \tag{5}$$

where f, K are given functions and u is the unknown function to be found. Let $J = [0,L]$, $I = [0,T]$, $R_+ = [0,\infty)$, $D = J \times I$; J, I, R_+ are the given subsets of R, the set of real numbers and denote by $C(A,B)$ the class of continuous functions from the set A to the set B. We assume that $f \in C(D,R)$, $K \in C(D \times I \times D \times R, R)$. The main objective of the present paper is to study the existence, uniqueness and other properties of solutions of equation (5) under various assumptions on the functions involved in equation (5). The Banach fixed point theorem and a new integral inequality with explicit estimate to be established here are used to obtain the results.

2. Existence and uniqueness

Let S be a space of those functions $\phi \in C(D,R)$ which fulfill the condition

$$|\phi(x,t)| = O(\exp(\mu(x+t))), \quad (x,t) \in D, \tag{6}$$

where $\mu > 0$ is a constant. In the space S, we define the norm (See [2])

$$|\phi|_S = \sup_{(x,t)\in D} [\,|\phi(x,t)| \exp(-\mu(x+t))] . \tag{7}$$

It is easily seen that S with norm defined in (7) is a Banach space. We note that the condition (6) implies that there exists a constant $M \geq 0$ such that $|\phi(x,t)| \leq M \exp(\mu(x+t))$, $(x,t) \in D$. Using this fact in (7) we observe that

$$|\phi|_S \leq M. \tag{8}$$

We assume that the kernel of problem (4) satisfies (6)-(7).

Now we are in a position to formulate the main result in this section.

Theorem 1. *Suppose that*

(i) the function K in equation (5) satisfies the condition

$$|K(x,t,s,y,\tau,u) - K(x,t,s,y,\tau,v)| \leq h(x,t,s,y,\tau)\,|u-v|, \tag{9}$$

where $h \in C(D \times I \times D, R_+)$,

(ii) for μ *as in (6),*

(a_1) *there exists a nonnegative constant* $\alpha < 1$ *such that*

$$\int_0^t \int_0^s \int_0^L h(x,t,s,y,\tau) \exp(\mu(y+\tau))\, dy d\tau ds \le \alpha \exp(\mu(x+t)), \tag{10}$$

(a_2) *there exists a nonnegative constant* β *such that*

$$|f(x,t)| + \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,0)| dy d\tau ds \le \beta \exp(\mu(x+t)), \tag{11}$$

where f, K *are as defined in equation (5). Then the equation (5) has a unique solution* $u(x,t)$ *in* S *on* D.

Proof. Let $u \in S$ and define the operator F by

$$(Fu)(x,t) = f(x,t) + \int_0^t \int_0^s \int_0^L K(x,t,s,y,\tau,u(y,\tau)) dy d\tau ds. \tag{12}$$

Now we shall show that Fu maps S into itself. Evidently Fu is continuous on D and $Fu \in R$. We verify that (6) is fulfilled. From (12) and using the hypotheses, we have

$$\begin{aligned} |(Fu)(x,t)| &\le |f(x,t)| + \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,u(y,\tau)) \\ &\quad -K(x,t,s,y,\tau,0)|\, dy d\tau ds + \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,0)| dy d\tau ds \\ &\le \beta \exp(\mu(x+t)) + \int_0^t \int_0^s \int_0^L h(x,t,s,y,\tau)\, |u(y,\tau)| dy d\tau ds \\ &\le \beta \exp(\mu(x+t)) + |u|_S \int_0^t \int_0^s \int_0^L h(x,t,s,y,\tau) \exp(\mu(y+\tau)) dy d\tau ds \\ &\le \beta \exp(\mu(x+t)) + |u|_S\, \alpha \exp(\mu(x+t)) \\ &\le [\beta + M\alpha] \exp(\mu(x+t)). \end{aligned} \tag{13}$$

From (13), it follows that $Fu \in S$. This proves that F maps S into itself.

