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PIECEWISE R&D DYNAMICS ON COSTS

Abstract. We consider an R&D cost reduction function in a Cournot competition model inspired by the logistic equation. We present the associated game and observe the existence of three different economical behaviors depending upon the firms' decisions in terms of investments. We exhibit the boundaries of these investment regions.

Key words: strategic R&D, Cournot duopoly model, patents.

AMS Mathematics Subject Classification: 37N40, 91A10, 91A40.

1. Introduction

We consider a Cournot competition model where two firms invest in R&D projects to reduce their production costs. This competition is modeled by a two-stage game (see d'Aspremont and Jacquemin [2]). In the first subgame, two firms choose, simultaneously, the R&D investment strategy to reduce their initial production costs. In the second subgame, the two firms are involved in a Cournot competition with production costs equal to the reduced cost determined by the R&D investment program.

We use an R&D cost reduction function inspired by the logistic equation (see Equation 2 in [6]) which was first introduced in Ferreira et al[6]. The main differences between this cost function and the standard R&D cost reduction function (see [2]) are explained in that same paper.

For the first subgame, consisting of an R&D investment program, we observe the existence of four different Nash investment equilibria regions that we define as follows (see [6]): a competitive Nash investment region C where both firms invest, a single Nash investment region S_1 for firm F_1, where only firm F_1 invests, a single Nash investment region S_2 for firm F_2, where only firm F_2 invests, and a nil Nash investment region N, where neither of the firms invest.

The nil Nash investment region N consists of four nil Nash investment regions, N_{LL}, N_{LH}, N_{HL} and N_{HH} where neither of the firms invest and so have constant production costs. The single Nash investment region S_i can be decomposed into two disjoint regions: a *single favorable Nash investment* *region* S_i^F where the production costs, after investment, are favorable to firm F_i; and a *single recovery Nash investment region* S_i^R where the production costs, after investment are, still, favorable to firm F_j but firm F_i recovers, slightly, from its initial disadvantage. The nil Nash investment region N determines the set of all production costs that are fixed by the dynamics. The competitive Nash investment region determines the region where the production costs of both firms evolve along the time. The single Nash investment region S_1 determines the set of production costs where the production cost of firm F_2 is constant, along the time, and just the production costs of firm F_1 evolve. Similarly, the single Nash investment region S_2 determines the set of production costs where the production cost of firm F_1 is constant, along the time, and just the production costs of firm F_2 evolve.

In this paper we exhibit the boundaries of each of these Nash investment regions.

2. The model

As in Ferreira et al[6] we consider an economy with a monopolistic sector with two firms, F_1 and F_2, each one producing a differentiated good, and assume that the representative consumer preferences are described by the following utility function

$$U(q_1, q_2) = \alpha q_1 + \alpha q_2 - \left(\beta q_1^2 + 2\gamma q_1 q_2 + \beta q_2^2\right)/2, \tag{1}$$

where q_i is the quantity produced by the firm F_i, and $\alpha, \beta > 0$. The inverse demands are linear and, letting p_i be the price of the good produced by the firm F_i, they are given, in the region of quantity space where prices are positive, by

$$p_i = \alpha - \beta q_i - \gamma q_j.$$

The goods can be substitutes $\gamma > 0$, independent $\gamma = 0$, or complements $\gamma < 0$.

Demand for good i is always downward sloping in its own price and increases (decreases) the price of the competitor, if the goods are substitutes (complements). The ratio γ^2/β^2 expresses the degree of product differentiation ranging from zero, when the goods are independent, to one, when the goods are perfect substitutes. When $\gamma > 0$ and γ^2/β^2 approaches one, we are close to a homogeneous market.

