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PATTERNS OF BOUNDEDNESS OF THE RATIONAL SYSTEM $x_{n+1} = \frac{\alpha_1+\beta_1 x_n}{A_1+C_1 y_n}$ and $y_{n+1} = \frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+B_2 x_n+C_2 y_n}$

Abstract. We establish the boundedness character of solutions of the rational system in the title, with the parameters α_1, β_1 positive and the remaining eight parameters nonnegative and with arbitrary nonnegative initial conditions such that the denominators are always positive. We present easily verifiable necessary and sufficient conditions, explicitly stated in terms of the parameters, which determine the boundedness character of the system.

Key words: boundedness, patterns of boundedness, rational equations, rational systems.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

We establish the boundedness character of solutions of the rational system in the plane,

$$\left\{ \begin{array}{l} x_{n+1} = \frac{\alpha_1+\beta_1 x_n}{A_1+C_1 y_n}, \\ \\ y_{n+1} = \frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+B_2 x_n+C_2 y_n}, \end{array} \right. \qquad n = 0, 1, \ldots \tag{1}$$

with the parameters α_1, β_1 positive and the remaining eight parameters nonnegative and with arbitrary nonnegative initial conditions such that the denominators are always positive.

System (1) contains

$$(2^2 - 1) \times (2^3 - 1) \times (2^3 - 1) = 147$$

special cases of systems, each with positive parameters.

We establish easily verifiable necessary and sufficient conditions, explicitly stated in terms of the parameters, under which the boundedness characterization of the system is:

$$\textbf{(B,B)},\ \textbf{(B,U)},\ \textbf{(U,B)} \text{ or } \textbf{(U,U)}.$$

A special case of System (1) has the boundedness characterization **(B,B)** when both components of every solution of the system, in this special case, are bounded.

A special case of System (1) has the boundedness characterization **(B,U)** when the first component of every solution in this special case of the system is always bounded and there exist solutions in which the second component is unbounded in some range of the parameters and for some initial conditions. Similarly, we define the boundedness characterizations **(U,B)** and **(U,U)** for a special case of System (1).

The boundedness character of solutions of a system is one of the main ingredients in understanding the global behavior of the system including its global stability. Boundedness is also essential in the study of most applications.

System (1) is a special case of the "full" rational system in the plane,

$$(2) \qquad \begin{cases} x_{n+1} = \frac{\alpha_1+\beta_1 x_n+\gamma_1 y_n}{A_1+B_1 x_n+C_1 y_n}, \\ \\ y_{n+1} = \frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+B_2 x_n+C_2 y_n}, \end{cases} \qquad n = 0, 1, \ldots$$

which contains

$$7 \times 7 \times 7 \times 7 = 2401$$

special cases each with positive parameters. A large number of open problems and conjectures about System (2) were posed in [9] and [11]. For some work on the boundedness character of System (2) see [1]-[8], and [11]-[12]. For the numbering system of the 2401 special cases contained in System (2), see Appendices 1 and 2 in [9]. Also for some basic results in the area of difference equations and systems see [10] and [13]-[15].

System (1) has the boundedness characterization **(B,B), if and only if:**

$$(3) \qquad B_2 = 0,\ C_1, \beta_2 \in (0, \infty) \quad \text{and} \quad (\gamma_2 = 0 \ \text{ or } \ C_2 > 0).$$

System (1), under condition (3) is restricted to the group of the following 20 special cases

$$(4) \qquad \begin{cases} (21,7),\ (21,8),\ (21,16),\ (21,22),\ (21,23), \\ (21,26),\ (21,31),\ (21,34),\ (21,41),\ (21,46), \\ (29,7),\ (29,8),\ (29,16),\ (29,22),\ (29,23), \\ (29,26),\ (29,31),\ (29,34),\ (29,41),\ (29,46). \end{cases}$$

and has the boundedness characterization **(B,B)**.

