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# ON CONNECTIONS BETWEEN OSCILLATORY SOLUTIONS OF FUNCTIONAL, DIFFERENCE AND DIFFERENTIAL EQUATIONS

Abstract. The paper contains connections between oscillation of solutions of iterative functional equations, difference equations and differential equations with advanced or delayed arguments. New oscillatory criteria for these equations are given.

Key words: oscillatory solutions, functional equations, difference equations, differential equations.

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## 1. Introduction

The aim of this paper is to present some dependence occurring between oscillation of solutions of iterative functional equations, difference equations and differential equations with deviating arguments. To present methods of investigation of this dependence we focus our attention on very simple forms of these equations.

First, let us observe that the development of the oscillation theory for differential equations has a history of more than a hundred years and it began from the classical results of Kneser [8] and Sturm [26]. In the second half of the 20th century we observed the development of the theory of differential equations with deviating arguments mostly in the area of oscillation of their solutions. It has appeared that oscillatory properties of solutions of ordinary differential equations and differential equations with deviating arguments have been essentially different. It was expressed, among others, by qualitative changes of solutions. Many papers have been devoted to these problems (see e.g. [2, 7, 10, 13, 15, 16, 18, 24, 25, 29] and the references cited therein). We approach this problem based on selected results for the very simple differential equation with the delay argument of the form

$$x'(t) + p(t)x(t-\tau) = 0, \tag{1}$$

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where $t \geq 0$, $p : \Re_+ \to \Re_+ = [0, \infty)$ is a continuous function, $\tau$ is a positive real number. A solution of this equation is called oscillatory if it has an infinite sequence of zeros tending to infinity. Otherwise it is called nonoscillatory. It is obvious that the ordinary differential equation

$$x'(t) + p(t)x(t) = 0,$$

where $t$ and $p$ are as previous, which corresponds to equation (1), has only nonoscillatory solution. But for equation (1) in 1950 Myshkis [18] proved the following result: *if*

$$\liminf_{t\to\infty} p(t) > \frac{1}{\tau e}, \tag{2}$$

*then all solutions of equation (1) are oscillatory.*

In 1972 Ladas, Lakshmikantham and Papadakis [15] obtained the following condition: *if*

(3) [15, 16]
$$\limsup_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)ds > 1,$$

*then all solutions of equation (1) oscillate.*

The most general result as (2) for equation (1) was given by Ladas [13] in 1979 and by Koplatadze and Chanturia [10] in 1982. It has the form: *if*

(4) [10, 13]
$$\liminf_{t\to\infty} \int\limits_{t-\tau}^{t} p(s)ds > \frac{1}{e}$$

*then all solutions of equation (1) oscillate.*

Some time later the oscillation theory of the difference equations has been developed.

Consider now the difference equation

$$x(n+1) - x(n) + p(n)x(n-m) = 0, \quad n = 0, 1, \ldots, \tag{5}$$

where $p : N \to \Re_+$ and $m$ is a positive integer, which represents a discrete analogue of the delay differential equation (1).

First it was studied for $m = 1$ by Domshlak [3] in 1981 .

In 1989 Erbe and Zhang [5] proved the following oscillation criteria for equation (5): *let*

$$\beta = \liminf_{n\to\infty} p(n) > 0 \quad \textit{and} \quad \limsup_{n\to\infty} p(n) > 1 - \beta, \tag{i}$$

*or*

$$\text{(ii)} \qquad \liminf_{n\to\infty} p(n) > \frac{m^m}{(m+1)^{m+1}},$$

*or*

$$\text{(6)} \quad \text{(iii)} \qquad \limsup_{n\to\infty} \sum_{i=0}^{m} p(n-i) > 1.$$

*Then all solutions of equation (5) oscillate.*

In the same year Ladas, Philos and Sficas [14] established the following result improving (ii): *if*

$$\liminf_{n\to\infty} \frac{1}{m} \sum_{i=1}^{m} p(n-i) > \frac{m^m}{(m+1)^{m+1}}, \tag{7}$$

*then every solution of equation (5) is oscillatory.*

Other interesting results concerning difference equations one can find in [1, 2, 7, 20, 25].

