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# OSCILLATION CRITERIA FOR HIGHER ORDER NEUTRAL DIFFERENCE EQUATIONS WITH OSCILLATING COEFFICIENT

Abstract. In this paper sufficient conditions for oscillation of all bounded solutions of the equation

$$\Delta^m(x_n + p_n x_{n-\tau}) + f(n, x_n, x_{n-\sigma}) = 0$$

where $m \geq 2$, $(p_n)$ is an oscillatory sequence of real numbers, $\lim_{n\to\infty} p_n = 0$, $\tau$ and $\sigma$ are positive integers, $f : N \times R \times R \longrightarrow R$ are established.

Key words: neutral difference equation, oscillating coefficient, oscillatory solution.

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## 1. Introduction

Recently, there has been an increasing interest in the study of oscillatory and asymptotic behavior of solutions higher-order neutral differential and difference equations. Such equations appear in a number of important applications including problems in population dynamics or in "cobweb" models in economics. A systematic development of the oscillation theory of neutral equations was initiated by Ladas and Sficas ([5], [6]).

In this paper we consider a higher order neutral type difference equation of the form

$$\Delta^m(x_n + p_n x_{n-\tau}) + f(n, x_n, x_{n-\sigma}) = 0 \tag{E}$$

where $m \geq 2$, $(p_n)$ is an oscillatory sequence of real numbers, $\tau$ and $\sigma$ are positive integers, $f : N \times R \times R \longrightarrow R$.
For all $k \in N$ we use the usual factorial notation

$$n^{\underline{k}} = n(n-1)\dots(n-k+1) \quad \text{with} \quad n^{\underline{0}} = 1.$$

The integer part of real number $t$ we denote by $\lfloor t \rfloor$.

By a solution of equation (E) we mean a real sequence $(x_n)$ which is defined for $n = 1, 2 \ldots$ and which satisfies equation (E) for $n > \max\{\tau, \sigma\}$. A nontrivial solution $(x_n)$ of equation (E) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise it is called oscillatory.

The problem of finding sufficient conditions which ensure that all solutions (or all bounded solutions) of certain classes of difference equations of neutral type are oscillatory has been studied by a number of authors, see for example, [2-4, 7-11, 13] and the references cited therein. Most of the authors consider the case when the sequence $(p_n)$ inside the neutral part is of constant sign. The results on oscillation of equation (E) when the sequence $(p_n)$ is an oscillatory sequence are relatively scarce, see [2, 3, 4, 14]. Some comparison results for higher order linear difference equation of the form

$$\Delta^m \left(y_n + p_n y_{\tau(n)}\right) + q_n y_{\sigma(n)} = 0$$

were obtained by Y. Bolat, O. Akin and H. Yildirim in recent paper [3] and for the higher-order sublinear difference equation

$$\Delta^m \left(y_n + p_n y_{\tau(n)}\right) + q_n y^{\alpha}_{\sigma(n)} = 0$$

where $\alpha \in (0, 1)$ is a ratio of positive odd integers, $q_n \geq 0$ by I. Kir and Y. Bolat in [4]. The purpose of this paper is to establish conditions under which all bounded solutions of equation (E) are oscillatory. Using the arguments developed by Zafer in [12], Sundaram in [9] obtained oscillation criteria for equation (E) when $m$ is even and $0 < p_n \leq 1$. We will use similar arguments for the oscillation of bounded solutions of equation (E) when $(p_n)$ is an oscillatory sequence. Similar problem for differential equations was considered in [13]. We also establish sufficient conditions for the oscillation of bounded solutions of equation (E) when $m$ is even and the equation is linear, i.e. when the function $f$ satisfies $f(n, u, v) = a_n v$.

## 2. Lemmas

In the proofs of our theorems we shall need the following lemmas.