System (1) has the boundedness characterization **(B,U)**, if and only if:

$$(5) \qquad B_2 = C_2 = 0 \quad \text{and} \quad C_1, \beta_2, \gamma_2 \in (0, \infty).$$

System (1) has the boundedness characterization **(U,B)**, if and only if:

$$(\beta_2 = 0 \text{ or } B_2 > 0),\ (\gamma_2 = 0 \text{ or } C_2 > 0), \text{ and } (\alpha_2 + \gamma_2 = 0, \text{ or } A_2 + C_2 > 0, \text{ or } C_1 = 0). \tag{6}$$

When none of the above three conditions (3), (5), and (6) is satisfied, System (1) has the boundedness characterization **(U,U)**.

One can see that System (1) contains 147 special cases of which, 20 special cases have the boundedness characterization **(B,B)**, 4 special cases have the boundedness characterization **(B,U)**, 77 special cases have the boundedness characterization **(U,B)**, and 46 special cases have the boundedness characterization **(U,U)**.

In Section 2 we will establish that Condition (3) is sufficient for every solution of System (1) to be bounded. The proof that Condition (3) is necessary for every solution of System (1) to be bounded will be presented in [7]. More precisely, we will show in [7] that when

$$C_1 = 0, \tag{7}$$

or when

$$C_1 > 0 \text{ and } \beta_2 = 0, \tag{8}$$

or when

$$C_1 > 0,\ \beta_2 > 0 \text{ and } B_2 > 0, \tag{9}$$

or when

$$C_1 > 0,\ \beta_2 > 0,\ \gamma_2 > 0 \text{ and } B_2 = C_2 = 0, \tag{10}$$

System (1) has unbounded solutions in a certain region of the parameters and for some initial conditions. In fact, we will prove that in each of the 127 special cases that correspond to Conditions (7)-(10) the component $\{x_n\}$ or the component $\{y_n\}$, of each solution, is unbounded in a certain region of the parameters and for some initial conditions. More precisely, we will obtain the boundedness characterization of each one of the 127 special cases. This will complete the proof that Condition (3) is a necessary and sufficient condition for every solution of System (1) to be bounded.

Finally, we should mention that in [7] we will give a detailed presentation of the boundedness characterizations of all 1029 special cases of the rational system

$$\left\{ \begin{aligned} x_{n+1} &= \frac{\alpha_1+\beta_1 x_n}{A_1+B_1 x_n+C_1 y_n}, \\ y_{n+1} &= \frac{\alpha_2+\beta_2 x_n+\gamma_2 y_n}{A_2+B_2 x_n+C_2 y_n}, \end{aligned} \right. \qquad n = 0, 1, \ldots . \tag{11}$$

The two special cases:

$$(12,31) \quad \text{and} \quad (30,31)$$

of System (11), will be shown to have the boundedness characterization (U,U) which refutes the conjectures given in [11], for these two special cases.

2. Necessary and sufficient conditions for the boundedness of solutions of system (1)

The main result in this section is the following theorem.

Theorem 1. *Every solution of System (1) is bounded, if and only if, Condition (3) is satisfied.*

As we mentioned in the Introduction, here we will establish that Condition (3) is sufficient for every solution of System (1) to be bounded. That is, we will show that each one of the 20 special cases listed in (4) has the boundedness characterization **(B,B)**.

The proof of boundedness of solutions of the four special cases $(21,7)$, $(21,8)$, $(21,22)$, and $(21,23)$ is straightforward and the details will be omitted. The special case $(21,26)$ can be reduced to the special case $(29,16)$ which we will be investigated in Lemma 3 of this paper. For the remaining two special cases, $(21,16)$ and $(21,31)$, we have the following lemma.

Lemma 1. *Assume that*

$$\alpha_1, \beta_1, A_2 \in (0,\infty) \quad \textit{and} \quad \alpha_2 \in [0,\infty).$$

