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A FIXED POINT RESULT FOR $\varphi-$CONTRACTIONS ON b-METRIC SPACES WITHOUT THE BOUNDEDNESS ASSUMPTION*

Abstract. Starting from a result in [V. Berinde, *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory (Preprint), "Babeş-Bolyai" University of Cluj-Napoca, **3** (1993), 3-9], we prove the existence and uniqueness of the fixed points for φ-contractions on b-metric spaces. We also build a theory of this fixed point result.

Key words: b-metric space, φ-contraction, fixed point, well-posedness, limit shadowing property.

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1. Introduction

In [3] two fixed point theorems for $\varphi-$contractions on b-metric spaces are proved. Based on a similar result for Banach contractions on b-metric spaces included in [1], these theorems require the boundedness of the Picard iteration in order to guarantee the existence and uniqueness of the fixed point.

The aim of this paper is to improve one of the above mentioned results from [3], by giving up the boundedness assumption. In this way we obtain the generalization of a theorem for $\varphi-$contractions in metric spaces, included in [5] as Theorem 1.5.1.

We shall also build o theory of the newly obtained theorem, following the model described in [10].

2. Preliminaries

We begin by recalling that:

Definition 1 ([1])**.** *A mapping $d : X \times X \to \mathbb{R}_+$ is called b-**metric** if there exists a real number $b \geq 1$ such that:*

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ι) *$d(x,y)=0$ if and only if $x=y$;*

$\iota\iota$) *$d(x,y)=d(y,x)$, for any $x,y\in X$;*

$\iota\iota\iota$) *$d(x,z)\le b[d(x,y)+d(y,z)]$, for any $x,y,z\in X$.*

*A nonempty set X endowed with a $b-$metric $d:X\times X\to\mathbb{R}_+$ is called b-**metric space**.*

For the theory of b-metric spaces see [6], [1], [7].

As known, a mapping $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is called a *comparison function* if it is increasing and $\varphi^n(t)\to 0, n\to\infty$, for any $t\in\mathbb{R}_+$ (see for example [8]). In [8] and [5] several results regarding comparison functions can be found. Among these we recall:

Lemma 1 ([8],[5]). *If $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a comparison function, then:*

1) *each iterate φ^k of φ, $k\ge 1$, is also a comparison function;*

2) *φ is continuous at zero;*

3) *$\varphi(t)<t$, for any $t>0$.*

For practical reasons, in [5] V. Berinde introduced the concept of *(c)-comparison function*:

Definition 2 ([5]). *A function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is called a **(c)-comparison function** if:*

ι) *φ is increasing;*

$\iota\iota$) *there exist $k_0\in\mathbb{N}$, $a\in(0,1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty}v_k$ such that*

$$\varphi^{k+1}(t)\le a\varphi^k(t)+v_k, \tag{1}$$

for $k\ge k_0$ and any $t\in\mathbb{R}_+$.

Regarding this concept we also mention the following result, proved in [5].

Lemma 2 ([5]). *If $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a $(c)-$comparison function, then the following hold:*

ι) *φ is a comparison function;*

$\iota\iota$) *the series $\sum\limits_{k=0}^{\infty}\varphi^k(t)$ converges for any $t\in\mathbb{R}_+$;*

$\iota\iota\iota$) *the function $s:\mathbb{R}_+\to\mathbb{R}_+$ defined by*

$$s(t)=\sum_{k=0}^{\infty}\varphi^k(t),\quad t\in\mathbb{R}_+, \tag{2}$$

is increasing and continuous at 0.

In the following we include the statement of a result proved in [5] as Theorem 1.5.1:

Theorem 1 ([5]). *Let (X, d) be a complete metric space and $f : X \to X$ a φ-contraction with φ a (c)-comparison function. Then:*

1) *f is a Picard operator, with $F_f = \{x^*\}$;*

2) *the rate of convergence of the Picard iteration is given by:*

$$d(x_n, x^*) \leq s(d(x_n, x_{n+1})), \quad n \geq 0,$$

where s is defined by (2) in Lemma 2;

The concept of (c)-comparison function was extended to *b-comparison functions* in [4], where the framework was that of a b-metric space.

Definition 3 ([4]). *Let $b \geq 1$ be a real number. A mapping $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a b-**comparison function** if:*

ι) *φ is monotone increasing;*

$\iota\iota$) *there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum\limits_{k=1}^{\infty} v_k$ such that*

$$b^{k+1}\varphi^{k+1}(t) \leq ab^k\varphi^k(t) + v_k, \tag{3}$$

for $k \geq k_0$ and any $t \in \mathbb{R}_+$.

