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ON ASYMPTOTICALLY PERIODIC SOLUTIONS OF LINEAR DISCRETE VOLTERRA EQUATIONS

Abstract. We show that a class of linear nonconvolution discrete Volterra equations has asymptotically periodic solutions. We also examine an example for which the calculations can be done explicitly. The results are established using theorems on the boundedness and convergence to a finite limit of solutions of linear discrete Volterra equations.

Key words: Volterra difference equation, asymptotically periodic solutions, asymptotic equilibria.

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1. Introduction

The problem of finding periodic and asymptotically periodic solutions of linear discrete Volterra equations has been investigated in several papers, including [3, 4, 6, 7, 8]. Each of these considers the equation

$$u(n+1) = A(n)u(n) + \sum_{j=0}^{n} B(n,j)u(j) + e(n),$$

and perturbations thereof, under the assumptions that

$$\begin{cases} A(n+N) = A(n), \qquad e(n+N) = e(n), \\ \qquad B(n+N, m+N) = B(n,m), \end{cases} \qquad n \in \mathbb{Z}^+,$$

for some integer $N > 0$.

In this paper we study the asymptotic behaviour of solutions of the initial-value problem

$$(1) \qquad x(n+1) = A(n)x(n) + \sum_{j=0}^{n} K(n,j)x(j) + f(n), \quad n \geq 0,$$

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assuming that the periodicity condition

$$A(n+N) = A(n), \quad n \in \mathbb{Z}^+,$$

holds for some integer $N > 0$. Here (1) is viewed as a perturbation of

$$u(n+1) = A(n)u(n).$$

Our method is to use a variation of constants to transform (1) into an equation of the form

$$z(n+1) = h(n) + \sum_{j=0}^{n} H(n,j)z(j), \quad n \in \mathbb{Z}^+,$$

and use results in [1, 5] on the boundedness and asymptotic constancy of solutions. Our main result establishes the existence of asymptotically periodic solutions

In the case of $A(n)$ and $K(n,j)$ being 2×2 real matrices, a different result on the existence of asymptotically periodic solutions of (9) was recently proved in [2] using fixed point arguments. We analyse fully an example found in [2] to show that our result can be less restrictive.

2. Preliminaries

Firstly we collect together some notation.

Let $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ denote the set of non-negative integers, and $\mathbb{K}$ either of the fields $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{K}^{d\times d}$ is the space of all $d\times d$ matrices with entries in $\mathbb{K}$, and the zero and identity matrices are denoted by 0 and I respectively. $\|A\| = (\sum_{j=1}^{d}\sum_{i=1}^{d} |A_{ij}|^2)^{1/2}$ is the Euclidean norm of $A = (A_{ij})$. $\mathbb{K}^{d\times d}$ can be endowed with many norms, but they are all equivalent. The *absolute value* of A is the matrix $|A|$ defined by $(|A|)_{ij} = |A_{ij}|$ for all $1 \le i \le d$ and $1 \le j \le d$. The matrix $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is *nonnegative* if $A_{ij} \ge 0$, in which case we write $A \ge 0$. A partial ordering is defined on $\mathbb{K}^{d\times d}$ by letting $A \le B$ if and only if $B - A \ge 0$, which is equivalent to $A_{ij} \le B_{ij}$ for all $1 \le i \le d$ and $1 \le j \le d$. The *spectral radius* of a matrix A is defined by $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, and is independent of the norm used: it equals the maximum of the absolute values of the eigenvalues of A. Note that $\rho(A) \le \rho(|A|)$, and $\rho(A) \le \rho(B)$ if $0 \le A \le B$.

Let $H : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ be a double sequence of matrices with $H(n,j) = 0$ for all $j > n$, and $h : \mathbb{Z}^+ \to \mathbb{K}^d$ a sequence of vectors. For each $\xi \in \mathbb{K}^d$, there is a unique solution $z : \mathbb{Z}^+ \to \mathbb{K}^d$ of the explicit discrete Volterra equation

$$z(n+1) = h(n) + \sum_{j=0}^{n} H(n,j)z(j), \quad n \in \mathbb{Z}^+, \tag{2}$$

which satisfies the initial condition

$$z(0) = \xi. \tag{3}$$

Sufficient conditions for the solution of (2) and (3) to be bounded can be found in [1, Theorem 5.1]. We state a variant of it.

