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ANALYSIS OF A STOCHASTIC DIFFERENCE EQUATION: EXIT TIMES AND INVARIANT DISTRIBUTIONS

Abstract. The mean return time of a discrete Markov chain to a point x is the reciprocal of the invariant probability $\pi(x)$. We revisit this classical theme to investigate certain exit times for stochastic difference equations of autoregressive type. More specifically, we will discuss the asymptotics, as $\varepsilon \to 0$, of the first time τ that the n-dimensional process

$$Y_t = f(Y_{t-1}) + \varepsilon\xi_t, \quad t = 1, 2, \dots$$

(where $\xi_1, \xi_2, \dots$ is a sequence of i.i.d. random n-vectors) leaves a given neighborhood of the fixed point of the contraction f.

Key words: autoregressive processes, recurrence, return times, multivariate normal.

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Mark Kac proved in 1947 that the mean return time of a discrete Markov chain to a point x is the reciprocal of the invariant probability $\pi(x)$. This result was extended to chains on general measure spaces (subject to some irreducibility conditions) by Cogburn (1975). We revisit this classical theme to investigate certain exit times for stochastic difference equations of an autoregressive type.

In the first section, we introduce some notation and recall the results for return times that were given by Cogburn (1975). In section 2, we use these results to analyse an exit time from a set for a stochastic difference equation. In the last section we study certain examples, among them the exit time from an interval for the autoregressive process.

1. Mean return times for recurrent sets

Consider a Markov process $X_0, X_1, X_2, \dots$ taking values in a measurable state space $(\mathcal{H}, \mathcal{A})$. Under the assumption that the chain is Harris recurrent, it has a σ-finite invariant measure π. Furthermore, a set $A \in \mathcal{A}$ is recurrent if and only if $\pi(A) > 0$. We assume that π is actually an invariant probability measure and consider return times to a recurrent set A for the process. The first return time is defined as

$$\tau_A^1 = \inf\{t \geq 1 : X_t \in A\}, \tag{1}$$

and the k:th return time, where $k \geq 2$, is defined recursively as

$$\tau_A^k = \inf\{t > \tau_A^{k-1} : X_t \in A\}. \tag{2}$$

Let π_A denote the restriction of the invariant probability measure π to the set A. Then, as Cogburn (1975) shows in his Corollary 3.1.,

$$E_{\pi_A}(\tau_A^1) = \frac{1}{\pi(A)}, \tag{3}$$

where the notation E_{π_A} refers to the assumption that the initial state X_0 of the chain is distributed according to π_A. For a similar result for the k:th return time, we may consider the chain restricted to A, that is, the process $X_0, X_{\tau_A^1}, X_{\tau_A^2}, \ldots$, where $X_0 \in A$. This process has the invariant probability measure π_A. Thus, by considering $X_{\tau_A^1}$ as a new starting point, one gets that $E_{\pi_A}(\tau_A^2) = 2/\pi(A)$, and that

$$E_{\pi_A}(\tau_A^k) = \frac{k}{\pi(A)} \tag{4}$$

for a general integer $k \geq 1$.

2. Exit times for a stochastic difference equation

We will now study the exit time from the set for the process $\{Y_t\}_{t\geq 1}$ in $\mathbb{R}^n$, defined by the stochastic difference equation

$$Y_t = f(Y_{t-1}) + \varepsilon\xi_t, \qquad Y_0 = y_0 \in \mathbb{R}^n.$$

Here, the function f is assumed to be continuous and contractive, the parameter ε is a positive real number and $\{\xi_t\}_{t\geq 1}$ is a sequence of independent and identically distributed random variables with mean 0 and finite covariance matrix. We want to consider the exit time from a set $\Gamma \subset \mathbb{R}^n$, so we assume that $y_0 \in \Gamma$ and define

$$T := \inf\{t \geq 1 : Y_t \notin \Gamma\}.$$

We assume that f and $\{\xi_t\}_{t\geq 1}$ are such that the process is Harris recurrent. The process has an invariant probability measure π (this follows from Theorem 12.3.4 in Meyn, Tweedie (1993), since the process is weak Feller and satisfies a drift condition). To apply Cogburn's result, let us compare the exit time T with a certain return time. For this, we define another process $\{X_t\}_{t\geq 1}$ in $\mathbb{R}^n$, where