Remark 1. It is easy to notice that, for $b = 1$, the concept of b-comparison function reduces to that of (c)-comparison function.

It has been proved that:

Lemma 3 ([2, 3]). *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a $b-$comparison function, then:*

1) *the series $\sum\limits_{k=0}^{\infty} b^k\varphi^k(t)$ converges for any $t \in \mathbb{R}_+$;*

2) *the function $s_b : \mathbb{R}_+ \to \mathbb{R}_+$ defined by*

$$s_b(t) = \sum_{k=0}^{\infty} b^k\varphi^k(t), \quad t \in \mathbb{R}_+, \tag{4}$$

is increasing and continuous at 0.

Using Lemma 3 it is easy to prove that:

Lemma 4. *Any b-comparison function is a comparison function.*

Proof. In case $b = 1$, since b-comparison functions coincide with (c)-comparison functions, the conclusion follows by Lemma 2.

In the following we suppose $b > 1$. Since the series $\sum_{k=0}^{\infty} b^k \varphi^k(t)$ converges for any $t > 0$, its general term satisfies

(5) $$b^n \varphi^n(t) \to 0 \quad \text{as} \quad n \to \infty, \quad t > 0.$$

Supposing $\varphi^n(t)$ converged to some $l > 0$, since $b > 1$ this would imply that $b^n \varphi^n(t) \to \infty$, which contradicts (5).

So clearly $\varphi^n(t) \to 0, n \to \infty$. This together with ι) in the definition of b-comparison functions guarantees that φ is also a comparison function. ■

In the recent paper [9] the following generalized Cauchy lemma was proved:

Lemma 5 ([9]). *Let $f_n, g_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$. We assume that:*

ι) *f_n is increasing, $f_n(0) = 0$ and f_n is continuous at 0, for any $n \in \mathbb{N}$;*

$\iota\iota$) *$\sum_{k=0}^{\infty} f_k(t) < \infty$, for any $t \in \mathbb{R}_+$;*

$\iota\iota\iota$) *$g_n(t) \to 0$ as $n \to \infty$, for any $t \in \mathbb{R}_+$.*

Then:

$$\sum_{k=0}^{n} f_{n-k}(g_k(t)) \to 0 \quad \textit{as} \quad n \to \infty, \quad \textit{for any} \quad t \in \mathbb{R}_+.$$

Using Lemma 5 it is easy to prove the following result, which is similar to the one proved in [9] for (c)-comparison functions, there called "strong comparison functions".

Lemma 6. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a b-comparison function with constant $b \geq 1$ and $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $a_n \to 0$ as $n \to \infty$. Then*

$$\sum_{k=0}^{n} b^{n-k} \varphi^{n-k}(a_k) \to 0 \quad \textit{as} \quad n \to \infty.$$

Proof. We take $f_n = b^n \varphi^n$ and $g_n(t) = a_n$, for any $t \in \mathbb{R}_+$. By Lemmas 4, 1 and 3, it is clear that $f_n = b^n \varphi^n$ fulfills ι) and $\iota\iota$) in Lemma 5. The conclusion follows immediately. ■

3. The main result

In [3] the following generalization of a result due to I.A. Bakhtin [1], originally for Banach contractions, was given:

Theorem 2 ([3]). *Let (X, d) be a complete b-metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a comparison function and $f : X \to X$ a φ-contraction.*

Then f has a unique fixed point if and only if there exists $x_0 \in X$ such that the Picard iteration $\{x_n\}_{n\geq 0}$ defined by

$$x_n = f(x_{n-1}), \quad n \geq 1, \tag{6}$$

is bounded.

By considering b-comparison functions instead of comparison functions, V. Berinde [3] obtained also an estimation of the rate of convergence, as one can see in the following result:

Theorem 3 ([3]). *Let (X, d) be a complete b-metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a b-comparison function and $f : X \to X$ a φ-contraction.*

If $x_0 \in X$ is such that the Picard iteration $\{x_n\}_{n\geq 0}$ is bounded and $F_f = \{x^*\}$, then:

$$d(x_n, x^*) \leq b\, s_b(d(x_n, x_{n+1})), \quad n \geq 0, \tag{7}$$

where s_b is given by (4) in Lemma 3.

In the following we prove the main result of this paper, which shows that the boundedness assumption from Theorem 3 is actually not necessary in order to obtain the existence and uniqueness of the fixed point. The same estimations are also obtained.