Then every solution of the system

$$\text{(12)} \qquad \begin{cases} x_{n+1} = \frac{\alpha_1+\beta_1 x_n}{y_n}\,, \\ \\ y_{n+1} = \frac{\alpha_2+\beta_2 x_n}{A_2+y_n}\,, \end{cases} \qquad n = 0,1,\dots$$

is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (12). Note that

$$\text{(13)} \qquad \frac{x_{n+1}}{y_{n+1}} = \frac{\alpha_1+\beta_1 x_n}{\alpha_2+\beta_2 x_n}\,\frac{y_n+A_2}{y_n}, \quad \text{for } n \geq 0,$$

together with the first equation of the system imply that the sequences $\frac{x_n}{y_n}$ and x_n are bounded from below by a positive constant. From this and

$$\text{(14)} \qquad y_{n+1} = \frac{\alpha_2}{A_2+y_n} + \frac{\beta_2 x_n}{A_2+y_n} = \frac{\alpha_2}{A_2+y_n} + \frac{x_n}{y_n}\,\frac{\beta_2 y_n}{A_2+y_n}$$

it follows that the component $\{y_n\}$ of the solution $\{x_n, y_n\}$ is bounded from below by positive constants. From this and in view of (13) it follows that the ratio $\frac{x_n}{y_n}$ is bounded and the result follows. ■

The next lemma establishes the boundedness character of solutions of the special case $(21, 34)$.

Lemma 2. *Assume that*

$$\beta_2, \gamma_2, A_2 \in (0, \infty).$$

Then every solution of the system

$$(15) \qquad \left\{ (21,34): \begin{array}{l} x_{n+1} = \frac{\alpha_1 + x_n}{y_n}, \\ \\ y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{A_2 + y_n}, \end{array} \right. \qquad n = 0, 1, \ldots$$

is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (15). Note that

$$(16) \qquad x_{n+2} = \frac{\alpha_1 + x_{n+1}}{y_{n+1}} = \frac{\alpha_1 + \alpha_1 y_n + x_n}{\beta_2 x_n + \gamma_2 y_n} \, (1 + \frac{A_2}{y_n}), \quad \text{for} \;\; n \geq 0,$$

implies that the component $\{x_n\}$ of the solution $\{x_n, y_n\}$ is bounded from below by a positive constant from which the result follows. ■

Lemma 3. *In each of the following nine special cases:*

$$(29,7),\ (29,8),\ (29,16),\ (29,22),$$

$$(29,26),\ (29,31),\ (29,34),\ (29,41),\ (29,46),$$

every solution is bounded.

Proof. The proof of boundedness for the special case $(29, 8)$ will be given separately. The remaining eight special cases are included in the system

$$(17) \qquad \left\{ \begin{array}{l} x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{1 + y_n}, \\ \\ y_{n+1} = \frac{\alpha_2 + x_n + \gamma_2 y_n}{A_2 + C_2 y_n}, \end{array} \right. \qquad n = 0, 1, \ldots$$

with

$$\beta_1 > 0, \;\; A_2 + \gamma_2 > 0 \;\; \text{and} \;\; (\gamma_2 = 0 \;\; \text{or} \;\; C_2 > 0).$$

First note that

$$(18) \qquad \frac{x_{n+1}}{y_{n+1}} = \frac{\alpha_1 + \beta_1 x_n}{\alpha_2 + x_n + \gamma_2 y_n} \, \frac{A_2 + C_2 y_n}{1 + y_n}, \;\; \text{for all} \;\; n \geq 0.$$

We divide the proof in five cases:

Case 1:

$$\alpha_2 > 0.$$

Clearly the quotient, $\frac{x_{n+1}}{y_{n+1}}$ is bounded from above by the positive number M

$$M = \begin{cases} (\frac{\alpha_1}{\alpha_2} + \beta_1)\ \max\{A_2, C_2\} & \text{if} \quad A_2 > 0 \ \text{ and } \ C_2 > 0 \\ \\ (\frac{\alpha_1}{\alpha_2} + \beta_1) A_2 & \text{if} \quad A_2 > 0 \ \text{ and } \ C_2 = 0 \\ \\ (\frac{\alpha_1}{\alpha_2} + \beta_1) C_2 & \text{if} \quad A_2 = 0 \ \text{ and } \ C_2 > 0. \end{cases}$$