Finally, in the last few years, oscillatory properties of solutions of functional equations, which are generalizations of recurrence equations, have been studied (for example in [6, 19, 21, 22, 23, 27]).

Forthright generalization of equation (5) is the functional equation of the form

$$x(g(t)) - x(t) = Q(t)x(g^{m+1}(t)), \quad m \geq 1, \tag{8}$$

where $t \in I$, $I$ is an unbounded subset of $\Re_+$. Functions $Q : I \to \Re_+$, $g : I \to I$ are given and $x$ is an unknown real-valued function. By $g^m$ we denote the m-th iterate of function $g$, i.e.

$$g^0(t) = t, \quad g^{m+1}(t) = g(g^m(t)), \quad t \in I, \quad m = 0, 1, \dots .$$

$g^{-1}$ is the inverse function to $g$ and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$. In each instance the relation $g^1(t) = g(t)$ is true.

A function $x : I \to \Re$ such that $\sup\{|x(s)| : s \in I_{t_0} = [t_0, \infty) \cap I\} > 0$ for any $t_0 \in \Re_+$, which satisfies (8) on $I$, we define as a solution of this equation.

A solution $x$ of equation (8) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}, t_n \in I$, such that $lim_{n\to\infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \leq 0$ for $n = 1, 2, \dots$. Otherwise it is called nonoscillatory.

In our considerations we assume that

$$g(t) \neq t \quad \text{and} \quad \lim_{t\to\infty} g(t) = \infty, \quad t \in I. \tag{9}$$

Moreover, we assume that $g$ has an inverse function.

As usual we take $\sum_{j=k}^{k-1} a_j = 0$.

The aim of this paper is to show that based on results for the functional equations it is possible to obtain new and interesting oscillatory criteria for difference equations and for the first order differential equations with delayed or advanced arguments. In our opinion in the available literature no such connections are known.

First we present sufficient conditions for the oscillation of all solutions of functional equation (8).

Next, from results obtained for the above functional equation we give conditions for the oscillation of all solutions of the difference equation of the form

$$x(n+\nu) - x(n) = Q(n)x(n+(m+1)\nu), \tag{10}$$

where $n, m \in N$, $Q : N \to \Re_+$ and $\nu \neq 0$ is any fixed integer. We show connections between conditions for functional equations and for difference equations.

In the end from results for equation (8) we obtain oscillatory criteria for differential equations with deviating arguments of the form

$$x'(t) = \lambda p(t)x(t+\lambda\tau) \tag{11}$$

where $t \geq 0$, $p : \Re_+ \to \Re_+$ is a continuous function, $\tau$ is a positive real number and $\lambda = \pm 1$.

## 2. Oscillation of functional equation

Let us observe that for functional equation (8) we can obtain similar conditions to (3) and (4) given for differential equation (1) and results (6) and (7) for recurrence equation (5). We start from the following

**Theorem 1.** *If*

$$\liminf_{I \ni t \to \infty} \sum_{i=0}^{m-1} Q(g^i(t)) > \left(\frac{m}{m+1}\right)^{m+1} \tag{12}$$
[19]

*or*

$$\limsup_{I \ni t \to \infty} \sum_{i=0}^{m} Q(g^i(t)) \left\{ 1 + \sum_{j=1}^{i} Q(g^{m+j}(t)) \right\} > 1, \tag{13}$$
[20]

*then*

*(i) the functional inequality*

$$x(g(t)) \geq x(t) + Q(t)x(g^{m+1}(t)), \tag{14}$$

*where* $m \geq 1, Q : I \to \Re_+$ *and* $g$ *is as previously, has not positive solutions,*
*(ii) the functional equation (8) has only oscillatory solutions.*

Now we present a new oscillatory criterion for equation (8). It could be applied when conditions (12) and (13) are not fulfilled.