**Lemma 1** ([1], Discrete analogue of Kiguradze's Lemma)**.** *Let $(u_n)$ be a sequence of real numbers and $u_n > 0$ with $(\Delta^m u_n)$ of constant sign and not eventually identically zero. Then, there exist integers $l \in \{0, 1, \ldots, m\}$ with $(m+l)$ odd for $\Delta^m u_n \leq 0$, and $(m+l)$ even for $\Delta^m u_n \geq 0$ and $N > 0$ such that*

$$\begin{aligned} &\Delta^j u_n > 0 \qquad \textit{for} \quad j = 0, 1, \ldots, l, \\ &(-1)^{j+l}\Delta^j u_n > 0 \quad \textit{for} \quad j = l+1, \ldots, m, \end{aligned} \tag{1}$$

*for $n \geq N$.*

**Lemma 2** ([1]). *Let* $(u_n)$ *be a sequence of real numbers and let* $(u_n)$ *and* $(\Delta^m u_n)$ *be of constant sign and such that* $u_n > 0$ *and* $\Delta^m u_n \leq 0$ *not identically equal to zero. Then there exists a large number* $N > 0$ *such that*

$$u_n \geq \frac{(n-N)^{\underline{m-1}}}{(m-1)!}\Delta^{m-1}u_{2^{m-l-1}n} \quad for \quad n \geq N.$$

**Lemma 3.** *Let* $m \geq 3$ *be an odd integer,* $\sigma$ *be a positive integer and* $(u_n)$ *be a sequence of real numbers such that*

$$(2) \qquad (-1)^j \Delta^j u_n > 0 \quad for \quad j = 0, 1 \ldots, m-1 \quad and \quad \Delta^m u_n \leq 0.$$

*Then*

$$u_{n-\sigma} \geq \frac{(m+\sigma-2)^{\underline{m-1}}}{(m-1)!}\Delta^{m-1}u_n \quad for \quad n \geq \sigma.$$

**Proof.** By discrete Taylor's formula (see [1], p. 43) we have

$$u_{n-\sigma} = \sum_{i=0}^{m-1} \frac{(i+\sigma-1)^{\underline{i}}}{i!}(-1)^i \Delta^i u_n - \frac{(-1)^{\underline{m-1}}}{(m-1)!}\sum_{l=n-\sigma}^{n-1}(l+m-1-n+\sigma)^{\underline{m-1}}\Delta^{m-1}u_l$$

for $n \geq \sigma$. Therefore, since $m$ is odd using (2) we get

$$u_{n-\sigma} \geq \frac{(m+\sigma-2)^{\underline{m-1}}}{(m-1)!}\Delta^{m-1}u_n.$$

Hence the lemma is proved. ∎

## 3. Main results

**Theorem 1.** *Let* $m \geq 2$ *be an even number. Assume that* $(p_n)$ *is an oscillatory sequence of real numbers such that* $\lim_{n\to\infty} p_n = 0$ *and*

(*i*) $f : N \times R \times R \longrightarrow R$ *and* $vf(n, u, v) > 0$ *for* $v \neq 0$;

(*ii*) *there exists a continuous function* $g : R_+ \longrightarrow R_+$ *and a sequence* $\phi : N \longrightarrow R_+$ *such that*

$$(3) \qquad |f(n,u,v)| \geq \phi_n g\left(\frac{|v|}{\left(\frac{n-\sigma}{2^{m-2}}\right)^{m-1}}\right)$$

*for all large* $n$*, where*

$$(4) \qquad \sum_{n=1}^{\infty} \phi_n = \infty$$

*and the function $g$ is nondecreasing and*

$$\int_0^\beta \frac{ds}{g(s)} < \infty \quad \text{for every} \quad \beta > 0. \tag{5}$$

*Then every bounded solution $(x_n)$ of the equation (E) is oscillatory.*

**Proof.** Suppose that equation (E) has a bounded nonoscillatory solution $(x_n)$. We may assume that $(x_n)$ is eventually positive (the proof when $(x_n)$ is eventually negative is similar). Then, there exists an integer $n_0 \geq 1$ such that $x_n > 0$, $x_{n-\tau} > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_0$. Set

$$z_n = x_n + p_n x_{n-\tau}. \tag{6}$$

From (E) we have

$$\Delta^m z_n = -f(n, x_n, x_{n-\sigma}) < 0, \quad n \geq n_0. \tag{7}$$

Therefore the sequences $(\Delta^i z_n), i = 0, 1, \ldots, m-1$ are strictly monotonic and of constant sign eventually. We claim that $z_n > 0$ for $n \geq n_0$. Otherwise $x_n < -p_n x_{n-\tau}$ which is a contradiction with $(p_n)$ being an oscillatory sequence.