The firm F_i invests an amount v_i in an R&D program $a_i : \mathbb{R}_0^+ \to [b_i, c_i]$ that reduces its production cost to

$$a_i(v_i) = c_i - \frac{\epsilon(c_i - c_L)v_i}{\lambda + v_i}. \tag{2}$$

Now, we explain the parameters of the R&D program: (*i*) the parameter c_i is the unitary production cost of firm F_i at the beginning of the period satisfying $c_L \leq c_i \leq \alpha$; (*ii*) the parameter c_L is the minimum attainable production cost; (*iii*) the parameter $0 < \epsilon < 1$ as the following meaning: since $b_i = a_i(+\infty) = c_i - \epsilon(c_i - c_L)$, the maximum reduction $\Delta_i = \epsilon(c_i - c_L)$ of the production cost is a percentage $0 < \epsilon < 1$ of the difference between the current cost c_i and the lowest possible production cost c_L; (*iv*) the parameter $\lambda > 0$ can be seen as a measure of the inverse of the quality of the R&D program for firm F_i, because a smaller λ will result in a bigger reduction of the production costs for the same investment. Note that, in particular, $c_i - a_i(\lambda)$ gives half $\Delta_i/2$ of the maximum possible reduction Δ_i of the production cost for firm F_i. Let us define, for simplicity of notation, $\eta_i = \epsilon(c_i - c_L)$.

The sets of possible new production costs for firms F_1 and F_2, given initial production costs c_1 and c_2 are, respectively,

$$A_1 = A_1(c_1, c_2) = [b_1, c_1] \quad \text{and} \quad A_2 = A_2(c_1, c_2) = [b_2, c_2],$$

where $b_i = c_i - \epsilon(c_i - c_L)$, for $i \in \{1, 2\}$.

The R&D programs a_1 and a_2 of the firms determine a bijection between the *investment region* $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ of both firms and the *new production costs region* $A_1 \times A_2$, given by the map

$$\begin{array}{rcccc} \mathbf{a} = (a_1, a_2) & : & \mathbb{R}_0^+ \times \mathbb{R}_0^+ & \longrightarrow & A_1 \times A_2 \\ & & (v_1, v_2) & \longmapsto & (a_1(v_1), a_2(v_2)) \end{array}$$

where

$$a_i(v_i) = c_i - \frac{\eta_i v_i}{\lambda + v_i}.$$

We denote by $W = (W_1, W_2) : \mathbf{a}\left(\mathbb{R}_0^+ \times \mathbb{R}_0^+\right) \to \mathbb{R}_0^+ \times \mathbb{R}_0^+$

$$W_i(a_i) = \frac{\lambda(c_i - a_i)}{a_i - c_i - \eta_i}$$

the inverse map of $\mathbf{a}$.

3. Output and R&D investment regions

The Cournot competition with R&D investment programs to reduce the production costs consists of two subgames in one period of time. The first subgame is an R&D investment program, where both firms have initial production costs and choose, simultaneously, their R&D investment strategies to obtain lower new production costs. The second subgame is a typical

Cournot competition on quantities with production costs equal to the reduced costs determined by the R&D investment program. As it is well known, the second subgame has a unique perfect Nash equilibrium. The analysis of the first subgame is of higher complexity and can be found with detail in Ferreira et al[6].

The new production costs region can be decomposed, at most, into three disconnected economical regions characterized by the optimal output level of the firms (see Figure 1):

M_1 The *monopoly region* M_1 of firm F_1 that is characterized by the optimal output level of firm F_1 being the monopoly output and, hence, the optimal output level of firm F_2 is zero;

D The *duopoly region* D that is characterized by the optimal output levels of both firms being non-zero and, hence, below their monopoly output levels;

M_2 The *monopoly region* M_2 of firm F_2 that is characterized by the optimal output level of firm F_2 being the monopoly output and, hence, the optimal output level of firm F_1 is zero.

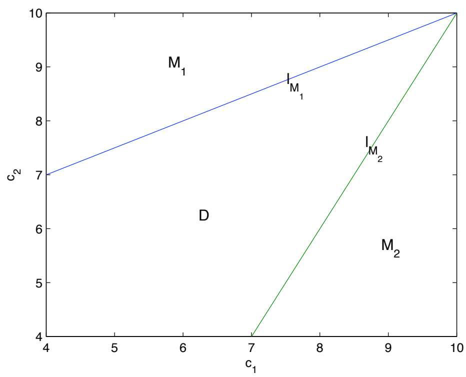

Figure 1. We exhibit the duopoly region D and the monopoly regions M_1 and M_2 for firms F_1 and F_2, respectively, in terms of their new production costs (a_1, a_2); l_{M_i} with $i \in \{1, 2\}$ are the boundaries between M_i and D. Reproduced from [6].