**Theorem 2.** *Let for sufficiently large* $t \in I_{t_1}$

$$\sum_{i=0}^{m-1} Q(g^i(t)) \geq M, \qquad 0 < M < \left(\frac{m}{m+1}\right)^{m+1}. \tag{15}$$

*If*

$$\limsup_{\Im \ni t \to \infty} \sum_{i=0}^{m} Q(g^i(t)) \left\{ 1 + \sum_{j=1}^{i} Q(g^{j+m}(t)) \right\} > 1 - \frac{M^{m+1}}{1 - M^m}, \tag{16}$$

*then the conclusion of Theorem 1 holds.*

**Proof.** Suppose that $x(t) > 0$, $t \in I_{t_1}$ is a nonoscillatory solution of inequality (14). Then in view of assumption (9) about function $g$ there exists a point $t_2 \geq t_1$ such that $x(g^i(t)) > 0$, $i \in \{1, 2, ..., m+1\}$ and $t \in I_{t_2}$. Thus from (14) for sufficiently large $t \in I_{t_3}$ the following inequality is satisfied

$$x(g^i(t)) \geq x(g^{i-1}(t)), \quad i = 1, 2, \ldots$$

and

$$x(g^{m+i+1}(t)) \geq x(g^{m+1}(t)), \quad i = 1, 2, \ldots . \tag{17}$$

From inequality (14) we have

$$x(g^{i+1}(t)) \geq x(g^i(t)) + Q(g^i(t))x(g^{m+i+1}(t)), \quad i = 0, 1, \ldots . \tag{18}$$

Summing now both sides of inequality (18) from $i = 1$ to $m$ we get

$$x(g^{m+1}(t)) \geq x(g(t)) + \sum_{i=1}^{m} Q(g^i(t))x(g^{m+i+1}(t)). \tag{19}$$

From (18) we obtain for $i \in \{1, 2, ..., m\}$ and $j \in \{0, 1, ..., m\}$

$$x(g^{m+i+1-j}(t)) \geq x(g^{m+i-j}(t)) + Q(g^{m+i-j}(t))x(g^{2m+i+1-j}(t))$$

and in view of (17)

$$x(g^{m+i+1-j}(t)) \geq x(g^{m+i-j}(t)) + Q(g^{m+i-j}(t))x(g^{m+1}(t)). \tag{20}$$

Applying now (14) and (20) for $j = 0$ in (19) we have

$$\begin{aligned} x(g^{m+1}(t)) \geq{} & x(t) + Q(t)x(g^{m+1}(t)) \\ & + \sum_{i=1}^{m} Q(g^i(t)) \left\{ x(g^{m+i}(t)) + Q(g^{m+i}(t))x(g^{m+1}(t)) \right\}. \end{aligned} \tag{21}$$

Hence

$$\begin{aligned} x(g^{m+1}(t)) \geq{} & x(t) + x(g^{m+1}(t)) \left\{ \sum_{i=0}^{1} Q(g^i(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t)) \right\} \\ & + \sum_{i=2}^{m} Q(g^i(t))x(g^{m+i}(t)). \end{aligned}$$

Using now (20) for $j = 1$ in the above inequality we obtain

$$\begin{aligned} & x(g^{m+1}(t)) \geq x(t) + x(g^{m+1}(t)) \\ & \times \left\{ \sum_{i=0}^{2} Q(g^i(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t)) + \sum_{i=2}^{m} Q(g^i(t))Q(g^{m+i-1}(t)) \right\} \\ & + \sum_{i=3}^{m} Q(g^i(t))x(g^{m+i-1}(t)). \end{aligned}$$

Similarly we have

$$\begin{aligned} x(g^{m+1}(t)) \geq{} & x(t) + x(g^{m+1}(t)) \\ & \times \left\{ \sum_{i=0}^{m} Q(g^i(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t)) \right. \\ & + \sum_{i=2}^{m} Q(g^i(t))Q(g^{m+i-1}(t)) \\ & \left. + \ldots + \sum_{i=m-1}^{m} Q(g^i(t))Q(g^{i+2}(t)) + Q(g^m(t))Q(g^{m+1}(t)) \right\}. \end{aligned} \tag{22}$$