Since $(p_n)$ is an oscillatory sequence with $\lim_{n\to\infty} p_n = 0$ and $(x_n)$ is bounded, we have $(z_n)$ is bounded, too. By Lemma 1, there exist an odd integer $l$ and $n_1 \geq n_0$ such that (1) is satisfied by $(z_n)$ for $n_1 \geq n_0$. Since $m$ is even and $z_n > 0$ and bounded it follows that $l = 1$ and hence $(z_n)$ is increasing.

Using the fact that $(x_n)$ is bounded and $\lim_{n\to\infty} p_n = 0$ we see that $\lim_{n\to\infty} p_n x_{n-\tau} = 0$. Then, by (6) it is easy to see that there exists $n_2 \geq n_0$ such that

$$x_n \geq \frac{1}{2} z_n, \quad n \geq n_2, \tag{8}$$

Using Lemma 2 (with $l = 1$) and the fact that $(z_n)$ is increasing, we have

$$z_n \geq z_{\left(\left\lfloor \frac{n}{2^{m-2}} \right\rfloor\right)} \geq \frac{1}{(m-1)!} \left( \left\lfloor \frac{n}{2^{m-2}} \right\rfloor - n_3 \right)^{\underline{m-1}} \Delta^{m-1} z_n, \; n \geq n_3. \tag{9}$$

It is easy to check that for $j > m$ holds $j^{\underline{m}} \geq \frac{j^m}{2(m-1)!}$. Hence, from (9) we get

$$\begin{aligned} z_n &\geq \frac{1}{2(m-1)!(m-2)!} \left( \left\lfloor \frac{n}{2^{m-2}} \right\rfloor - n_3 \right)^{m-1} \Delta^{m-1} z_n \\ &\geq \frac{1}{2(m-1)!(m-2)!} \left( \frac{n}{2^{m-2}} - 1 - n_3 \right)^{m-1} \Delta^{m-1} z_n. \end{aligned}$$

Therefore, by choosing $n_4 \geq n_3$, arbitrary large, we have

$$z_n \geq \frac{1}{2(m-1)!(m-2)!}\left(\frac{n}{2^{m-2}}\right)^{m-1}\Delta^{m-1}z_n, \quad n \geq n_4. \tag{10}$$

Hence, by (8) and using the fact that $\Delta^{m-1}z_n$ is decreasing, we have

$$\frac{x_{n-\sigma}}{\left(\frac{n-\sigma}{2^{m-2}}\right)^{m-1}} \geq c\Delta^{m-1}z_n, \quad n \geq n_4$$

where $c = \frac{1}{4(m-1)!(m-2)!}$ Therefore, using (3) we get

$$\Delta^m z_n + \phi_n g(c\Delta^{m-1}z_n) \leq 0.$$

Setting $u_n = c\Delta^{m-1}z_n$ we have $\Delta u_n = c\Delta^m z_n$ and

$$\frac{\Delta u_n}{g(u_n)} + c\phi_n \leq 0.$$

Since

$$c\phi_i \leq -\frac{\Delta u_i}{g(u_i)} \leq -\int_{u_i}^{u_{i+1}} \frac{ds}{g(s)}$$

summing the above inequality from $n$ to $n_4$ we obtain

$$c\sum_{i=n_4}^{n-1}\phi_i \leq -\sum_{i=n_4}^{n-1}\int_{u_i}^{u_{i+1}}\frac{ds}{g(s)} = \int_{u_n}^{u_{n_4}}\frac{ds}{g(s)}.$$