Theorem 4. *Let (X, d) be a complete b-metric space with constant $b \geq 1$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a b-comparison function and $f : X \to X$ a φ-contraction. Then:*

1) *f is a Picard operator;*

2) *the following estimates hold:*

$$d(x_n, x^*) \leq b\, s_b(\varphi^n(d(x_0, x_1))), \quad n \geq 0, \tag{8}$$

$$d(x_n, x^*) \leq b\, s_b(d(x_n, x_{n+1})), \quad n \geq 0, \tag{9}$$

where s_b is given by Lemma 3;

3) *for any $x \in X$ we have that:*

$$d(x, x^*) \leq b\, s_b(d(x, f(x))). \tag{10}$$

Proof.

1) Let $x_0 \in X$ and $x_n = f(x_{n-1})$, $n \geq 1$. For $n \geq 1$ we have that:

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \varphi(d(x_{n-1}, x_n)),$$

which by induction yields

$$d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)). \tag{11}$$

As d is a b-metric, for $n \geq 0$, $p \geq 1$ we obtain:

$$\begin{aligned} d(x_n, x_{n+p}) \leq b\, d(x_n, x_{n+1}) + b^2 d(x_{n+1}, x_{n+2}) \\ + \ \dots\ + b^p d(x_{n+p-1}, x_{n+p}). \end{aligned} \tag{12}$$

By (11) it follows that:

$$\begin{aligned} d(x_n, x_{n+p}) \leq b\, \varphi^n(d(x_0, x_1)) + b^2 \varphi^{n+1}(d(x_0, x_1)) \\ + \cdots + b^p \varphi^{n+p-1}(d(x_0, x_1)), \end{aligned} \tag{13}$$

which can also be written as

$$\begin{aligned} d(x_n, x_{n+p}) \leq \frac{1}{b^{n-1}} \left[b^n \varphi^n(d(x_0, x_1)) \right. \\ \left. + \cdots + b^{n+p-1} \varphi^{n+p-1}(d(x_0, x_1)) \right]. \end{aligned} \tag{14}$$

Denoting

$$S_n = \sum_{k=0}^{n} b^k \varphi^k(d(x_0, x_1)), \quad n \geq 1,$$

(14) becomes:

$$d(x_n, x_{n+p}) \leq \frac{1}{b^{n-1}} [S_{n+p-1} - S_{n-1}], \quad n \geq 1, \ \ p \geq 1. \tag{15}$$

Supposing $d(x_0, x_1) > 0$, by Lemma 3 the series $\sum\limits_{k=0}^{\infty} b^k \varphi^k(d(x_0, x_1))$ converges, so there is

$$S = \lim_{n \to \infty} S_n \in \mathbb{R}_+.$$

Since $b \geq 1$, by (15) we obtain that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the complete metric space (X, d). So there is $x^* \in X$ such that

$$x^* = \lim_{n \to \infty} x_n.$$

In the following we prove that x^* is a fixed point for f. For $n \geq 0$ we have:

$$d(x_{n+1}, f(x^*)) = d(f(x_n), f(x^*)) \leq \varphi(d(x_n, x^*)). \tag{16}$$

But d is continuous, and by Lemmas 4 and 1 φ is also continuous at 0. Letting $n \to \infty$ in (16) we obtain that:

$$d(x^*, f(x^*)) = 0,$$

that is, $x^* \in F_f$. Supposing there would be $y^* \in X$ such that $y^* = f(y^*)$ and $y^* \neq x^*$, by Lemmas 4 and 1, 3), we have:

$$d(x^*, y^*) = d(f(x^*), f(y^*)) \leq \varphi(d(x^*, y^*)) < d(x^*, y^*),$$

which is a contradiction. So f is a Picard operator.

2) Inequality (13) can also be written as

$$d(x_n, x_{n+p}) \leq b\,[\varphi^0(\varphi^n(d(x_0, x_1))) + b\varphi(\varphi^n(d(x_0, x_1))) + \cdots + b^{p-1}\varphi^{p-1}(\varphi^n(d(x_0, x_1)))], \tag{17}$$

where $n \geq 0$, $p \geq 1$. Letting $p \to \infty$ in (17) we obtain the a priori estimate

$$d(x_n, x^*) \leq b\, s_b(\varphi^n(d(x_0, x_1))), n \geq 0.$$

On the other hand, for $n \geq 1$, $k \geq 0$ we have that:

$$d(x_{n+k}, x_{n+k+1}) = d(f(x_{n+k-1}), f(x_{n+k})) \leq \varphi(d(x_{n+k-1}, x_{n+k})),$$

which by induction yields

$$d(x_{n+k}, x_{n+k+1}) \leq \varphi^k(d(x_n, x_{n+1})), \quad n \geq 1, \ \ k \geq 0. \tag{18}$$