$$X_t = f(X_{t-1}) + \varepsilon\xi_t, \qquad X_0 = x_0.$$

Here, the function f and the sequence $\{\xi_t\}_{t\geq 1}$ are the same as in the definition of the process $\{Y_t\}_{t\geq 1}$. This process has the same stationary distribution π as $\{Y_t\}_{t\geq 1}$. Since f is contractive, the sequences $\{Y_t\}_{t\geq 1}$ and $\{X_t\}_{t\geq 1}$ will be arbitrarily close together after a sufficient number of steps, that is, for an arbitrary $\eta > 0$ there is an $M > 0$ such that $t \geq M \Rightarrow ||Y_t - X_t|| < \eta$. Now, for fixed $\eta, h > 0$, we define the set A_η as

$$A_\eta = \{x \in \mathbb{R}^n : \eta < \inf_{y\in\Gamma} ||y - x|| \leq \eta + h\}$$

and consider return times to this set. Let $\tau^1_{A_\eta} = \inf\{t \geq 1 : X_t \in A_\eta\}$ be the first return time, and $\tau^k_{A_\eta} = \inf\{t > \tau^{k-1}_{A_\eta} : X_t \in A_\eta\}$, where $k \geq 2$, the time of the k:th return to the set A_η. Assume also that x_0 follows the distribution π_{A_η}, that is, the stationary distribution of the process restricted to A_η. Now, consider the time of the M:th return to A_η. If $\tau^M_{A_\eta} = t$, then $X_t \in A_\eta$. Also, $t \geq M$, so $||Y_t - X_t|| < \eta$. This implies that $Y_t \notin \Gamma$, so $T \leq t$. Thus, $T \leq \tau^M_{A_\eta}$. The result (4) stated that $E_{\pi_{A_\eta}} \tau^M_{A_\eta} = M/\pi(A_\eta)$ and this implies that

$$ET \leq \frac{M}{\pi(A_\eta)}. \tag{5}$$

Thus, we have an upper bound for the expected exit time from the set Γ.

3. Exit times for a process of autoregressive type

We illustrate the result (5) by applying it to a process of autoregressive type. Let $\{Y_t\}_{t\geq 1}$ in $\mathbb{R}^n$ be such that

$$Y_t = RY_{t-1} + \varepsilon S\xi_t, \qquad Y_0 = y_0,$$

where $y_0 \in \Gamma \subset \mathbb{R}^n$, R is an $n \times n$-matrix, S is an $n \times p$-matrix for some $p \leq n$ and $\{\xi_t\}_{t\geq 1}$ is a sequence of independent and identically distributed multivariate normal random variables with mean 0 and covariance matrix I, taking values in $\mathbb{R}^p$. It has been shown in Meyn, Tweedie (1993) that under the assumptions that all eigenvalues of R fall within the open unit disk in the complex plane and that the matrix $[R^{n-1}S|R^{n-2}S|\ldots|RS|S]$ has rank n, the process is Harris recurrent and has an invariant probability distribution π. This π is a multivariate normal distribution with mean 0 and covariance matrix $\varepsilon^2\Sigma$, where Σ is the solution of

$$\Sigma = R\Sigma R^T + SS^T.$$

We consider the exit time T from a set $\Gamma \in \mathbb{R}^n$. The result (5) states that $ET \le M/\pi(A_\eta)$, where A_η is defined as before and M is a constant that depends on η. In this case,

$$\pi(A_\eta) = \int_{A_\eta} \frac{1}{(2\pi)^{n/2}\varepsilon^n|\Sigma|^{1/2}} \exp(-\frac{1}{2\varepsilon^2}u^T\Sigma^{-1}u)\, du.$$

When ε is small, this integral is of the same order of magnitude as

$$\sup_{u\in A_\eta} \frac{1}{(2\pi)^{n/2}\varepsilon^n|\Sigma|^{1/2}} \exp(-\frac{1}{2\varepsilon^2}u^T\Sigma^{-1}u),$$

which implies that

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \pi(A_\eta) = -\inf_{u\in A_\eta} \frac{1}{2}u^T\Sigma^{-1}u.$$