From (19) it follows that

$$x(g^m(t)) \geq x(t) + \sum_{i=0}^{m-1} Q(g^i(t))x(g^{m+i+1}(t))$$

and in view of (17)

$$x(g^m(t)) \geq x(t) + x(g^{m+1}(t)) \sum_{i=0}^{m-1} Q(g^i(t)).$$

Using assumption (15) we have

$$x(g^m(t)) \geq x(t) + Mx(g^{m+1}(t)) \tag{23}$$

and

$$x(g^m(t)) \geq Mx(g^{m+1}(t)).$$

Hence

$$x(t) \geq M^i x(g^i(t)), \quad i = 1, 2, ..., m.$$

From the last inequality for $i = m$ and (23) we have

$$\frac{x(t)}{M^m} \geq x(t) + Mx(g^{m+1}(t)).$$

Thus

$$x(t) \geq \frac{M^{m+1}}{1 - M^m} x(g^{m+1}(t)).$$

Applying now the above inequality in (22) we get

$$\begin{aligned} x(g^{m+1}(t)) \geq{} & \frac{M^{m+1}}{1 - M^m} x(g^{m+1}(t)) + x(g^{m+1}(t)) \\ & \times \left\{ \sum_{i=0}^{m} Q(g^i(t)) + \sum_{i=1}^{m} Q(g^i(t))Q(g^{m+i}(t)) + \sum_{i=2}^{m} Q(g^i(t))Q(g^{m+i-1}(t)) \right. \\ & \left. + ... + \sum_{i=m-1}^{m} Q(g^i(t))Q(g^{i+2}(t)) + Q(g^m(t))Q(g^{m+1}(t)) \right\}. \end{aligned}$$

Dividing now both sides of the above inequality by $x(g^{m+1}(t))$ we obtain a contradiction to (16). Thus the proof is complete. ■

## 3. Oscillatory solutions of difference equations

Let us observe that functional equations are some kind of generalization of recurrence equations and from conditions (12) and (13) we can obtain oscillation criteria for difference equations. For example, if we assume that $I = N = \{1, 2, ...\}$ and $g(n) = n + \nu$, $\nu \neq 0$ then from equation (8) we get the difference equation of form (10). Thus from Theorem 1 we obtain the following

**Corollary 1.** *If one of the following conditions is satisfied*

$$\liminf_{n\to\infty} \sum_{i=0}^{m-1} Q(n+\nu i) > \left(\frac{m}{m+1}\right)^{m+1} \tag{24}$$

*or*

$$\limsup_{n\to\infty} \sum_{i=0}^{m} Q(n+\nu i) \left\{1 + \sum_{j=1}^{i} Q(n+(m+j)\nu)\right\} > 1, \tag{25}$$

*then all solutions of equation (10) are oscillatory.*

Now, in view of Theorem 2, we present a new oscillation criterion for equation (10). If conditions (24) and (25) from Corollary 1 are not fulfilled we could apply the following

**Corollary 2.** *Let*

$$\sum_{i=0}^{m-1} Q(n+\nu i) \geq M, \qquad 0 < M < \left(\frac{m}{m+1}\right)^{m+1}. \tag{26}$$

*If*

$$\limsup_{n\to\infty} \sum_{i=0}^{m} Q(n+\nu i) \left\{1 + \sum_{j=1}^{i} Q(n+(m+j)\nu)\right\} > 1 - \frac{M^{m+1}}{1-M^m}, \tag{27}$$

*then equation (10) has only oscillatory solutions.*

**Remark 1.** Notice that from condition (24) for $\nu = -1$ we get condition (7) and for $\nu = 1$ the result of Györi and Ladas [7]. On the other hand, condition (25) is better than (6). Let us note that condition (27) for the oscillation of solutions of difference equation (10) is new.