The boundary between the duopoly region D and the monopoly region M_i is l_{M_i} with $i \in \{1, 2\}$. The explicit expression characterizing l_{M_i}, the boundary between the monopoly region M_i and the duopoly region D, is presented in [6].

To determine the *best investment response function* $V_1(v_2)$ of firm F_1 to a given investment v_2 of firm F_2, we study, separately, the cases where the new production costs $(a_1(v_1, v_2), a_2(v_1, v_2))$ belong to (i) the monopoly region M_1; (ii) the duopoly region D; (iii) the monopoly region M_2.

Let c_L be the minimum attainable production cost and α the market saturation. Given production costs $(c_1, c_2) \in [c_L, \alpha] \times [c_L, \alpha]$, the *Nash investment equilibria* $(v_1, v_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ are the solutions of the system

$$\begin{cases} v_1 = V_1(v_2) \\ v_2 = V_2(v_1) \end{cases}$$

where V_1 and V_2 are the best investment response functions computed in the previous sections.

All the results presented, consistently with [6], hold in an open region of parameters $(c_L, \epsilon, \alpha, \lambda, \beta, \gamma)$ containing the point $(4, 0.2, 10, 10, 0.013, 0.013)$.

The Nash investment equilibria consists of a unique, or two, or three points depending upon the pair of initial production costs. The set of all Nash investment equilibria form the *Nash investment equilibrium set* (see Figure 2):

C the *competitive Nash investment region* C that is characterized by both firms investing;

S_i the *single Nash investment region* S_i that is characterized by only one of the firms investing;

N the *nil Nash investment region* N that is characterized by neither of the firms investing.

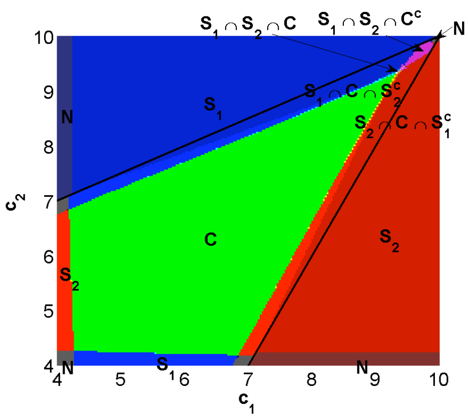

Figure 2. Full characterization of the Nash investment regions in terms of the firms' initial production costs (c_1, c_2). The monopoly lines l_{M_i} are colored black. The nil Nash investment region N is colored grey. The single Nash investment regions S_1 and S_2 are colored blue and red, respectively. The competitive Nash investment region C is colored green. The region where S_1 and S_2 intersect are colored pink, the region where S_1 and C intersect are colored lighter blue and the region where S_2 and C intersect are colored yellow. The region where the regions S_1, S_2 and C intersect are colored lighter grey. Reproduced from [6].

In Figure 2, the Nil Nash investment region is the union of N_{LL}, N_{LH}, N_{HL} and N_{HH} and the Single Nash investment region is the union of S_i^F and S_i^R. The economical meaning of the subregions of N and S_i is explained in the next subsections.

Denote by $R = [c_L, \alpha] \times [c_L, \alpha]$ the region of all possible pairs of production costs (c_1, c_2). Let $A^c = R - A$ be the complementary of A in R and let $R_{A \cap B}$ be the intersection between the Nash investment region A and the Nash investment region B.

4. Single Nash investment region

The *single Nash investment region* S_i consists of the set of production costs (c_1, c_2) with the property that the Nash investment equilibrium set contains a pair (v_1, v_2) with the Nash investment $v_i = V_i(0) > 0$ and the Nash investment $v_j = V_j(v_i) = 0$, for $j \neq i$.

The single Nash investment region S_i can be decomposed into two disjoint regions: a *single favorable Nash investment region* S_i^F where the production costs, after investment, are favorable to firm F_i, and in a *single recovery Nash investment region* S_i^R where the production costs, after investment are, still, favorable to firm F_j but firm F_i recovers a little from its disadvantageous (see Figure 3).