Lemma 1. *Suppose that*

$$W_H := \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} |H(n,j)| < \infty, \quad \rho(W_H) < 1, \tag{4}$$

$$\sup_{n\geq j} |H(n,j)| < \infty \quad \text{for each } j \geq 0. \tag{5}$$

If h is bounded, then z is also bounded.

Sufficient conditions for the solution of (2) and (3) to converge to a finite limit are given in [5, Theorem 3.1], a variant of which is now given.

Lemma 2. *Suppose that (4) holds, and that*

$$H(n,k) \to H_\infty(k) \quad \text{as } n \to \infty. \tag{6}$$

Assume also that there is V_H in $\mathbb{K}^{d\times d}$ such that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \left| \sum_{j=m}^{n} H(n,j) - V_H \right| = 0, \tag{7}$$

and $h(n) \to h(\infty)$ as $n \to \infty$. Then $z(n) \to z(\infty)$ as $n \to \infty$, and this limit obeys

$$z(\infty) = (I - V_H)^{-1} \left[h(\infty) + \sum_{j=0}^{\infty} H_\infty(j) z(j) \right]. \tag{8}$$

The hypotheses of this theorem ensure that all the terms in (8) are well-defined. Since

$$|V_H| \leq \left| V_H - \sum_{j=m}^{n} H(n,j) \right| + \sum_{j=m}^{n} |H(n,j)|,$$

we see that $|V_H| \leq W_H$ by firstly taking the limit superior as $n \to \infty$, followed by the limit superior as $m \to \infty$. Hence $\rho(V_H) \leq \rho(|V_H|) \leq \rho(W_H) < 1$. Therefore $I - V_H$ is invertible and $(I - V_H)^{-1} \geq 0$.

Also for large enough $m \in \mathbb{Z}^+$,

$$\limsup_{n\to\infty} \sum_{j=0}^{n} |H(n,j)| = \sum_{j=0}^{m-1} |H_\infty(j)| + \limsup_{n\to\infty} \sum_{j=m}^{n} |H(n,j)|$$

is certainly finite. Hence

$$\sum_{j=0}^{k} |H_\infty(j)| = \limsup_{n\to\infty} \sum_{j=0}^{k} |H(n,j)| \le \limsup_{n\to\infty} \sum_{j=0}^{n} |H(n,j)|,$$

is uniformly bounded for $k \in \mathbb{Z}^+$, and the series $\sum_{j=0}^{\infty} |H_\infty(j)|$ is summable. Therefore if z is bounded,

$$\left|\sum_{j=0}^{\infty} H_\infty(j) z(j)\right| \le \sum_{j=0}^{\infty} |H_\infty(j)| \sup_{j\ge 0} |z(j)| < \infty.$$

3. Formulation of problem

We study the asymptotic behaviour of the solution $x(\cdot;\xi) : \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ of the initial-value problem

$$x(n+1;\xi) = A(n)x(n;\xi) + \sum_{j=0}^{n} K(n,j)x(j;\xi) + f(n), \quad n \ge 0, \tag{9a}$$

$$x(0;\xi) = \xi, \tag{9b}$$

assuming that the periodicity condition

$$A(n+N) = A(n), \quad n \in \mathbb{Z}^+, \tag{10}$$

holds for some integer $N > 0$.

We regard (9) as a perturbation of

$$\begin{cases} u(n+1) = A(n)u(n), \\ u(0) = \xi, \end{cases} \tag{11}$$

which has solution

$$u(n) = \Phi(n)\xi, \quad n \ge 0,$$

where $\Phi : \mathbb{Z} \to \mathbb{K}^{d\times d}$ is the principal matrix solution defined by

$$\begin{cases} \Phi(n+1) = A(n)\Phi(n), \quad n \ge 0, \\ \Phi(0) = I. \end{cases}$$

The matrix $\Phi(N)$ is termed a monodromy matrix, and its eigenvalues are called Floquet multipliers.

The following standing hypotheses are assumed to hold throughout the remainder of the paper.

(A1) $A : \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ is a sequence of invertible matrices, which satisfies the periodicity condition (10) for some positive integer N.

