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EULER APPROXIMATIONS CAN DESTROY UNBOUNDED SOLUTIONS *

Abstract. We show that there are ordinary differential equations in $\mathbb{R}^d$ with unbounded solutions, for which the difference equations obtained by using the forward Euler method have all solutions bounded.

Key words: differential equations, difference equations, Euler method, unbounded solutions.

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1. Introduction

The most straightforward way of numerically solving an ordinary differential equation in $\mathbb{R}^d$ is to use the *Euler method* (called also the *forward Euler method*). The right-hand side of a differential equation is a vector field; with the Euler method we move consecutively by those vectors multiplied by a small constant ε, called *step*. That is, we approximate the differential equation

$$\dot{\mathbf{x}} = F(\mathbf{x}) \tag{1}$$

with the difference equation

$$\mathbf{x}_n = \mathbf{x}_{n-1} + \varepsilon F(\mathbf{x}_{n-1}). \tag{2}$$

We will call (2) the *Euler approximation of* (1) with step ε.

If ε is very small, the solutions of (2) approximate well the solution of (1) in the sense that if $\mathbf{x}(t)$ is a solution to (1) with the initial condition $\mathbf{x}(0) = \mathbf{y}$ and $(\mathbf{x}_n)$ is the solution to (2) with the initial condition $\mathbf{x}_0 = \mathbf{y}$, then $\mathbf{x}_n$ is close to $\mathbf{x}(n\varepsilon)$. However, this closeness works only for finite pieces of trajectories. The smaller ε is, the larger $n\varepsilon$ we can consider, but we can

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never claim (in the general case) that the solutions are close to each other forever.

From the point of view of the Dynamical Systems Theory the long-term behavior of the trajectories is the most interesting thing. Thus, it is tempting to try to show that even though the quantitative behaviors of the solutions to a differential equation and its Euler approximations are different, the qualitative behavior (for small $\varepsilon > 0$) is maybe similar. The standard example of the harmonic oscillator

$$(\dot{x}, \dot{y}) = (-y, x)$$

shows that this is not the case. The solutions of the differential equation are all bounded (they are circles), while the solutions of the Euler approximations are all (except the fixed point $(0, 0)$) unbounded "spirals."

The mechanism of the above example is simple, but it produces only solutions of the difference equation "wider" than the solutions of the differential equation.Therefore the question arises: is it possible to construct a differential equation in $\mathbb{R}^d$ that has an unbounded (in forward time) solution, but its Euler approximations for sufficiently small ε have only bounded solutions? In this paper we construct such examples.

We are of course not the first authors comparing differential equations and their discretizations. The literature on this subject is substantial; let us mention for instance [4], [6], [7] and [8] and the references therein. Implications of differences in the behavior of differential and difference equations in some models in economy are discussed in [1] (see also [3] and [2]). However, all those papers concentrate on a more or less local behavior of the solutions – what happens in some compact set. Here we investigate a global behavior of solutions – are they bounded or not? This type of questions has been studied for instance in [5], but for a different dependence between the differential and difference equations.

Let us introduce notation that will be used throughout the paper. We will consider equations (1) and (2) with the function F different in different examples. For the example in $\mathbb{R}^2$, we will set $\mathbf{x} = (x, y)$ and $\mathbf{x}_n = (x_n, y_n)$; similarly, for the examples in $\mathbb{R}^3$ we will set $\mathbf{x} = (x, y, z)$ and $\mathbf{x}_n = (x_n, y_n, z_n)$.

The solutions of the difference equation (2) are the trajectories of a dynamical system with discrete time, given by the map $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$, defined by

$$f_\varepsilon(\mathbf{x}) = \mathbf{x} + \varepsilon F(\mathbf{x}). \tag{3}$$

Note that there is no reason for the map f_ε to be one-to-one. We would also like to consider the solutions of the differential equation (1) as the trajectories of a dynamical system with continuous time. For this we need the function F to be sufficiently regular. It should be at least continuous, and if it is not smooth, we will have to prove that there is uniqueness of solutions of (1). Moreover, we will have to show for each example that F treated as a vector field is *complete*, that is, the solutions of (1) do not escape to infinity in finite time.