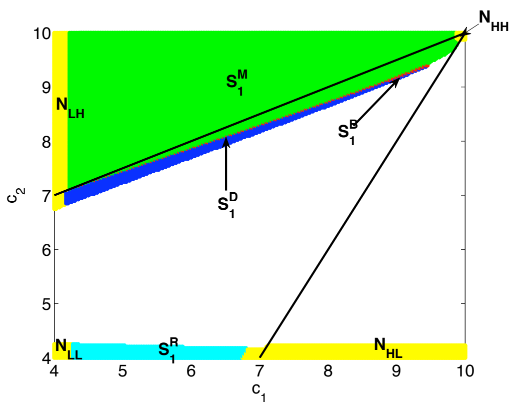

Figure 3. Full characterization of the single Nash investment region S_1 and of the nil Nash investment region N in terms of the firms' initial production costs (c_1, c_2). The subregions N_{LL}, N_{LH}, N_{HL} and N_{HH} of the nil Nash investment region N are colored yellow. The subregion S_1^R of the single Nash investment region S_1 is colored lighter blue. The subregion S_1^F of the single Nash investment region S_1 is decomposed in three subregions: the *single Duopoly region* S_i^D colored blue, the *single Monopoly region* S_i^M colored green and the *single Monopoly boundary region* S_i^B colored red. Reproduced from [6].

The single favorable Nash investment region S_i^F can be decomposed into three regions: the *single Duopoly region* S_i^D, the *single Monopoly region* S_i^M and the *single Monopoly boundary region* S_i^B (see Figure 3). For every cost $(c_1, c_2) \in S_i^F$, let $(a_1(v_1), a_2(v_2))$ be the Nash new investment costs obtained by the firms F_1 and F_2 choosing the Nash investment equilibrium (v_1, v_2) with $v_2 = 0$. The single duopoly region S_i^D consists of all production costs (c_1, c_2) such that for the Nash new investment costs $(a_1(v_1), a_2(v_2))$ the firms are in the duopoly region D (see Figure 3). The single monopoly region S_i^M consists of all production costs (c_1, c_2) such that for the Nash new costs $(a_1(v_1), a_2(v_2))$ the Firm F_i is in the interior of the Monopoly region M_i. The single monopoly boundary region S_i^B consists of all production costs (c_1, c_2) such that the Nash new investment costs $(a_1(v_1), a_2(v_2))$ are in the boundary of the Monopoly region l_{M_i}.

We are going to characterize the boundary of the single monopoly region S_1^M (due to symmetry, a similar characterization holds for S_2^M). In the next subsections, we present the boundaries of S_1^M by separating them into four distinct boundaries: the *upper boundary* $U_{S_1}^M$, that is the union of a vertical segment line $U_{S_1}^l$ with a curve $U_{S_1}^c$, the *intermediate boundary* $I_{S_1}^M$, the *lower boundary* $L_{S_1}^M$ and the *left boundary* $Le_{S_1^M}$ (see Figure 4). The left boundary of the single monopoly region $Le_{S_1}^M$ is the right boundary d_1 of the nil Nash investment region N_{LH} that will be characterized in Section 5.

The boundary of the single monopoly boundary region S_1^B is the union of a *upper boundary* $U_{S_1}^B$ and a *lower boundary* $L_{S_1}^B$(see Figure 5).

The boundary of the single duopoly region S_1^D is the union of a *upper boundary* $U_{S_1}^D$, a *lower boundary* $L_{S_1}^D$ and a *left boundary* $Le_{S_1}^D$ (see Figure 6). The left boundary of the single duopoly region $Le_{S_1}^D$ is the right boundary d_3 of the nil Nash investment region N_{LH} that will be characterized in Section 5.

The single recovery Nash investment region S_1^R has three boundaries: the *upper boundary* $U_{S_1}^R$, the *left boundary* $Le_{S_1}^R$, and the *right boundary* $R_{S_1}^R$ (see Figure 7).

4.1. Boundary of the single monopoly region S_1^M

In this subsection we exhibit the boundary of the single monopoly region S_1^M.