(A2) The monodromy matrix $\Phi(N)$ is assumed to satisfy

$$\Phi(N) = I. \tag{12}$$

(A3) $K : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ is a double sequence of matrices with $K(n, j) = 0$ for all $j > n$, and

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} |K(n, j)| < \infty. \tag{13}$$

(A4) $f : \mathbb{Z}^+ \to \mathbb{K}^d$ is a sequence with

$$\sum_{j=0}^{\infty} |f(j)| < \infty. \tag{14}$$

Remark 1. (12) is equivalent to

$$\Phi(n + N) = \Phi(n), \quad n \in \mathbb{Z}^+. \tag{15}$$

Due to this periodicity,

$$\Phi(n) = \Phi(n - [\tfrac{n}{N}]N), \quad n \geq 0, \tag{16}$$

where $[\cdot]$ is the greatest integer part function, though it is cumbersome to employ this expression in succeeding formula.

4. Main results and discussion

Our main result clearly exhibits the asymptotic behaviour of all solutions of (9).

The theorem refers to a matrix M and vector m, each of which is defined to be the limit of the solution of a discrete Volterra equation. Let $Y : \mathbb{Z}^+ \to \mathbb{K}^{d\times d}$ be the solution of

$$\begin{cases} Y(n+1) = I + \displaystyle\sum_{j=0}^{n} \sum_{k=j}^{n} \Phi(k+1)^{-1} K(k, j)\Phi(j)Y(j), \\ \qquad\qquad Y(0) = I. \end{cases}$$

Then it is proved in Section 5 that

$$M = \lim_{n\to\infty} Y(n) \tag{17}$$

exists. Clearly M *is independent of the initial-value* ξ. Also let $w : \mathbb{Z}^+ \to \mathbb{K}^d$ be the solution of

$$w(n+1) = \sum_{j=0}^{n}\sum_{k=j}^{n} \Phi(k+1)^{-1}K(k,j)\Phi(j)w(j) + \sum_{k=0}^{n} \Phi(k+1)^{-1} f(k),$$

which satisfies $w(0) = 0$. It is also shown that in Section 5 that

$$m = \lim_{n\to\infty} w(n) \tag{18}$$

exists. Observe that because $\{w(n)\}$ *is independent of* ξ, m *is independent of* ξ. In Section 6 an example is given in which M and m are calculated explicitly.

Theorem 1. *Suppose that (A1)-(A4) hold, and let M be defined by (17) and m by (18). Then for every $\xi \in \mathbb{K}^d$, the solution of (9) satisfies*

$$x(n;\xi) = \Phi(n)[M\xi + m] + v(n;\xi), \quad n \in \mathbb{Z}^+, \tag{19}$$

where $v(n;\xi) \to 0$ as $n \to \infty$.

The sequence $\{x(n;\xi)\}$ is *asymptotically periodic* if there is a nontrivial periodic $\{p(n)\}$ such that

$$x(n;\xi) = p(n) + o(1) \quad \text{as} \quad n \to \infty.$$

Under the conditions of Theorem 1, $x(\cdot;\xi)$ is asymptotically periodic if and only if $M\xi + m \neq 0$. There are three cases:

(a) $M = 0$ and $m = 0$, implying that $\{x(n;\xi)\}$ is not asymptotically periodic; the zero solution is a global attractor.

(b) $M = 0$ but $m \neq 0$, implying that $\{x(n;\xi)\}$ is asymptotically periodic with respect to the $\{\Phi(n)m\}$ for all initial values.

(c) If $M \neq 0$ and $c - m$ is in the range of M, then $\{x(n;\xi)\}$ is asymptotically periodic with respect to the $\{\Phi(n)c\}$ for all initial-values obeying $M\xi + m = c$.

It is clearly desirable to know what are the null space and range of the linear mapping associated with M.

In order to compare our result with [2, Theorem 1], we state the following.