Next we verify that the operator F is a contraction map. Let $u, v \in S$. From (12) and using the hypotheses, we have

$$
\begin{aligned}
&\left|(Fu)(x,t) - (Fv)(x,t)\right| \\
&\le \int_0^t \int_0^s \int_0^L \left|K(x,t,s,y,\tau,u(y,\tau)) - K(x,t,s,y,\tau,v(y,\tau))\right| dy d\tau ds \\
&\le \int_0^t \int_0^s \int_0^L h(x,t,s,y,\tau) \left|u(y,\tau) - v(y,\tau)\right| dy d\tau ds \\
&\le |u-v|_S \int_0^t \int_0^s \int_0^L h(x,t,s,y,\tau) \exp(\mu(y+\tau)) \, dy d\tau ds \\
&\le |u-v|_S \, \alpha \exp(\mu(x+t)).
\end{aligned}
$$

Consequently, we have

$$
|Fu - Fv|_S \le \alpha \, |u-v|_S \, .
$$

Since $\alpha < 1$, it follows from Banach fixed point theorem (see [3, p. 37]) that F has a unique fixed point in S. The fixed point of F is however a solution of equation (5) in S. The proof is complete. ■

Remark 1. We note that the norm defined by (7) is a variant of Bielecki's norm [2], first used in 1956 for the study of ordinary differential equations. For the study of handling directly with the equations of the forms (1)-(3) by using different techniques, see [1, 4-8, 13-16].

3. Properties of solutions

First we establish the following new integral inequality which is useful to study various properties of solutions of equation (5). For detailed account on such inequalities, see [9-11].

Lemma 1. *Let $u, p, q, g \in C(D, R_+)$. If*

$$
u(x,t) \le p(x,t) + q(x,t) \int_0^t \int_0^s \int_0^L g(y,\tau) \, u(y,\tau) \, dy d\tau ds, \tag{14}
$$

for $(x,t) \in D$, *then*

$$u(x,t) \le p(x,t) + q(x,t)\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,p(y,\tau)\,dy d\tau ds\right) \times \exp\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,q(y,\tau)\,dy d\tau ds\right), \tag{15}$$

for $(x,t) \in D$.

Proof. Introducing the notation

$$e(\tau) = \int_0^L g(y,\tau)\,u(y,\tau)\,dy, \tag{16}$$

in (14), we get

$$u(x,t) \le p(x,t) + q(x,t)\int_0^t\int_0^s e(\tau)\,d\tau ds, \tag{17}$$

for $(x,t) \in D$. Define

$$z(t) = \int_0^t\int_0^s e(\tau)\,d\tau ds, \tag{18}$$

for $t \in I$, then, it is easy to see that $z(0) = 0$, $z'(0) = 0$ and

$$u(x,t) \le p(x,t) + q(x,t)\,z(t)\,. \tag{19}$$

From (18), (16), (19), we observe that

$$\begin{aligned} z''(t) &= e(t) = \int_0^L g(y,t)\,u(y,t)\,dy \\ &\le \int_0^L g(y,t)\left[p(y,t) + q(y,t)\,z(t)\right]dy \\ &= \int_0^L g(y,t)\,p(y,t)\,dy + z(t)\int_0^L g(y,t)\,q(y,t)\,dy. \end{aligned} \tag{20}$$

From (20) and the fact that $z(t)$ is nondecreasing for $t \in I$, it is easy to see that

$$z(t) \leq \int_0^t \int_0^s \int_0^L g(y,\tau)\, p(y,\tau)\, dy d\tau ds + \int_0^t z(s) \left\{ \int_0^s \int_0^L g(y,\tau)\, q(y,\tau)\, dy d\tau \right\} ds. \tag{21}$$

Clearly, the first integral on the right hand side in (21) is nonnegative and nondecreasing in $t \in I$. Now a suitable application of the inequality in Theorem 1.3.1 given in [9] to (21) yields

$$z(t) \leq \left(\int_0^t \int_0^s \int_0^L g(y,\tau)\, p(y,\tau)\, dy d\tau ds \right) \times \exp \left(\int_0^t \int_0^s \int_0^L g(y,\tau)\, q(y,\tau)\, dy d\tau ds \right), \tag{22}$$

for $t \in I$. Using (22) in (19), we get the required inequality in (15). ■

The following theorem is true concerning the uniqueness of solution of equation (5) on D.

Theorem 2. *Suppose that the function K in equation (5) satisfies the condition*

$$|K(x,t,s,y,\tau,u) - K(x,t,s,y,\tau,v)| \leq q(x,t)\, g(y,\tau)\, |u-v|, \tag{23}$$

where $q, g \in C(D, R_+)$. Then the equation (5) has at most one solution on D.

Proof. Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (5) on D. Using these facts and hypotheses, we have

$$\begin{aligned} &|u(x,t) - v(x,t)| \\ &\leq \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,u(y,\tau)) - K(x,t,s,y,\tau,v(y,\tau))| dy d\tau ds \\ &\leq q(x,t) \int_0^t \int_0^s \int_0^L g(y,\tau)\, |u(y,\tau) - v(y,\tau)| dy d\tau ds. \end{aligned} \tag{24}$$

Now a suitable application of Lemma 1 (when $p(x,t) = 0$) to (24) yields $|u(x,t) - v(x,t)| \leq 0$, which implies $u(x,t) = v(x,t)$. Thus there is at most one solution to equation (5) on D. ■

The next theorem deals with the estimate on the solution of equation (5).