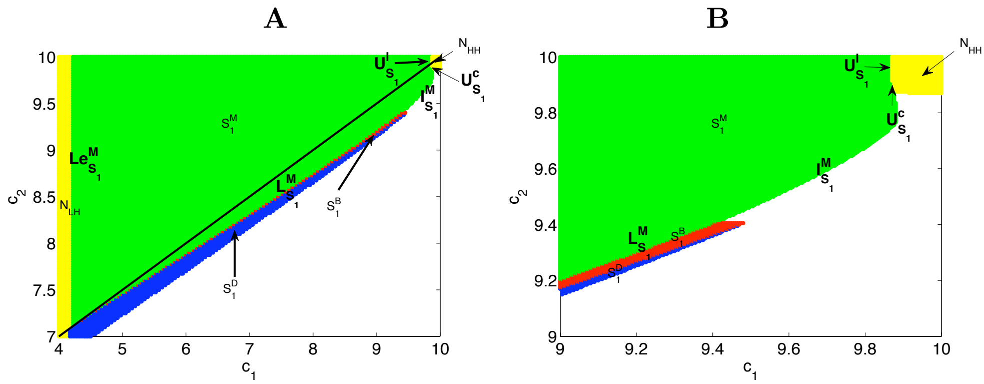

Figure 4. (**A**) Full characterization of the boundaries of the single monopoly region S_1^M: the upper boundary $U_{S_1}^C$ is the union of a vertical segment line $U_{S_1}^l$ with a curve $U_{S_1}^c$; the lower boundary $L_{S_1}^M$; and the left boundary $Le_{S_1}^M$; (**B**) Zoom of the upper part of figure (**A**) where the boundaries $U_{S_1}^C$ and $U_{S_1}^l$ can be seen in more detail. Reproduced from [6].

4.2. Boundary of the single monopoly boundary region S_1^B

In this subsection we exhibit the boundary of the single monopoly boundary region S_1^B.

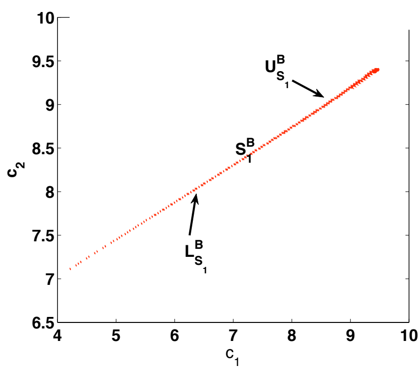

Figure 5. Full characterization of the boundaries of the single monopoly boundary region S_1^B: the upper boundary $U_{S_1}^B$ and the lower boundary $L_{S_1}^B$. Reproduced from [6].

Note that the upper boundary of the single monopoly boundary region $U^B_{S_1}$ is the lower boundary of the single monopoly region $L^M_{S_1}$.

4.3. Boundary of the single duopoly region S^D_1

In this subsection we exhibit the boundary of the single duopoly region S^D_1.

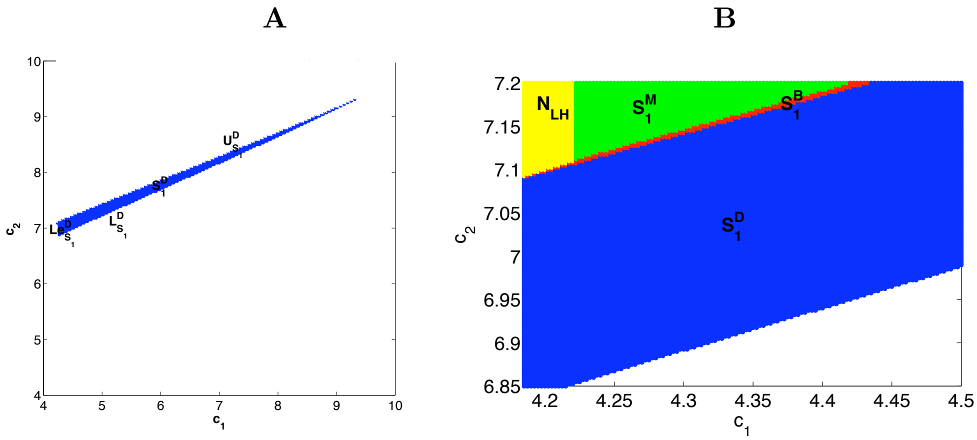

Figure 6. (**A**) Full characterization of the boundaries of the single duopoly region S^D_1: the upper boundary $U^D_{S_1}$; the lower boundary $L^D_{S_1}$; and the left boundary $Le^D_{S_1}$; (**B**) Zoom of the lower part of $Le^D_{S_1}$. Reproduced from [6].