Of course, the higher the smoothness of F, the stronger the example. However, we have to take also into account other properties of the solutions. Similarly, the lower the dimension, the stronger the example. Increasing the dimension of the examples is simple; just take $G(\mathbf{x}, \mathbf{y}) = (F(\mathbf{x}), -\mathbf{y})$, where $\mathbf{y} \in \mathbb{R}^m$. Therefore, we will provide more than one example.

Observe that in dimension 1 the criterion for the existence of unbounded solutions is the same for both (1) and (2): $F(x)$ has to be positive for sufficiently large x or negative for sufficiently small x. Therefore we start with dimension 2. In Section we provide an example with F continuous in $\mathbb{R}^2$. In Section we give an example with F of class C^∞ in $\mathbb{R}^3$, having an additional property that f_ε has a fixed point that is globally attracting. Finally, in Section we give an example also in $\mathbb{R}^3$, but with F real analytic (in fact, rational).

2. First example

In this section we give an example in dimension 2. Set

$$F(x,y) = \begin{cases} (-x, 1) & \text{for} \quad x \le -1/e \\ \left(x \ln|x|, \frac{1}{1+\ln|\ln|x||}\right) & \text{for} \quad x \in (-1/e, 0) \\ (-x, 0) & \text{for} \quad x \ge 0 \end{cases} \tag{4}$$

and consider the differential equation (1) and the associated difference equation (2).

Since the way the function F is defined is a little complicated, we have to prove some properties of this function and the equation (1).

Lemma 1. *The function F is continuous.*

Proof. If $x \notin \{-1/e, 0\}$ then continuity at (x, y) follows from the fact that the components of F are compositions of continuous functions. Continuity when $x = 0$ follows from the limits

$$\lim_{x\to 0} x \ln|x| = 0 \quad \text{and} \quad \lim_{x\to 0} \ln\big|\ln|x|\big| = \infty.$$

Continuity when $x = -1/e$ follows from the equalities

$$\ln|-1/e| = -1 \quad \text{and} \quad \ln\big|\ln|-1/e|\big| = 0.$$

This completes the proof. ■

Lemma 2. *For every* $\mathbf{x} \in \mathbb{R}^2$ *there is a unique solution of the differential equation (1) with the initial condition* $\mathbf{x}(0) = \mathbf{x}$ *with* F *given by (4).*

Proof. The function F is locally Lipschitz continuous except at the line $x = 0$ from the left. Therefore the only danger that there is no uniqueness of solutions is that the solutions starting to the left of this line approach it in finite time. However, this is not the case, because for every $\delta \in (-1/e, 0)$ the integral

$$\int_\delta^0 \frac{dx}{x \ln |x|}$$

is infinite (use the substitution $u = \ln |x|$ to see it). ∎

Lemma 3. *The vector field* F *given by (4) is complete.*

Proof. Looking at the definition of F, we see that the only possible problem is whether the solutions do not escape to infinity in the y-direction when x approaches 0 from the left. However, as we established in Lemma 1, the corresponding limit of the second component of F is 0, so the solutions do not escape to infinity there. ∎

Now we prove the main result of this section.

Theorem 1. *For* F *given by (4), equation (1) has an unbounded (in forward time) solution, but all solutions to the associated Euler approximation (2) with* $\varepsilon \in (0, 1)$ *are bounded.*

Proof. We note that the line $\{\mathbf{x} : x = 0\}$ is the set of fixed points of both (1) and (2).

The solution of (1) with the initial condition

$$\mathbf{x}(0) = (-1/e, 0)$$

can be written explicitly for $t \geq 0$ as

$$\mathbf{x}(t) = (-e^{-e^t}, \ln(1 + t)).$$

Since $\ln(1 + t)$ goes to ∞ as $t \to \infty$, this solution is unbounded.

We now consider the Euler approximation (2) for a fixed positive time step $\varepsilon < 1$. It is helpful to consider several cases for the first variable.

(a) If $x_n \geq 0$ then $x_{n+1} = (1 - \varepsilon)x_n$, the second component is a fixed constant y and the solution asymptotically approaches $(0, y)$.

(b) If $-e^{-1/\varepsilon} < x_n < 0$ then $\ln |x_n| < -1/\varepsilon$, so

$$x_{n+1} = x_n(1 + \varepsilon \ln |x_n|) > 0$$

and (a) describes the remainder of the trajectory.