Corollary 1. *Suppose that* $\{a_1(n)\}$ *and* $\{a_2(n)\}$ *are* N*-periodic sequences in* $\mathbb{K} \setminus \{0\}$*, satisfying*

$$\prod_{j=0}^{N-1} a_s(j) = 1, \quad s = 1, 2. \tag{20}$$

Let

$$A(n) = \begin{pmatrix} a_1(n) & 0 \\ 0 & a_2(n) \end{pmatrix}, \quad n \in \mathbb{Z}^+, \tag{21}$$

and suppose that (A3)-(A4) hold with $d = 2$*. Then the solution* $x(\cdot, \xi)$ *of* (9) *obeys* (19)*.*

Note that in this case

$$\Phi(n) = \begin{pmatrix} \prod_{j=0}^{n-1} a_1(j) & 0 \\ 0 & \prod_{j=0}^{n-1} a_2(j) \end{pmatrix}, \quad n \in \mathbb{Z}^+.$$

There are important differences between [2, Theorem 1] and Corollary 1. Firstly the conclusion of [2, Theorem 1] says that for each c in a precisely defined region of $\mathbb{R}^2$, there is a solution of (9) such that

$$x(n; \xi) = \Phi(n)c + o(1) \quad \text{as} \quad n \to \infty. \tag{22}$$

Secondly, additional hypotheses to those in Corollary 1 are required in [2] to prove Theorem 1.

In Section 6 an example is discussed for which M and m are explicitly calculated. In the example M is invertible, and hence for each $c \in \mathbb{R}^2$, there is an initial-value $\xi = M^{-1}(c - m)$ for which (22) holds; also $x(\cdot; \xi)$ is asymptotically periodic as long as $\xi \neq -M^{-1}m$. This shows that conditions on the vector c in [2, Theorem 1], though sufficient, are not necessary for the existence of asymptotically periodic solutions.

5. Proof of main theorem

We define $z(\cdot; \xi) : \mathbb{Z}^+ \to \mathbb{K}^d$ by

$$z(n; \xi) = \Phi(n)^{-1} x(n; \xi), \quad n \geq 0, \tag{23}$$

and investigate the auxiliary problem it satisfies.

Lemma 3. *x is a solution of (9) if and only if z solves*

$$z(n+1;\xi) = \xi + f^*(n) + \sum_{j=0}^{n} H(n,j)z(j;\xi), \tag{24a}$$

$$z(0;\xi) = \xi, \tag{24b}$$

where

$$H(n,j) = \begin{cases} \sum_{k=j}^{n} \Phi(k+1)^{-1}K(k,j)\Phi(j), & 0 \le j \le n, \\ 0, & j > n, \end{cases} \tag{25}$$

$$f^*(n) = \sum_{k=0}^{n} \Phi(k+1)^{-1} f(k), \quad n \ge 0. \tag{26}$$

Proof. Clearly $x(0;\xi) = \xi$ if and only if $z(0;\xi) = \Phi(0)^{-1}x(0;\xi) = \xi$. By substituting (23) into (9a) , we obtain

$$z(n+1;\xi) = z(n;\xi) + \sum_{j=0}^{n} \Phi(n+1)^{-1}K(n,j)\Phi(j)z(j;\xi) + \Phi(n+1)^{-1}f(n).$$

A variation of constants formula is easily obtained by observing that

$$\begin{aligned} z(n+1;\xi) - z(0;\xi) &= \sum_{k=0}^{n} \{z(k+1;\xi) - z(k;\xi)\} \\ &= \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \Phi(k+1)^{-1}K(k,j)\Phi(j) \right) z(j;\xi) \\ &\quad + f^*(n). \end{aligned}$$

Hence we obtain the equation in (24a). The converse is proved similarly. ∎

It is important to know what is the asymptotic behaviour of $\{f^*(n)\}$.

Lemma 4. *$f^*(n) \to f^*(\infty)$ as $n \to \infty$, where*

$$f^*(\infty) := \sum_{j=0}^{N-1} \Phi(j)^{-1} \sum_{i=0}^{\infty} f(iN + j - 1). \tag{27}$$

Proof. Recall that $f : \mathbb{Z}^+ \to \mathbb{K}^d$ is in ℓ^1. Extending f to $\{-1, 0, 1, \ldots\}$ by setting $f(-1) = 0$, it follows from (26) that

$$f^*(n) = \sum_{k=0}^{n+1} \Phi(k)^{-1} f(k-1),$$

We write $n+1 = pN+r$ where $p = [(n+1)/N]$, so that r is in $\{0,\ldots,N-1\}$, and $p \to \infty$ if $n \to \infty$. Then

$$f^*(n) = \sum_{k=0}^{pN-1} \Phi(k)^{-1} f(k-1) + \sum_{k=pN}^{pN+r} \Phi(k)^{-1} f(k-1). \tag{28}$$