Hence

$$c\sum_{i=n_4}^{n-1}\phi_i \leq \int_{u_n}^{u_{n_4}}\frac{ds}{g(s)}. \tag{11}$$

Since $(u_n)$ is positive and decreasing, it follows that there exists $\lim_{n\to\infty} u_n = L \geq 0$. If $L \neq 0$ then by (11) we must have

$$\sum_{i=1}^{\infty}\phi_i < \infty \tag{12}$$

which contradicts (4). In the case when $L = 0$, letting $n \to \infty$ in (11), we again obtain (12). This completes the proof. ■

Theorem 1 applied to the generalized Emden-Fowler difference equation

$$\Delta^m(x_n + p_n x_{n-\tau}) + a_n |x_{n-\sigma}|^\alpha sgn(x_{n-\sigma}) = 0, \quad 0 < \alpha < 1 \tag{E1}$$

where $m \geq 2$, $(p_n)$ is an oscillatory sequences with $\lim_{n\to\infty} p_n = 0$, $\tau$ and $\sigma$ are positive integers, $(a_n)$ is a sequences of real numbers leads to the following corollary.

**Corollary 1.** *Assume that*

$$\sum_{j=1}^{\infty} j^{\alpha(m-1)}|a_j| = \infty.$$

*If m is even then every bounded solution* $(x_n)$ *of equation (E1) is oscillatory.*

**Proof.** The conclusion of Corollary 1 follows from Theorem 1 with

$$\phi_n = \left(\frac{n-\sigma}{2^{m-2}}\right)^{\alpha(m-1)} |a_n| \quad \text{and} \quad g(u) = u^\alpha.$$

■

**Example 1.** Consider the difference equation

$$\Delta^2\left(x_n + \frac{(-1)^{n+1}}{n} x_{n-1}\right) + \left(\frac{4n^2+8n+2}{(n+2)^{\underline{3}}} + \frac{(-1)^n 6}{(n+2)^{\underline{4}}}\right) \times (n-3)^{\frac{1}{2}} |x_{n-3}|^{\frac{1}{2}} sgn x_{n-3} = 0. \tag{13}$$

It is easy to see that $\sum\limits_{j=1}^{\infty} j^{\frac{1}{2}}(j-3)^{\frac{1}{2}}|\left(\frac{4j^2+8j+2}{(j+2)^{\underline{3}}} + \frac{(-1)^j 6}{(j+2)^{\underline{4}}}\right)| = \infty$. From Corollary 1 it follows that every bounded solution of equation (13) is oscillatory. One such solution is $x_n = \frac{(-1)^n}{n}$.

Note, that Theorem 3.1 from [4] for equation (13) can not be applied.

**Theorem 2.** *Let* $m \geq 3$ *be an odd integer. Assume that* $(p_n)$ *is an oscillatory sequence of real numbers such that* $\lim_{n\to\infty} p_n = 0$ *and*

(*i*) $f : N \times R \times R \longrightarrow R$ *and* $vf(n,u,v) > 0$ *for* $v \neq 0$;

(*ii*) *there exists a continuous function* $g : R_+ \longrightarrow R_+$ *and a sequence* $\phi : N \longrightarrow R_+$ *such that*

$$|f(n,u,v)| \geq \phi_n g\left(\frac{|v|}{(m+\sigma-2)^{\underline{m-1}}}\right) \tag{14}$$

*for all large n, where*

$$\sum_{n=1}^{\infty} \phi_n = \infty \tag{15}$$

*and the function g is nondecreasing and*

$$\int_0^\beta \frac{ds}{g(s)} < \infty \quad \text{for every} \quad \beta > 0. \tag{16}$$