Note that the upper boundary of the single duopoly region $U^D_{S_1}$ is the lower boundary of the single monopoly boundary region $L^B_{S_1}$. The left boundary of the single duopoly region $Le^D_{S_1}$ is the right boundary d_3 of the nil Nash investment region N_{LH}.

4.4. Boundary of the single recovery region S^R_1

In this subsection we exhibit the boundary of the single recovery region S^R_1.

The single recovery region S^R_1 (because of the symmetry, a similar characterization holds for S^R_2) has three boundaries: the *upper boundary* $U^R_{S_1}$, the *left boundary* $L^R_{S_1}$, and the *right boundary* $R^R_{S_1}$.

5. Nil Nash investment region

The *nil Nash investment region* N is the set of production costs $(c_1, c_2) \in N$ with the property that $(0, 0)$ is a Nash investment equilibrium. Hence, the nil Nash investment region N consists of all production costs (c_1, c_2) with

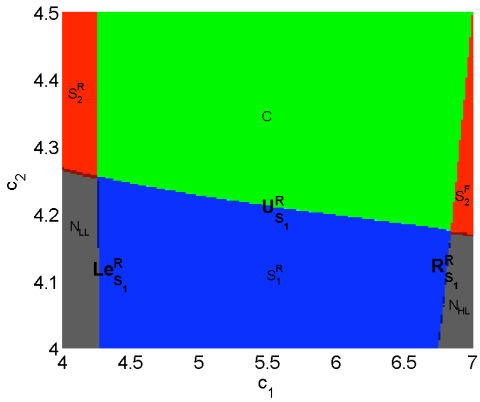

Figure 7. Full characterization of the boundaries of the single recovery region S_1^R: the upper boundary $U^R_{S_1}$; the right boundary $R^R_{S_1}$; and the left boundary $Le^R_{S_1}$. In green the competitive Nash investment region C, in grey the nil Nash investment region N, in red the single Nash investment region S_2 for firm F_2 and in blue the single recovery region S_1^R for firm F_1. Reproduced from [6].

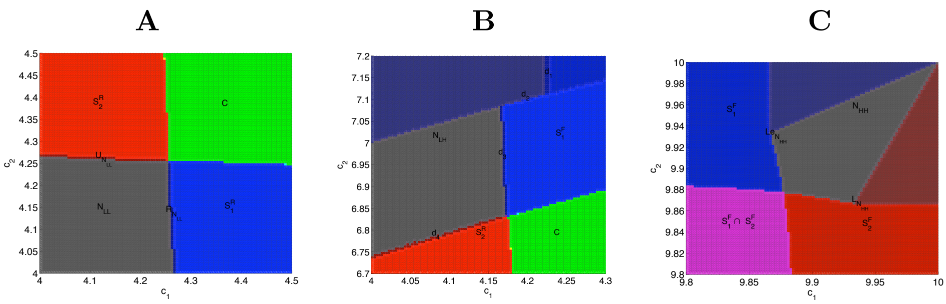

Figure 8. Full characterization of the nil Nash investment region N in terms of the firms' initial production costs (c_1, c_2): (**A**) The subregion N_{LL} of the nil Nash investment region N is colored grey corresponding to initial production cost such that the firms do not invest and do not produce; (**B**) The subregion N_{LH} of the nil Nash investment region N is colored grey corresponding to initial production cost such that the firms do not invest and do not produce and dark blue corresponding to cases where the firms do not invest but firm F_1 produces a certain amount q_1 greater than zero; (**C**) The subregion N_{HH} of the nil Nash investment region N is colored grey corresponding to initial production cost such that the firms do not invest and do not produce; dark blue corresponding to cases where the firms do not invest but firm F_1 produces a certain amount q_1 greater than zero and dark red corresponding to cases where the firms do not invest but firm F_2 produces a certain amount q_2 greater than zero. Reproduced from [6].

the property that the new production costs $(a_1(v_1), a_2(v_2))$, with respect to the Nash investment equilibrium $(0, 0)$, are equal to the production costs (c_1, c_2).