Using the periodicity of Φ, we can manipulate the first summation and deduce that

$$\begin{aligned}
\sum_{k=0}^{pN-1} \Phi(k)^{-1} f(k-1) &= \sum_{i=0}^{p-1} \sum_{k=iN}^{(i+1)N-1} \Phi(k)^{-1} f(k-1) \\
&= \sum_{i=0}^{p-1} \sum_{j=0}^{N-1} \Phi(iN+j)^{-1} f(iN+j-1) \\
&= \sum_{j=0}^{N-1} \Phi(j)^{-1} \sum_{i=0}^{p-1} f(iN+j-1) \\
&\to \sum_{j=0}^{N-1} \Phi(j)^{-1} \sum_{i=0}^{\infty} f(iN+j-1) \quad \text{as } p \to \infty,
\end{aligned}$$

since $f \in \ell^1$. To complete the proof, it suffices to notice that the second term in (28) satisfies

$$\begin{aligned}
\sum_{k=pN}^{pN+r} \Phi(k)^{-1} f(k-1) &= \sum_{j=0}^{r} \Phi(j+pN)^{-1} f(pN+j-1) \\
&= \sum_{j=0}^{r} \Phi(j)^{-1} f(pN+j-1) \\
&\to 0 \quad \text{as } p \to \infty,
\end{aligned}$$

again since $f \in \ell^1$. ∎

Our method is to apply Lemma 2 to (24). It must therefore be verified that H satisfies the hypotheses of that lemma.

Lemma 5. *The kernel H defined in (25) has the property (4) with $W_H = 0$, (6) with*

$$H_\infty(j) = \sum_{k=j}^{\infty} \Phi(k+1)^{-1} K(k,j) \Phi(j), \tag{29}$$

and (7) with $V_H = 0$.

Proof. By the periodicity of $n \mapsto \Phi(n)$ and $n \mapsto \Phi(n)^{-1}$ there are positive matrices C_1 and C_2 such that

$$|\Phi(n)| \le C_1, \qquad |\Phi(n)^{-1}| \le C_2, \tag{30}$$

for all $n \ge 0$. Also by (13)

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} |K(n,j)| = \sum_{j=0}^{\infty} S(j) < \infty,$$

where $S(j) = \sum_{n=j}^{\infty} |K(n,j)|$.

For an integer $m > 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \sum_{j=m}^{n} |H(n,j)| &\le \limsup_{n\to\infty} \sum_{j=m}^{n} \sum_{k=j}^{n} |\Phi(k+1)^{-1}||K(k,j)||\Phi(j)| \\
&\le C_2 \limsup_{n\to\infty} \sum_{j=m}^{n} \sum_{k=j}^{n} |K(k,j)| C_1 \\
&\le C_2 \limsup_{n\to\infty} \sum_{j=m}^{n} \sum_{k=j}^{\infty} |K(k,j)| C_1 \\
&\le C_2 \limsup_{n\to\infty} \sum_{j=m}^{n} S(j) C_1 \\
&\le C_2 \sum_{j=m}^{\infty} S(j) C_1.
\end{aligned}$$

Since $\sum_{j=0}^{\infty} S(j)$ is finite, $\sum_{j=m}^{\infty} S(j) \to 0$ as $m \to \infty$, and

$$W_H = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{j=m}^{n} |H(n,j)| = 0.$$

Consequently (7) also holds with $V_H = 0$.

Lastly we demonstrate that (6) is true. We fix $j \in \mathbb{Z}^+$. For $n \ge j$,

$$H(n,j) \to \sum_{k=j}^{\infty} \Phi(k+1)^{-1} K(k,j) \Phi(j) \quad \text{as } n \to \infty.$$

Because

$$\sum_{k=j}^{\infty} |\Phi(k+1)^{-1} K(k,j) \Phi(j)| \le C_2 S(j) C_1,$$

the limit is finite. ∎

It is a consequence of Lemma 2 that $z(n;\xi)$ converges to a limit $z(\infty;\xi)$ as $n \to \infty$, which satisfies

$$z(\infty;\xi) = \xi + f^*(\infty) + \sum_{j=0}^{\infty} H_\infty(j) z(j;\xi), \tag{31}$$

where $f^*(\infty)$ is given by (27) and $H_\infty(j)$ by (29) .