Theorem 3. *Suppose that the function K in equation (5) satisfies the condition*

$$|K(x,t,s,y,\tau,u)| \leq q(x,t)\, g(y,\tau)\, |u|, \tag{25}$$

where $q, g \in C(D, R_+)$. If $u(x,t)$ is any solution of equation (5) on D, then

$$|u(x,t)| \leq |f(x,t)| + q(x,t)\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,|f(y,\tau)|\,dyd\tau ds\right) \times \exp\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,q(y,\tau)\,dyd\tau ds\right), \tag{26}$$

for $(x,t) \in D$.

Proof. Using the fact that $u(x,t)$ is a solution of equation (5) and hypotheses, we have

$$\begin{aligned} |u(x,t)| &\leq |f(x,t)| + \int_0^t\int_0^s\int_0^L |K(x,t,s,y,\tau,u(y,\tau))|\,dyd\tau ds \\ &\leq |f(x,t)| + q(x,t)\int_0^t\int_0^s\int_0^L g(y,\tau)\,|u(y,\tau)|\,dyd\tau ds. \end{aligned} \tag{27}$$

Now an application of Lemma 1 to (27) yields (26). ■

The following theorem deals with a slight variant of Theorem 3, assuming that the function K in equation (5) satisfies the Lipschitz type condition.

Theorem 4. *Suppose that the function K in equation (5) satisfies the condition (23). If $u(x,t)$ is any solution of equation (5) on D, then*

$$|u(x,t) - f(x,t)| \leq Q(x,t) + q(x,t)\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,Q(y,\tau)\,dyd\tau ds\right) \times \exp\left(\int_0^t\int_0^s\int_0^L g(y,\tau)\,q(y,\tau)\,dyd\tau ds\right), \tag{28}$$

for $(x,t) \in D$, *where*

$$Q(x,t) = \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,f(y,\tau))|\, dy d\tau ds, \tag{29}$$

for $(x,t) \in D$.

Proof. From the fact that $u(x,t)$ is a solution of equation (5) and the condition (23), we have

$$\begin{aligned} |u(x,t) - f(x,t)| &\le \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,u(y,\tau)) \\ &\quad - K(x,t,s,y,\tau,f(y,\tau))|\, dy d\tau ds \\ &\quad + \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,f(y,\tau))|\, dy d\tau ds \\ &\le Q(x,t) + q(x,t) \int_0^t \int_0^s \int_0^L g(y,\tau)\, |u(y,\tau) - f(y,\tau)|\, dy d\tau ds, \end{aligned} \tag{30}$$

for $(x,t) \in D$. Now an application of Lemma 1 to (30) gives the required estimate in (28).

We next consider the equation (5) and the following integral equation

$$w(x,t) = \bar{f}(x,t) + \int_0^t \int_0^s \int_0^L \bar{K}(x,t,s,y,\tau,w(y,\tau))\, dy d\tau ds, \tag{31}$$

where $\bar{f} \in C(D,R)$, $\bar{K} \in C(D \times I \times D \times R, R)$. ■

The following theorem holds.

Theorem 5. *Suppose that the function* K *in equation (5) satisfies the condition (23). Then for every given solution* $w \in C(D,R)$ *of equation (31) and every solution* $u \in C(D,R)$ *of equation (5), we have the estimation*

$$\begin{aligned} |u(x,t) - w(x,t)| &\le h(x,t) + q(x,t) \left(\int_0^t \int_0^s \int_0^L g(y,\tau)\, h(y,\tau)\, dy d\tau ds \right) \\ &\quad \times \exp\left(\int_0^t \int_0^s \int_0^L g(y,\tau)\, q(y,\tau)\, dy d\tau ds \right), \end{aligned} \tag{32}$$

for $(x,t) \in D$, *in which*

$$h(x,t) = \left|f(x,t) - \bar{f}(x,t)\right| + \int_0^t \int_0^s \int_0^L \left|K(x,t,s,z,\sigma,w(z,\sigma)) - \bar{K}(x,t,s,z,\sigma,w(z,\sigma))\right| dz d\sigma ds, \tag{33}$$

for $(x,t) \in D$.