The nil Nash investment region N is the union of four disjoint sets: the set N_{LL} consisting of all production costs that are low for both firms (see Figure **A**); the set N_{LH} (resp. N_{HL}) consisting of all production costs that are low for firm F_1 (resp. F_2) and high for firm F_2 (resp. F_1) (see Figure **B**); and the set N_{HH} consisting of all production costs that are high for both firms (see Figure 8**C**).

6. Competitive Nash investment region

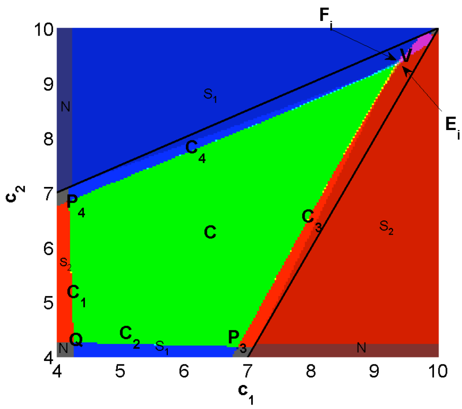

Figure 9. Firms' investments in the competitive Nash investment region. The competitive Nash investment region is colored green, the single Nash investment region S_1 (respectively S_2) is colored blue (respectively red) and the nil Nash investment region N is colored grey. Reproduced from [6].

The *competitive Nash investment region* C consists of all production costs (c_1, c_2) with the property that there is a Nash investment equilibrium (v_1, v_2) with the property that $v_1 > 0$ and $v_2 > 0$. Hence, the new production costs $a_1(v_1, v_2)$ and $a_2(v_1, v_2)$ of firms F_1 and F_2 are smaller than the actual production costs c_1 and c_2 of the firms F_1 and F_2, respectively.

In Figure 2, the boundary of region C consists of four piecewise smooth curves: The curve C_1 is characterized by $a_1(v_1) = c_1$ i.e $v_1 = 0$; the curve C_2 is characterized by $a_2(v_2) = c_2$ i.e $v_2 = 0$; the curve C_3 corresponds to points (c_1, c_2) such that the Nash investment equilibrium $(a_1(v_1), a_2(v_2))$ has the property that $\pi_1(a_1, a_2) = \pi_1(a_1, c_2)$; and the curve C_4 corresponds to points (c_1, c_2) such that the Nash investment equilibrium $(a_1(v_1), a_2(v_2))$ has the property that $\pi_1(a_1, a_2) = \pi_1(c_1, a_2)$.

The curve C_2 (respectively C_1) is the common boundary between the competitive region C and the single recovery region S_2^R (respectively S_1^R). The boundary C_3 can be decomposed in three parts C_3^D, C_3^B and C_3^M. The boundary C_3^D consists of all points in C_3 between the points P_3 and E_3 (see Figure 9). The boundary $C_3^D - \{P_3\}$ has the property of being contained in the lower boundary of the single duopoly region S_2^D of firm F_2. The boundary C_3^B consists of all points in C_3 between the points E_3 and F_3 (see Figure 9). The boundary C_3^B has the property of being contained in the lower boundary of the single monopoly boundary region S_2^B of firm F_2. The boundary C_3^M consists of all points in C_3 between the points F_3 and V (see Figure 9). The boundary C_3^M has the property of being contained in the lower boundary of the single monopoly boundary region S_2^B of firm F_2. Because of the symmetry, a similar characterization holds for the boundary C_4. The points P_3, P_4, Q and V are the corners of the competitive region C (see Figure 9). The point Q is characterized by being in the intersection between the competitive region C and the nil Nash region N_{LL}. The point P_3 (respectively P_4) is characterized by being in the intersection between the competitive region C and the nil region N_{HL}^D (respectively N_{LH}^D). The point E_3 in the boundary of the competitive region C is characterized by belonging to the boundaries of the single duopoly region S_2^D and the single monopoly boundary region S_2^B (see Figure 9). The point F_3 in the boundary of the competitive region C is characterized by belonging to the boundaries of the single monopoly boundary region S_2^B and the single monopoly region S_2^M (see Figure 9).