We now explicitly exhibit the dependence of the limit $z(\infty;\xi)$ on the initial value ξ. For this reason, we examine (24) in the case that $f(n) \equiv 0$, which corresponds to the homogeneous problem associated with (9):

$$y(n+1;\xi) = \xi + \sum_{j=0}^{n} H(n,j) y(j;\xi), \tag{32a}$$

$$y(0;\xi) = \xi, \tag{32b}$$

But $\xi \mapsto y(\cdot;\xi)$ is linear; indeed $y(n) = Y(n)\xi$, where

$$\begin{cases} Y(n+1) = I + \sum_{j=0}^{n} H(n,j) Y(j), \\ Y(0) = I. \end{cases} \tag{33}$$

Lemma 2 says that

$$M = \lim_{n\to\infty} Y(n) \tag{34}$$

exists, and $y(n;\xi) \to M\xi$ as $n \to \infty$. Also the matrix M satisfies the implicit limit formula

$$M = I + \sum_{j=0}^{\infty} H_\infty(j) Y(j).$$

We summarise what has been discovered about (32).

Lemma 6. *The solution y of (32) satisfies $y(n;\xi) \to M\xi$ as $n \to \infty$, where M is given by (34).*

Finally we introduce the sequence $\{w(n)\}$ by

$$w(n) = z(n;\xi) - Y(n)\xi, \quad n \geq 0,$$

observing that it solves

$$\begin{cases} w(n+1) = f^*(n) + \sum_{j=0}^{n} H(n,j) w(j), \\ w(0) = 0. \end{cases} \tag{35}$$

We infer from Lemma 2 that

$$m = \lim_{n\to\infty} w(n) \tag{36}$$

exists, and that $m = f^*(\infty) + \sum_{j=0}^{\infty} H_\infty(j)w(j)$.

In this section the following theorem has been proved.

Theorem 2. *Suppose that (A1)-(A4) hold. For every $\xi \in K^d$, the solution z of (24) satisfies*

$$\lim_{n\to\infty} z(n;\xi) = M\xi + m, \tag{37}$$

where the matrix M is defined by (34) and the vector m by (36).

Theorem 1 is now consequence of this and (23).

6. Example

To illustrate our results we examine an example from [2], which concerned (9) the case that

$$A(n) = -I, \qquad f(n) = \frac{1}{2^{n+1}3^n}e, \qquad K(n,j) = \frac{(-3)^j}{2^n 3^{n+1}}Q,$$

where

$$Q = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Note that $Q^2 = 0$ and $Qe = 0$. Since $A(0)A(1) = I$, $N = 2$ and $\Phi(n) = (-1)^n I$. Also

$$\sum_{j=0}^{\infty} |f(j)| = \frac{3}{5}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \sum_{n=0}^{\infty}\sum_{j=0}^{n} |K(n,j)| = \frac{4}{5}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

so that (A1)-(A4) hold with $d = 2$ and $\mathbb{K} = \mathbb{R}$.

Let $x(n;\xi) = (-1)^n z(n;\xi)$ for all $n \ge 0$. Then $\{z(n;\xi)\}$ satisfies

$$\begin{cases} z(n+1;\xi) = \xi + \displaystyle\sum_{j=0}^{n} H(n,j)z(j;\xi) + f^*(n), \\ \qquad\qquad z(0;\xi) = \xi, \end{cases} \tag{38}$$

where

$$H(n,j) = \frac{2}{7}\frac{(-1)^{j+1}}{2^j}\left[1 - \left(\frac{-1}{6}\right)^{n-j+1}\right]Q,$$

$$f^*(n) = -\frac{3}{7}\left[1 - \left(\frac{-1}{6}\right)^{n+1}\right]e.$$

These expressions can be written more concisely as

$$H(n,j) = \alpha(n-j)\beta(j)Q, \qquad f^*(n) = \gamma(n)e,$$

where

$$\alpha(j) = 1 - \left(\frac{-1}{6}\right)^{j+1}, \qquad \beta(j) = \frac{2}{7}\frac{(-1)^{j+1}}{2^j},$$
$$\gamma(n) = -\frac{3}{7}\left[1 - \left(\frac{-1}{6}\right)^{n+1}\right].$$