*Then every bounded solution* $(x_n)$ *of the equation (E) is oscillatory.*

**Proof.** Suppose that equation (E) has a bounded nonoscillatory solution $(x_n)$. We proceed as in the of Theorem 1 and obtain that $(z_n)$ is eventually positive and bounded and $\Delta^m z_n < 0$ eventually. Therefore, since $m$ is odd, by Lemma 1 it follows that (1) is satisfied with $l = 0$. Hence, applying Lemma 3 we get

$$z_{n-\sigma} \geq \frac{(m+\sigma-2)^{\underline{m-1}}}{(m-1)!} \Delta^{m-1} z_n \tag{17}$$

for sufficiently large $n$. From (8) and (17) it follows

$$\frac{x_{n-\sigma}}{(m+\sigma-2)^{\underline{m-1}}} \geq c\Delta^{m-1} z_n$$

where $c = \frac{1}{2(m-1)!}$. Therefore, using (14) we get

$$\Delta^m z_n + \phi_n g(c\Delta^{m-1} z_n) \leq 0.$$

The rest of the proof is similar to that of Theorem 1 and thus we omit it. ■

**Example 2.** Consider the difference equation

$$\Delta^3 \left( x_n + \left(-\frac{1}{2}\right)^n x_{n-2} \right) + \left( -8 - \frac{1}{8} \left(-\frac{1}{2}\right)^n \right) x_{n-3}^{\frac{1}{3}} = 0. \tag{18}$$

Here $g(x) = x^{\frac{1}{3}}$, $\phi_n = 8 + \frac{1}{8}\left(-\frac{1}{2}\right)^n$. Since $q_n = -8 - \frac{1}{8}\left(-\frac{1}{2}\right)^n < 0$, Theorem 3.2 from [4] does not apply here. But one can see that all conditions of Theorem 2 are satisfied. Thus every bounded solution of equation (18) is oscillatory. One such solution is $x_n = (-1)^n$.

**Corollary 2.** *Assume that*

$$\sum_{j=1}^{\infty} |a_j| = \infty.$$

*If $m$ is odd then every bounded solution $(x_n)$ of equation (E1) is oscillatory.*

**Proof.** Apply Theorem 2 with

$$\phi_n = \left((m+\sigma-2)^{\underline{m-1}}\right)^\alpha |a_n| \quad \text{and} \quad g(u) = u^\alpha.$$

■

Note, that Theorem 1 and Theorem 2 are not applicable to linear equations.

Let us consider linear equation

$$\Delta^m (x_n + p_n x_{n-\tau}) + a_n x_{n-\sigma} = 0 \tag{E2}$$

where $m \geq 2$, $(p_n)$ is an oscillatory sequences with $\lim_{n\to\infty} p_n = 0$, $\tau$ and $\sigma$ are positive integers, $(a_n)$ is a sequences of positive real numbers. For even order equations of type (E2) we have following result.

**Theorem 3.** *Let $m$ be even. If*

$$\liminf_{n\longrightarrow\infty} \sum_{j=n-\sigma}^{n-1} \left(\frac{j-\sigma}{2^{m-2}}\right)^{m-1} a_j > 4(m-1)!(m-2)! \left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{19}$$

*then every bounded solution $(x_n)$ of the equation (E2) is oscillatory.*

**Proof.** Proceeding as in the proof of Theorem 1, from (7)-(10) we obtain

$$\Delta^m z_n + \frac{1}{4(m-1)!(m-2)!} \left(\frac{n-\sigma}{2^{m-2}}\right)^{m-1} a_n \Delta^{m-1} z_{n-\sigma} \leq 0.$$

Setting in the above inequality $u_n = \Delta^{m-1} z_n$ we get

$$\Delta u_n + \frac{1}{4(m-1)!(m-2)!} \left(\frac{n-\sigma}{2^{m-2}}\right)^{m-1} a_n u_{n-\sigma} \leq 0. \tag{20}$$

Now, an application of the result in [1], Theorem 6.20.5, implies that if (19) is satisfied then (20) cannot have an eventually positive solution. Hence, every bounded solution of equation (E2) must be oscillatory. This completes the proof. ∎

Note, that from the proof of Theorem 3 it follows that every bounded solution of equation (E2) is oscillatory if the inequality (20) has not any eventually positive bounded solution. The resuts obtained in Theorem 3.1 in [3] and in Theorem 3 are similar.

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