Using (18) back in (12) we obtain that

$$d(x_n, x_{n+p}) \leq b\ \left[d(x_n, x_{n+1}) + b\varphi(d(x_n, x_{n+1})) + \cdots + b^{p-1}\varphi^{p-1}(d(x_n, x_{n+1}))\right], \quad n \geq 0, \ \ p \geq 1. \tag{19}$$

Letting $p \to \infty$ in (19) we obtain the a posteriori estimate

$$d(x_n, x^*) \leq b\, s_b(d(x_n, x_{n+1})), \quad n \geq 0.$$

3) Let $x_n := x$ in (9), for an arbitrary $x \in X$. Then

$$d(x, x^*) \leq b\, s_b(d(x, f(x))).$$

■

Remark 2. All the conclusions in Theorem 1 can be obtained from Theorem 4 for $b = 1$.

4. A theory of the main result

Following the direction suggested in [10] of how to establish a so-called *theory of a fixed point theorem* and using the terminology therein, we prove the results below:

Theorem 5. *Let $f : X \to X$ be as in Theorem 4. Then f is a good Picard operator.*

Proof. Let $x_0 \in Y$. By (11) in the proof of Theorem 4, we know that

$$d(f^n(x_0), f^{n+1}(x_0)) = d(x_n, x_{n+1}) \leq \varphi^n(d(x_0, x_1)), \quad n \geq 0,$$

which also holds for the case $b = 1$. Then by $\iota\iota\iota)_b$ in the definition of a b-metric we obtain:

$$\sum_{n=0}^{\infty} d(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=0}^{\infty} b^n \varphi^n(d(x_0, x_1)) = s_b(d(x_0, x_1)).$$

So, by Lemma 3, $\sum_{n=0}^{\infty} d(f^n(x_0), f^{n+1}(x_0)) < \infty$, and consequently f is a good Picard operator. ■

Remark 3. *An open problem is to check whether $f : X \to X$ as in Theorem 4 is a special Picard operator or not.*

Theorem 6. *Let $f : X \to X$ be as in Theorem 4. Then the fixed point problem for f is well posed.*

Proof. Let $\{z_n\}_{n \in \mathbb{N}} \subset X$ be a sequence such that

$$d(z_n, f(z_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{20}$$

Applying (10) for $x = z_n$, $n \in \mathbb{N}$, we have:

$$d(z_n, x^*) \leq b\, s_b(d(z_n, f(z_n))), \quad n \in \mathbb{N}. \tag{21}$$

From Lemma 3 we know that s_b is continuous at 0. Then letting $n \to \infty$ in (21), by (20) we obtain that

$$d(z_n, x^*) \to 0, \quad n \to \infty,$$

so the fixed point problem for f is well posed. ■

Theorem 7. *Let $f : X \to X$ be as in Theorem 4. If φ satisfies:*

$$\varphi(a_1 t_1 + a_2 t_2) \leq a_1 \varphi(t_1) + a_2 \varphi(t_2), \tag{22}$$

for any $a_1, a_2, t_1, t_2 \in \mathbb{R}_+$, then f has the limit shadowing property.

Proof. Let $\{z_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying

$$d(z_{n+1}, f(z_n)) \to 0 \quad \text{as} \quad n \to \infty. \tag{23}$$

For $n \geq 0$ we have:

$$d(z_{n+1}, x^*) \leq b\, d(z_{n+1}, f(z_n)) + b\, d(f(z_n), f(x^*)). \tag{24}$$

As f is a φ-contraction, inequality (24) becomes:

$$d(z_{n+1}, x^*) \leq b\, d(z_{n+1}, f(z_n)) + b\, \varphi(d(z_n, x^*)), \quad n \geq 0. \tag{25}$$

In the same way we get:

$$d(z_n, x^*) \leq b\, d(z_n, f(z_{n-1})) + b\, \varphi(d(z_{n-1}, x^*)), \quad n \geq 1,$$

which applied back in (25), by (22) yields

$$d(z_{n+1}, x^*) \leq b\, d(z_{n+1}, f(z_n)) + b^2\varphi(d(z_n, f(z_{n-1}))) + b^2\varphi^2(d(z_{n-1}, x^*)).$$

By induction we obtain:

$$\begin{aligned} d(z_{n+1}, x^*) \leq{}& b\, d(z_{n+1}, f(z_n)) + b^2\varphi(d(z_n, f(z_{n-1}))) \\ &+ \cdots + b^{n+1}\varphi^n(d(z_1, f(z_0))) + b^{n+2}\varphi^{n+1}(d(z_0, x^*)), \end{aligned}$$

which can also be written as

$$\begin{aligned} d(z_{n+1}, x^*) \leq{}& b\sum_{k=0}^{n} b^k\varphi^k(d(z_{n-k+1}, f(z_{n-k}))) \\ &+ b^{n+2}\varphi^{n+1}(d(z_0, x^*)), \end{aligned} \tag{26}$$