Here $\{\alpha(n)\}$ and $\{\gamma(n)\}$ are convergent, and $\{\beta(n)\}$ is summable with

$$\lim_{n\to\infty} \alpha(n) = 1, \qquad \sum_{j=0}^{\infty} \beta(j) = -\frac{4}{21}, \qquad \lim_{n\to\infty} \gamma(n) = -\frac{3}{7}.$$

Note that

$$H_\infty(j) = \lim_{n\to\infty} H(n,j) = \beta(j)Q, \qquad f^*(\infty) = \lim_{n\to\infty} f^*(n) = -\frac{3}{7}e.$$

The solution of (38) can be split up as

$$z(n;\xi) = Y(n)\xi + w(n),$$

where Y is the solution of

$$\left\{\begin{array}{c} Y(n+1) = I + \displaystyle\sum_{j=0}^{n} \alpha(n-j)\beta(j)QY(j), \\ Y(0) = I, \end{array}\right. \tag{39}$$

and w is the solution of

$$\left\{\begin{array}{c} w(n+1) = \gamma(n)e + \displaystyle\sum_{j=0}^{n} \alpha(n-j)\beta(j)Qw(j), \\ w(0) = 0. \end{array}\right. \tag{40}$$

We attempt to find a solution of (39) under the condition that $QY(n) = Q$ for all $n \geq 0$. This is equivalent to Y having the form

$$Y(n) = \begin{pmatrix} Y_{11}(n) & Y_{12}(n) \\ Y_{21}(n) & Y_{22}(n) \end{pmatrix} = \begin{pmatrix} Y_{11}(n) & Y_{12}(n) \\ 1 - Y_{11}(n) & 1 - Y_{12}(n) \end{pmatrix}.$$

By substituting this into (39), we obtain

$$Y_{11}(n+1) = 1 + \sum_{j=0}^{n} \alpha(n-j)\beta(j), \qquad Y_{12}(n+1) = \sum_{j=0}^{n} \alpha(n-j)\beta(j),$$

leading to

$$Y(n+1) = I + \sum_{j=0}^{n} \alpha(n-j)\beta(j)Q. \tag{41}$$

The solution of (40) is found in a similar fashion. We look for its solution in the form

$$w(n) = \begin{pmatrix} w_1(n) \\ w_2(n) \end{pmatrix} = \begin{pmatrix} w_1(n) \\ -w_1(n) \end{pmatrix} = w_1(n)e,$$

so that $Qw(n) = 0$ for all $n \geq 0$. The equation becomes $w_1(n+1) = \gamma(n)$, and

$$w(n+1) = \gamma(n)e. \tag{42}$$

The exact solution of (38) is then $z(0,\xi) = \xi$ and

$$z(n+1;\xi) = \xi + \sum_{j=0}^{n} \alpha(n-j)\beta(j)Q\xi + \gamma(n)e, \quad n \geq 0. \tag{43}$$

Since

$$\lim_{n\to\infty} \sum_{j=0}^{n} \alpha(n-j)\beta(j) = -\frac{4}{21}, \qquad \lim_{n\to\infty} \gamma(n)e = -\frac{3}{7}e = -\frac{3}{7}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

it follows from (43) that

$$z(n+1;\xi) \to \left(I - \frac{4}{21}Q\right)\xi + \left(-\frac{3}{7}e\right) \quad \text{as } n \to \infty. \tag{44}$$

We can calculate the matrix M defined by (34) from (41), and the vector m given by (36) from (42), and obtain

$$M = I - \frac{4}{21}Q = \frac{1}{21}\begin{pmatrix} 17 & -4 \\ 4 & 25 \end{pmatrix}, \qquad m = -\frac{3}{7}e = -\frac{3}{7}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Observe that $\det M = 1$ so that M is invertible. It follows from (44) that $z(n+1;\xi) = c + o(1)$ as $n \to \infty$, where $c = M\xi + m$. Since M is invertible, there is a solution with this property for every $c \in \mathbb{R}^2$. Therefore (9) has asymptotically periodic solutions of form $x(n) = (-1)^n c + o(1)$ for every nonzero c in $\mathbb{R}^2$.

We remark that if $\xi = \begin{pmatrix} 2 & 1 \end{pmatrix}^T$, then

$$c = \frac{1}{21}\begin{pmatrix} 17 & -4 \\ 4 & 25 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{3}{7}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is the case examined in [2].

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