(c) If $x_n \in (-1/e, -e^{-1/\varepsilon}]$ then

$$x_n \ln |x_n| > \frac{e^{-1/\varepsilon}}{\varepsilon} > 0$$

since the function $x \mapsto x \ln |x|$ is decreasing in $[-1/e, 0]$, and thus there is an $N > 0$ such that $x_N > -e^{-1/\varepsilon}$.

(d) If $x_n \leq -1/e$ then $x_{n+1} = (1-\varepsilon)x_n > x_n$ and there is an $N > 0$ such that $x_N > -1/e$.

In each case, solutions approach a fixed point. ∎

3. Second example

In this section we provide a stronger example, but to do it we have to go to dimension 3. By "stronger" we mean that it is of class C^∞ and that the associated Euler approximation has not only all orbits bounded, but it has a globally attracting fixed point.

Let us fix a function $\psi : \mathbb{R} \to [0,1]$ of class C^∞ such that $\psi(t) = 0$ for $t \leq 0$ and $\psi(t) = 1$ for $t \geq 1$. Existence of such functions is proved in practically every textbook in mathematical analysis.

Now we define a function $\varphi : \mathbb{R} \to [-1,1]$ by

$$\varphi(t) = 2\psi(t)\psi(2-t) - 1.$$

This function is of class C^∞, $\varphi(0) = -1$, $\varphi(1) = 1$ and $\varphi(t) = -1$ for $t \geq 2$.

Finally, we define a function $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F_1(x,y,z) = -y\psi(z-1) - x(1-\psi(z-1)), \tag{5}$$

$$F_2(x,y,z) = \quad x\psi(z-1) - y(1-\psi(z-1)), \tag{6}$$

$$F_3(x,y,z) = \varphi(x^2+y^2)\psi(z-2) - z(1-\psi(z-2)). \tag{7}$$

Clearly, F is of class C^∞.

We will consider a differential equation (1) and the difference equation (2).

Lemma 4. *The vector field F given by (5)-(7) is complete.*

Proof. We have

$$(F_1(x,y,z), F_2(x,y,z)) \cdot (x,y) = -(x^2+y^2)(1-\psi(z-1)) \leq 0,$$

so the projections of the solutions of (1) to the (x,y)-plane are bounded. Moreover, $F_3(x,y,z) = -z$ if $z \leq 0$, and $F_3(x,y,z) \leq 1$ if $z \geq 3$. Therefore the third component of the solutions also cannot escape to infinity in finite time. ∎

Theorem 2. *For F given by (5)-(7), the differential equation (1) has an unbounded (in forward time) solution, while the difference equation (2) with any $\varepsilon > 0$ has a globally attracting fixed point.*

Proof. Let us start by listing several properties of the function F that follow immediately from the formulas (5)-(7) and the properties of the functions ψ and φ:
(a) if $z \le 1$ then $(F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)) = (-x,-y,-z)$,
(b) if $z \ge 2$ then $(F_1(x,y,z), F_2(x,y,z)) = (-y,x)$,
(c) if $z \le 2$ then $F_3(x,y,z) = -z$,
(d) if $z \ge 2$ and either $(x,y) = (0,0)$ or $x^2+y^2 \ge 2$ then $F_3(x,y,z) \le -1$,
(e) if $z \ge 3$ and $x^2+y^2 = 1$ then $F_3(x,y,z) = 1$.

By (b) and (e), the solution of (1) with the initial condition

$$(x(0), y(0), z(0)) = (1,0,3)$$

can be written explicitly for $t \ge 0$ as

$$(x(t), y(t), z(t)) = (\cos t, \sin t, t+3).$$

Since $z(t)$ goes to ∞ as $t \to \infty$, this solution is unbounded.

Let us fix $\varepsilon > 0$ and consider the solutions of the equation (2). If $z_0 \le 2$ then by (c) and (a) (x_n, y_n, z_n) converges to $(0,0,0)$ as $n \to \infty$. Thus, for any (x_0, y_0, z_0), if there is $k \ge 0$ such that $z_k \le 2$, then (x_n, y_n, z_n) converges to $(0,0,0)$ as $n \to \infty$. We will show that this happens for all $(x_0, y_0, z_0) \in \mathbb{R}^3$.