Hence,

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{1 + y_n} < \alpha_1 + \beta_1 \frac{x_n}{y_n} < \alpha_1 + \beta_1 M, \quad \text{for } \ n \geq 1$$

and so the component $\{x_n\}$ of every solution is bounded. Then, eventually,

$$y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + C_2 y_n} + \frac{x_n}{A_2 + C_2 y_n} \leq \frac{\max(\alpha_2, \gamma_2)}{\min(A_2, C_2)} + \frac{1}{C_2}\frac{x_n}{y_n} \leq \frac{\max(\alpha_2, \gamma_2)}{\min(A_2, C_2)} + \frac{M}{C_2}$$

when

$$A_2 > 0 \quad \text{and} \quad C_2 > 0,$$

or

$$y_{n+1} \leq \frac{\gamma_2}{C_2} + \frac{M}{C_2}$$

when

$$\alpha_2 = A_2 = 0 \quad \text{and} \quad C_2 > 0,$$

or

$$y_{n+1} \leq \frac{\alpha_2 + \alpha_1 + \beta_1 M}{A_2}$$

when

$$\gamma_2 = C_2 = 0 \quad \text{and} \quad A_2 > 0.$$

Therefore, the component $\{y_n\}$ of every solution is also bounded.

Case 2:

$$\alpha_2 = 0 \ \text{ and } \ \gamma_2 > 0.$$

Observe that the sequence $\{x_n + \gamma_2 y_n\}$ is bounded from below by a positive constant, namely m. Therefore, the quotient, $\frac{x_{n+1}}{y_{n+1}}$ is bounded from above by the positive number M

$$M = \begin{cases} (\frac{\alpha_1}{m} + \beta_1) \cdot \max\{A_2, C_2\} & \text{if} \quad A_2 > 0 \ \text{ and } \ C_2 > 0 \\ \\ (\frac{\alpha_1}{m} + \beta_1) A_2 & \text{if} \quad A_2 > 0 \ \text{ and } \ C_2 = 0 \\ \\ (\frac{\alpha_1}{m} + \beta_1) C_2 & \text{if} \quad A_2 = 0 \ \text{ and } \ C_2 > 0 \end{cases}$$

from which the result follows.

Case 3:

$$\alpha_2 = \gamma_2 = 0 \quad \text{and} \quad A_2C_2 > 0.$$

Observe that the ratio $\frac{x_{n+1}}{y_{n+1}}$ is bounded from below by a positive constant, and so consequently, the sequence x_n is bounded from below by a positive constant, namely m. Therefore, the quotient, $\frac{x_{n+1}}{y_{n+1}}$ is bounded from above by the positive number M

$$M = (\frac{\alpha_1}{m} + \beta_1)\ \max\{A_2, C_2\}$$

from which the result follows.

Case 4:

$$\alpha_2 = \gamma_2 = A_2 = 0 \quad \text{and} \quad C_2 > 0.$$

The proof in this case has been established in [3].

Case 5:

$$\alpha_2 = \gamma_2 = C_2 = 0 \quad \text{and} \quad A_2 > 0.$$

In this case the component $\{x_n\}$ of the solution satisfies a second-order rational equation for which it is known that every solution is bounded and the result follows.

To complete the proof of the lemma we need to establish the boundedness of all solutions of the system

$$(19) \qquad \left\{ (29,8) : \begin{array}{l} x_{n+1} = \frac{\alpha_1+\beta_1 x_n}{1+y_n}\,, \\ \\ y_{n+1} = \frac{x_n}{y_n}\,, \end{array} \right. \qquad n = 0, 1, \ldots$$

with positive parameters. Note that

$$x_{n+1} = \frac{\alpha_1}{1+y_n} + \frac{y_{n-1}}{x_{n-1}+y_{n-1}}\,\frac{\beta_1\alpha_1}{1+y_{n-1}} + \frac{x_{n-1}}{x_{n-1}+y_{n-1}}\,\frac{\beta_1^2 y_{n-1}}{1+y_{n-1}}$$

from which it follows that the component $\{x_n\}$ of the solution is bounded.