(This can also be deduced by using a large deviation principle for the multivariate normal distribution.) Since M is constant,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log ET \le -\lim_{\varepsilon\to 0} \varepsilon^2 \log \pi(A_\eta) = \inf_{u\in A_\eta} \frac{1}{2}u^T\Sigma^{-1}u.$$

Since this holds for any choice of η, we can let $\eta \to 0$, and get

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log ET \le \inf_{u\in A_0} \frac{1}{2}u^T\Sigma^{-1}u, \tag{6}$$

where $A_0 = \{x \in \mathbb{R}^n : 0 < \inf_{y\in\Gamma} ||y-x|| \le h\}$. The infimum on the right hand side is then attained in the point where the level curve of the density function of the stationary distribution touches the boundary of the set Γ.

A simple example of the kind of process considered is the autoregressive process of order one, where $\{Y_t\}_{t\ge 1}$ in $\mathbb{R}$ is such that

$$Y_t = rY_{t-1} + \varepsilon\xi_t, \quad Y_0 = y_0,$$

where $\{\xi_t\}_{t\ge 1}$ is a sequence of independent and identically distributed normal random variables, each with mean 0 and variance σ^2. If $|r| < 1$, the process is Harris recurrent and has a stationary distribution which is normal with mean 0 and variance $\sigma^2/(1-r^2)$. If Γ is the interval

$(-1,1)$ and T is the exit time from this interval, the set A_0 is defined as $A_0 = \{x \in \mathbb{R} : 1 \leq |x| \leq 1+h\}$. The upper bound in (6) is then

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log ET \leq \inf_{1 \leq |u| \leq 1+h} \frac{u^2(1-r^2)}{2\sigma^2} = \frac{1-r^2}{2\sigma^2}. \tag{7}$$

In this case, this is the smallest possible upper bound, since it has been shown by other methods in Ruths (2008) that $\lim_{\varepsilon \to 0} \varepsilon^2 \log ET = (1-r^2)/(2\sigma^2)$.

As another example we can consider the more general case of the autoregressive process of order n, where $\{Y_t\}_{t \geq 1}$ in $\mathbb{R}$ is such that

$$Y_t = r_1 Y_{t-1} + \ldots + r_n Y_{t-n} + \varepsilon \xi_t, Y_0 = y_0,$$

where $\{\xi_t\}_{t \geq 1}$ is again a sequence of independent and identically distributed normal random variables with mean 0 and variance σ^2. By introducing the new notations

$$\tilde{Y}_t := \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-n+1} \end{pmatrix}, \quad R := \begin{pmatrix} r_1 & r_2 & \ldots & r_n \\ 1 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \ldots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\xi}_t := \begin{pmatrix} \xi_t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

we can write the autoregressive process of order n in multivariate form as

$$\tilde{Y}_t = R\tilde{Y}_{t-1} + \varepsilon \tilde{\xi}_t.$$

It has been shown in Meyn, Tweedie (1993) that this R is such that the matrix $[R^{n-1}S|R^{n-2}S|\ldots|RS|S]$, where $S = (1,0,\ldots,0)^T$, has rank n. If all eigenvalues of R fall within the open unit disk in the complex plane, the process has an invariant probability distribution π which is multivariate normal with mean 0 and covariance matrix $\varepsilon^2\Sigma$, where Σ is the solution of

$$\Sigma = R\Sigma R^T + SS^T.$$

Let us consider exits from the interval $\Gamma = (-1,1)$ for the autoregressive process $\{Y_t\}_{t \geq 1}$, and thus define

$$T = \inf\{t \geq 1 : |Y_t| \geq 1\}.$$

We then define $A_0 = \{x \in \mathbb{R}^n : 1 \leq |x_n| \leq 1+h\}$ and the upper bound in (6) is

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log ET \leq \inf_{1 \leq |u_n| \leq 1+h} \frac{1}{2} u^T \Sigma^{-1} u = \frac{1}{2\Sigma_{11}}, \tag{8}$$

where one can determine the infimum by using Lagrange multipliers. Here, Σ_{11} is the variance of the stationary distribution for the process $\{Y_t\}_{t \geq 1}$.

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