Suppose that there is a point $(x_0, y_0, z_0) \in \mathbb{R}^3$ such that $z_n > 2$ for all $n \ge 0$. If $(x_0, y_0) = (0,0)$ then by (b) and (d) $(x_n, y_n) = (0,0)$ and $z_{n+1} \le z_n - \varepsilon$ for all $n \ge 0$, a contradiction. Hence, $(x_0, y_0) \ne (0,0)$. Now by (b) we get $x_{n+1}^2 + y_{n+1}^2 = (1+\varepsilon^2)(x_n^2 + y_n^2)$ for all $n \ge 0$, so there is $N \ge 0$ such that $x_n^2 + y_n^2 \ge 2$ for all $n \ge N$. Then by (d) $z_{n+1} \le z_n - \varepsilon$ for all $n \ge N$, a contradiction. This proves that all solutions of (2) converge to $(0,0,0)$. ■

4. Third example

In this section we modify the example from the preceding section. In one sense, we make it stronger. Namely, the vector field is real analytic, and even rational. However, we pay for it by weakening the properties of the corresponding difference equations. There will be no longer a global attracting fixed point; however, there will be no unbounded solutions. Moreover, this will work not for every $\varepsilon > 0$, but only for $\varepsilon \in (0,1)$.

We define rational functions $\sigma, \tau : [0,\infty) \to \mathbb{R}$ by

$$\sigma(t) = \frac{(t-1)^2(t-2)}{(t+1)^3}, \qquad \tau(t) = \frac{1-2(t-1)^2}{(t+1)^2}.$$

Lemma 5. *The functions σ and τ have the following properties:*
(a) $\sigma(1) = \sigma(2) = 0$,
(b) if $t \in [0,1) \cup (1,2)$ *then* $\sigma(t) \in [-2,0)$,
(c) if $t > 2$ *then* $\sigma(t) \in (0,1)$,
(d) the function σ *is increasing on* $[2,\infty)$,
(e) $\tau(0) = -1$ *and* $\tau(1) = 1/4$,
(f) if $t \geq 2$ *then* $\tau(t) \in (-2,-1/9]$,
(g) if $t \geq 0$ *then* $\tau(t) \leq 1$.

Proof. Properties (a) and (e) follow immediately from the definitions of σ and τ.

Assume that $t \in [0,1) \cup (1,2)$. Then $\sigma(t)$ is negative, the distance of t from 1 is smaller than or equal to the distance of t from -1, and the distance of t from 2 is smaller than or equal to 2 times the distance of t from -1. This proves (b).

Assume that $t > 2$. Then $\sigma(t)$ is positive and the distances of t from 1 and 2 are smaller than the distance of t from -1. This proves (c).

The functions $(t-1)/(t+1)$ and $(t-2)/(t+1)$ are increasing on $[2,\infty)$, so σ is also increasing there. This proves (d).

Simple calculations show that $\tau(t) \leq -1/9$ is equivalent to the inequality

$$(t-2)(17t-4) \geq 0,$$

which holds for $t \geq 2$. Moreover, for $t \geq 2$ we have

$$\tau(t) = \frac{1-2(t-1)^2}{(t+1)^2} > -2\left(\frac{t-1}{t+1}\right)^2 > -2.$$

This proves (f).

If $t \geq 0$ then $1 - 2(t-1)^2 \leq 1$ and $(t+1)^2 \geq 1$, so $\tau(t) \leq 1$. This proves (g). ■

Now we define a function $G = (G_1, G_2) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G_1(x,y) = -y - \sigma(x^2+y^2)x, \tag{8}$$

$$G_2(x,y) = \quad x - \sigma(x^2+y^2)y, \tag{9}$$

and a function $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F_1(x,y,z) = G_1(x,y) = -y - \sigma(x^2+y^2)x, \tag{10}$$

$$F_2(x,y,z) = G_2(x,y) = \quad x - \sigma(x^2+y^2)y, \tag{11}$$

$$F_3(x,y,z) = \tau(x^2+y^2)z. \tag{12}$$

As in the preceding sections, we consider the differential equation (1), difference equations (2) and maps (3).