Also,

$$y_{n+1} = \frac{\alpha_1 y_{n-1}}{(1+y_{n-1})(\alpha_1+\beta_1 x_{n-2})} + \frac{\alpha_1 y_{n-1}}{1+y_{n-1}}\,\frac{1}{\frac{\alpha_1}{y_{n-2}}+\beta_1 y_{n-1}} + \frac{\beta_1 y_{n-1}}{1+y_{n-1}}$$

implies that the component $\{x_n\}$ of the solution is bounded. ■

The next lemma establishes the boundedness of solutions in the special case:

$$(20) \qquad \left\{ (29,23) : \begin{array}{l} x_{n+1} = \frac{\alpha_1+\beta_1 x_n}{1+y_n}\,, \\ \\ y_{n+1} = \frac{\alpha_2+\beta_2 x_n}{y_n}\,, \end{array} \right. \quad , \quad n = 0, 1, \ldots .$$

Lemma 4. *Assume that*

$$\beta_1, \alpha_2 \in (0, \infty).$$

Then every solution of System (20) is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (20).
Then, clearly

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\alpha_1 + \beta_1 x_n}{\alpha_2 + x_n} \frac{y_n}{1 + y_n} < \frac{\max\{\alpha_1, \beta_1\}}{\min\{\alpha_2, \beta_2\}}, \quad n = 0, 1, \ldots \tag{21}$$

and

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{1 + y_n} < \alpha_1 + \beta_1 \frac{\max\{\alpha_1, \beta_1\}}{\min\{\alpha_2, \beta_2\}}, \quad n = 1, 2, \ldots$$

which implies that the sequence $\{x_n\}$ is bounded. Assume for the sake of the contradiction that there exists a sequence of indices $\{n_i\}$ such that

$$\lim_{n\to\infty} y_{n_i+1} = \infty \quad \text{and} \quad y_{n_i+1} > y_n, \quad \text{for} \quad n < n_i + 1.$$

From

$$y_{n+1} = \frac{\alpha_2 + x_n}{y_n}$$

it follows that

$$y_{n_i}, y_{n_i-2} \to 0 \quad \text{and} \quad y_{n_i-1}, y_{n_i-3} \to \infty.$$

Clearly,

$$x_{n_i} \to 0 \quad \text{and} \quad x_{n_i-1} \to \alpha_2.$$

Then, eventually,

$$y_{n_i+1} = \frac{\alpha_2 + \beta_2 x_{n_i}}{\alpha_2 + \beta_2 x_{n_i-1}} y_{n_i-1} < y_{n_i-1}$$

which is a contradiction and the proof is complete. ∎

Next, we establish the boundedness of solutions in the remaining two special cases which are listed in (4):

$$(21, 41) \quad \text{and} \quad (21, 46).$$

The change of variables

$$y_n = \gamma_2 + Y_n$$

transforms system $(21, 41)$ to a system of the form $(29, 31)$, whose boundedness was established in Lemma 3. Finally, the following lemma establishes the boundedness of solutions of the special case $(21, 46)$.

Lemma 5. *Assume that*

$$\beta_1, \alpha_2, \beta_2, \gamma_2, A_2 \in (0, \infty).$$

Then every solution of the system

$$(22) \qquad \begin{cases} (21, 46): & \begin{array}{l} x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{y_n}, \\ \\ y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n}, \end{array} \end{cases} \qquad n = 0, 1, \ldots$$

is bounded.

Proof. Let $\{x_n, y_n\}$ be a solution of System (22). Clearly,

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + y_n} \geq \frac{\min\{\alpha_2, \gamma_2\}}{\max\{A_2, 1\}}$$

and so the component $\{y_n\}$ of the solution is bounded from below by the positive number

$$m = \frac{\min\{\alpha_2, \gamma_2\}}{\max\{A_2, 1\}}.$$

From this and in view of

$$\frac{x_{n+1}}{y_{n+1}} = \frac{\alpha_1 + \beta_1 x_n}{\alpha_2 + \beta_2 x_n + \gamma_2 y_n} \left(\frac{A_2}{y_n} + 1\right) \leq \left(\frac{\alpha_1}{\alpha_2} + \frac{\beta_1}{\beta_2}\right) \left(\frac{A_2}{m} + 1\right), \quad \text{for all } n \geq 1,$$

we see that the component $\{x_n\}$ of the solution is bounded. By the second equation of the system it follows that the component $\{y_n\}$ of the solution is also bounded and the proof is complete. ■

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