Proof. Using the facts that $u(x,t)$ and $w(x,t)$ are respectively the solutions of equations (5) and (31) and hypotheses, we have

$$\begin{aligned}
|u(x,t) - w(x,t)| &\leq \left|f(x,t) - \bar{f}(x,t)\right| + \int_0^t \int_0^s \int_0^L \left|K(x,t,s,y,\tau,u(y,\tau)) - \bar{K}(x,t,s,y,\tau,w(y,\tau))\right| dy d\tau ds \\
&\leq \left|f(x,t) - \bar{f}(x,t)\right| + \int_0^t \int_0^s \int_0^L |K(x,t,s,y,\tau,u(y,\tau)) - K(x,t,s,y,\tau,w(y,\tau))| dy d\tau ds \\
&\quad + \int_0^t \int_0^s \int_0^L \left|K(x,t,s,y,\tau,w(y,\tau)) - \bar{K}(x,t,s,y,\tau,w(y,\tau))\right| dy d\tau ds \\
&\leq h(x,t) + q(x,t) \int_0^t \int_0^s \int_0^L g(y,\tau) |u(y,\tau) - w(y,\tau)| dy d\tau ds.
\end{aligned} \tag{34}$$

Now an application of Lemma 1 to (34) yields (32). ■

Remark 2. *We note that, one can use the inequality in Lemma 1 to establish the results on continuous dependence of solutions of equations of the form (5) by closely looking at the results given in [12]. The generality of the equation (5), allow us to include the study of equations (1)-(3). We hope that our approach and results given here will serve as a model for future investigations.*

4. Discrete analogues

Let N denote the set of natural numbers, $M_{\alpha,\beta} = \{\alpha, \alpha+1, ..., \alpha+n = \beta\}$ and $N_{a,b} = \{a, a+1, ..., a+n = b\}$; $\alpha, a \in N_0, n \in N$. Let $H = M_{\alpha,\beta} \times N_{a,b}$ and denote by $E(A,B)$ the class of discrete functions from the set A to the set B. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. The sum-difference equation which constitutes the discrete analogue of equation (5) can be written as

$$v(x,m) = f(x,m) + \sum_{s=0}^{m-1}\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} k(x,m,s,y,\tau,v(y,\tau)), \tag{35}$$

for $x \in M_{\alpha,\beta}$, $m \in N_{a,b}$, where f, k are given functions and v is the unknown function to be found. We assume that $f \in E(H,R)$, $k \in E(H \times N_{a,b} \times H \times R, R)$. In this section, we formulate in brief the discrete analogues of Lemma 1 and Theorems 2 and 3 only. One can formulate results similar to those in Theorems 1, 4 and 5 for the solutions of equation (35).

Lemma 2. *Let $v,p,q,g \in E(H,R_+)$. If*

$$v(x,m) \le p(x,m) + q(x,m)\sum_{s=0}^{m-1}\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} g(y,\tau)\,v(y,\tau), \tag{36}$$

for $(x,m) \in H$, then

$$\begin{aligned} v(x,m) \le p(x,m) + q(x,m)&\left(\sum_{s=0}^{m-1}\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} g(y,\tau)\,p(y,\tau)\right) \\ &\times \prod_{s=0}^{m-1}\left[1+\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} g(y,\tau)\,q(y,\tau)\right], \end{aligned} \tag{37}$$

for $(x,m) \in H$.

Theorem 6. *Suppose that the function k in equation (35) satisfies the condition*

$$|k(x,m,s,y,\tau,v) - k(x,m,s,y,\tau,w)| \le q(x,m)\,g(y,\tau)\,|v-w|, \tag{38}$$

where $q,g \in E(H,R_+)$. Then the equation (35) has at most one solution on H.

Theorem 7. *Suppose that the function k in equation (35) satisfies the condition*

$$|k(x,m,s,y,\tau,v)| \le q(x,m)\,g(y,\tau)\,|v|, \tag{39}$$

where $q, g \in E(H, R_+)$. If $v(x,m)$ is any solution of equation (35) on H, then

$$|v(x,m)| \le |f(x,m)| + q(x,m)\left(\sum_{s=0}^{m-1}\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} g(y,\tau)|f(y,\tau)|\right) \times \prod_{s=0}^{m-1}\left[1+\sum_{\tau=0}^{s-1}\sum_{y=\alpha}^{\beta} g(y,\tau)\,q(y,\tau)\right], \tag{40}$$

for $(x,m) \in H$.

The proof of Lemma 2 can be completed by following the proof of Lemma 1 given above and closely looking at the similar results given in [10, 11]. The proofs of Theorems 6 and 7 follows by the similar arguments as in the proofs of Theorems 2 and 3 given above. We omit the details.

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