Now applying Lemma 6 for $a_n = d(z_{n+1}, f(z_n))$, it follows that

$$\sum_{k=0}^{n} b^k\varphi^k(d(z_{n-k+1}, f(z_{n-k}))) \to 0, \quad n \to \infty.$$

If $z_0 = x^*$, obviously $b^n\varphi^n(d(z_0, x^*)) = 0$. If $z_0 \neq x^*$, we also have that $b^n\varphi^n(d(z_0, x^*)) \to 0$ as $n \to \infty$, by Lemma 3. Thus letting $n \to \infty$ in (26), we obtain that

$$d(z_{n+1}, x^*) \to 0, \quad n \to \infty. \tag{27}$$

By Theorem 4 we know that for any $x \in X$ the Picard iteration $\{f^n(x)\}_{n\geq 0}$ converges to x^*. So, for some fixed $x \in Y$, we may write:

$$d(z_{n+1}, f^n(x)) \leq d(z_{n+1}, x^*) + d(x^*, f^n(x)), \quad n \geq 0. \tag{28}$$

Now letting $n \to \infty$ in (28), by (27) we obtain that

$$d(z_{n+1}, f^n(x)) \to 0, \quad n \to \infty,$$

so f has the limit shadowing property. ■

We can also state a result regarding the data dependence of the fixed point in the case of φ-contractions on b-metric spaces with φ a b-comparison function:

Theorem 8. *Let $f : X \to X$ be as in Theorem 4 and $g : X \to X$ such that:*

ι) g has at least one fixed point, say $x_g^* \in F_g$;

$\iota\iota$) there exists $\eta > 0$ such that

$$d(f(x), g(x)) \le \eta, \quad \textit{for any} \quad x \in X. \tag{29}$$

Then

$$d(x_f^*, x_g^*) \le b\, s_b(\eta),$$

where $F_f = \{x_f^*\}$ and s_b is like in Lemma 3.

Proof. Applying (10) from Theorem 4 for $x := x_g^*$, we have:

$$d(x_f^*, x_g^*) \le b\, s_b(d(x_g^*, f(x_g^*))) = b\, s_b(d(g(x_g^*), f(x_g^*))).$$

From Lemma 3, s_b is increasing, so by ($\iota\iota$) it follows that

$$d(x_f^*, x_g^*) \le b\, s_b(\eta).$$

■

A Nadler type result regarding sequences of operators converging to a φ-contraction defined on a b-metric space, where φ is a b-comparison function, was proved in [4].

Remark 4. *A theory of Theorem 1.5.1 from [5] in metric spaces, here included as Theorem 1, can easily be derived from the above results, for $b = 1$.*

References

[1] Bakhtin I.A., The contraction principle in quasimetric spaces (Russian), *Func.An., Unianowsk, Gos.Ped.Ins.,* 30(1989), 26-37.

[2] Berinde V., Une generalization de critere du d'Alembert pour les series positives, *Bul. St. Univ. Baia Mare,* 7(1991), 21-26.

[3] BERINDE V., Generalized contractions in quasimetric spaces, *Seminar on Fixed Point Theory (Preprint), "Babeş-Bolyai" University of Cluj-Napoca,* 3(1993), 3-9.

[4] BERINDE V., Sequences of operators and fixed points in quasimetric spaces, *Stud. Univ. "Babeş-Bolyai", Math.,* 16(4)(1996), 23-27.

[5] BERINDE V., *Contracţii generalizate şi aplicaţii,* Editura Cub Press 22, Baia Mare, 1997.

[6] BOURBAKI N., *Topologie gńérale,* Herman, Paris, 1961.

[7] CZERWIK S., Nonlinear set-valued contraction mappings in b-metric spaces, *Atti. Sem. Mat. Univ. Modena,* 46(1998), 263-276.

[8] RUS I.A., *Generalized Contractions and Applications,* Cluj University Press, Cluj-Napoca, 2001.

[9] RUS I.A., SERBAN M.A., Some generalizations of a Cauchy lemma and applications, *Topics in Mathematics, Computer Science and Philosophy. St. Cobzas (Ed.), Cluj University Press, Cluj-Napoca,* (2008), 173-181.

[10] RUS I.A., The theory of a metrical fixed point theorem: theoretical and applicative relevances, *Fixed Point Theory,* 9(2)(2008), 541-559.

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