7. Conclusions

The following conclusions are valid in some parameter region of our model. We described four main economic regions for the R&D deterministic dynamics corresponding to distinct perfect Nash equilibria: a competitive Nash investment region C where both firms invest, a single Nash investment region for firm F_1, S_1, where only firm F_1 invests, a single Nash investment region for firm F_2, S_2, where only firm F_2 invests, and a nil Nash investment region N where neither of the firms invest.

The nil Nash investment region has four subregions: N_{LL}, N_{LH}, N_{HL} and N_{HH}. The single Nash investment region can be divided into four subregions: the single favorable region for firm F_1, S_1^F, the single recovery region for firm F_1, S_1^R, the single favorable region for firm F_2, S_2^F, the single recovery region for firm F_2, S_2^R. The single favorable region S_1^F (due to the symmetry the same characterization holds for S_2^F) is the union of three disjoint regions: the single duopoly region S_1^D where the production costs, after the investments, belong to the duopoly region D; the single monopoly boundary region S_1^B where the production costs, after the investments, belong to the boundary of the monopoly region l_{M_1}; and the single monopoly region S_1^M where the production costs, after the investments, belong to the monopoly region M_1.

From Section to Section , we exhibited the boundaries of the different Nash investment regions.

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References

[1] AMIR R., EVSTIGNEEV I., WOODERS J., *Noncooperative versus cooperative R&D with endogenous spillover rates,* Core Discussion Paper 2001/50, Louvain-la-Neuve, Belgium.

[2] D'ASPREMONT C., JACQUEMIN A., Cooperative and noncooperative R&D in duopoly with spillovers, *American Economic Review,* 78(1988), 1133-1137. Erratum. In *American Economic Review* 80: 641:642.

[3] BRANDER J.A., SPENCER B.J., Strategic commitment with R&D: the symmetric case, *The Bell Journal of Economics,* 14(1983), 225-235.

[4] COURNOT A., *Recherches sur les Principes Mathématiques de la Théorie des Richesses,* Paris, 1838.English edition:*Researches into the Mathematical Principles of the Theory of Wealth,* Edited by N. Bacon. New York: Macmillan, 1897.

[5] DEBONDT R., Spillovers and innovative activities, *International Journal of Industrial Organization,* 15(1997), 1-28.

[6] FERREIRA M., OLIVEIRA B.M.P.M., PINTO A.A., Patents in new technologies, *Journal of Difference equations and Applications,* 15(2009), 1135-1149.

[7] FERREIRA F.A., FERREIRA F., FERREIRA M., PINTO A.A., *Quantity competition in a differentiated duopoly,* Intelligent Engineering Systems and Computational Cybernetics, Springer Netherlands, Chapter 30, 2008.

[8] KAMIEN M., MULLER E., ZANG I., Research joint ventures and R&D cartels, *American Economic Review,* 82(1992), 1293-1306.

[9] KAMIEN M., ZANG I., Competing research joint ventures, *Journal of Economics and Management Strategy,* 2(1993), 23-40.

[10] KATZ M., An analysis of cooperative research and development, *Rand Journal of Economics,* 17(1986), 527-543.

[11] PINTO A.A., *Game Theory and Duopoly Models,* (Postgraduate and Research Level) Interdisciplinary Applied Mathematics, Springer-Verlag, 2008, 1-275.

[12] Pinto A.A., Oliveira B., Ferreira F.A., Ferreira M., *Investing to survive in a duopoly model,* Intelligent Engineering Systems and Computational Cybernetics, Springer Netherlands, Chapter 23, 2008.

[13] Tirole J., *The Theory of Industrial Organization,* MIT Press, MA, 1988.

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