Lemma 6. *The vector field F given by (10)-(12) is complete.*

Proof. Similarly as in the preceding section, we have

$$(F_1(x,y,z), F_2(x,y,z)) \cdot (x,y) = -(x^2+y^2)\sigma(x^2+y^2).$$

By Lemma 5, this is negative if $x^2+y^2 > 2$. Therefore the projections of the solutions of (1) to the (x,y)-plane are bounded. Moreover, $F_3(x,y,z) = \tau(x^2+y^2)z$, and by Lemma 5 (c), $\tau(x^2+y^2)$ is bounded from above by 1. Therefore the third component of the solutions also cannot escape to infinity in finite time. ■

In order to investigate what happens with the two first components of the trajectories of f_ε, we define a map $g_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$g_\varepsilon(x,y) = (x,y) + \varepsilon G(x,y) = (1 - \varepsilon\sigma(x^2+y^2))(x,y) + \varepsilon(-y,x).$$

Observe that the norm of $g_\varepsilon(x,y)$ depends only on the norm of (x,y). Therefore it makes sense to introduce one more map, h_ε, defined by

$$h_\varepsilon(t) = t\big((1-\varepsilon\sigma(t))^2 + \varepsilon^2\big).$$

Then

$$\|g_\varepsilon(x,y)\|^2 = h_\varepsilon(\|(x,y)\|^2). \tag{13}$$

Note that $h_\varepsilon(0) = 0$ and $h_\varepsilon((0,\infty)) \subset (0,\infty)$.

Lemma 7. *Assume that $\varepsilon \in (0,1)$. Then the map h_ε has in $(0,\infty)$ a globally attracting point $t_\varepsilon > 2$.*

Proof. Let $t > 0$. Elementary computations show that the sign of $h_\varepsilon(t) - t$ is the same as the sign of $\varepsilon(\sigma(t))^2 - 2\sigma(t) + \varepsilon$. The function $s \mapsto \varepsilon s^2 - 2s + \varepsilon = 0$ has two zeros, $s_- \in (0,1)$ and $s_+ > 1$. Hence, this function is positive for $s < s_-$ and negative for $s \in (s_-, 1)$. Therefore, by Lemma 5, the function $t \mapsto \varepsilon(\sigma(t))^2 - 2\sigma(t) + \varepsilon$ has in $(0,\infty)$ one zero t_ε, this zero is located in $(2,\infty)$, and the function is positive for $t < t_\varepsilon$ and negative for $t > t_\varepsilon$. This proves that the point t_ε is a globally attracting point of h_ε in $(0,\infty)$. ■

Theorem 3. *For the function F defined by (10)-(12), the differential equation (1) has an unbounded (in forward time) solution, while the difference equation (2) with any $\varepsilon \in (0,1)$ has all solutions bounded.*

Proof. Let us start by listing several properties of the functions F and G that follow immediately from the formulas (8)-(12) and Lemma 5:

(a) if $x^2 + y^2 = 1$ then $F(x, y, z) = (-y, x, z/4)$,
(b) if $(x, y) = (0, 0)$ then $F(x, y, z) = (0, 0, -z)$,
(c) if $x^2 + y^2 \geq 2$ then $F_3(x, y, z)$ is between $-\frac{1}{9}z$ and $-2z$.

By (a), the solution of (1) with the initial condition

$$(x(0), y(0), z(0)) = (1, 0, 1)$$

can be written explicitly for $t \geq 0$ as

$$(x(t), y(t), z(t)) = (\cos t, \sin t, e^{t/4}).$$

Since $z(t)$ goes to ∞ as $t \to \infty$, this solution is unbounded.

Let us fix $\varepsilon \in (0, 1)$ and consider the solutions of the equation (2). Consider first the case when $(x_0, y_0) = (0, 0)$. By (b), (x_n, y_n) stays $(0, 0)$, while $z_{n+1} = (1 - \varepsilon)z_n$ for all $n \geq 0$, so z_n converges to 0 as $n \to \infty$. Thus, those solutions are bounded.

Assume now that $(x_0, y_0) \neq (0, 0)$. By (13) and Lemma 7, $x_n^2 + y_n^2$ converges to t_ε as $n \to \infty$. Since $t_\varepsilon > 2$, by Lemma 5 z_n converges to 0 as $n \to \infty$. Therefore those solutions are also bounded. ■

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