## 4. Oscillation of solutions of differential equations with deviating arguments

Results obtained for functional equations could be applied to get oscillatory criteria for differential equations with deviating arguments. We present results for the first order differential equation of form (11). The problem of existence of oscillatory solutions of the first order differential equations with deviating arguments has been considered by many authors (see e.g. [2, 7, 10, 13, 15, 16, 18, 24, 25, 29] and the references cited therein), but we succeeded in obtaining some new oscillation criteria for equation (11).

We give sufficient conditions for the oscillation of equation (11) in the case $\lambda = -1$, i.e. for the delay differential equation of the form

$$x'(t) + p(t)x(t-\tau) = 0. \tag{1}$$

Analogous results we could obtain also for advanced differential equations, i.e. $\lambda = 1$.

**Theorem 3.** *If for some $m \geq 1$ one of the following conditions is satisfied*

$$\liminf_{t\to\infty} \int_{t-\frac{m}{m+1}\tau}^{t} p(s)ds > \left(\frac{m}{m+1}\right)^{m+1} \tag{28}$$

*or*

$$\limsup_{t\to\infty} \left\{ \int_{t-\tau}^{t} p(s)ds + \sum_{i=0}^{m} \int_{t-\frac{i+1}{m+1}\tau}^{t-\frac{i}{m+1}\tau} p(s)ds \int_{t-\frac{m+i+1}{m+1}\tau}^{t-\tau} p(u)du \right\} > 1, \tag{29}$$

*then all solutions of equation (1) are oscillatory.*

Now we give a criterion for the oscillation of all solutions of equation (1), which could be applied when conditions (28) and (29) are not satisfied.

**Theorem 4.** *Let for some $m \geq 1$*

$$\int_{t-\frac{m}{m+1}\tau}^{t} p(s)ds \geq M, \qquad 0 < M < \left(\frac{m}{m+1}\right)^{m+1}. \tag{30}$$

*If*

$$\limsup_{t\to\infty} \left\{ \int_{t-\tau}^{t} p(s)ds + \sum_{i=0}^{m} \int_{t-\frac{i+1}{m+1}\tau}^{t-\frac{i}{m+1}\tau} p(s)ds \int_{t-\frac{m+i+1}{m+1}\tau}^{t-\tau} p(u)du \right\} > 1 - \frac{M^{m+1}}{1-M^m}, \tag{31}$$

*then equation (1) has only oscillatory solutions.*

**Proof of Theorems 3 and 4.** Suppose that equation (1) has a nonoscillatory solution $x(t) > 0$ for $t \geq t_0$. Let $m \geq 1$ be a natural number.

Integrating now both sides of the considered equation from $t - \frac{\tau}{m+1}$ to $t$ we obtain

$$x(t - \frac{\tau}{m+1}) - x(t) = \int_{t-\frac{\tau}{m+1}}^{t} p(s)x(s-\tau)ds.$$

Since $x'(t) < 0$ for $t \geq t_1$, we have

$$x(t - \frac{\tau}{m+1}) \geq x(t) + x(t-\tau) \int_{t-\frac{\tau}{m+1}}^{t} p(s)ds.$$

In view of assumptions (28), (29), (31) and Theorems 1 and 2 the last inequality cannot possess positive solutions. We obtained a contradiction with the fact that $x(t)$ is a positive solution of (1). Thus the proof is complete. ■

**Remark 2.** Let us remark that "classic" conditions for the oscillation of all solutions of equation (1) have forms (3) and (4). Compare now conditions (4) and (28). It is obvious that

$$\int_{t-\tau}^{t} p(s)ds > \int_{t-\frac{m}{m+1}\tau}^{t} p(s)ds \tag{32}$$

but

$$\frac{1}{e} > \left(\frac{m}{m+1}\right)^{m+1} \quad \text{for} \quad m \geq 1. \tag{33}$$

Notice that conditions (4) and (28) are asymptotically equivalent in such a sense that the left hand sides of inequalities (32) and (33) are equal to limits of the right hand sides of those inequalities with $m \to \infty$.

Moreover, notice that condition (29) is better than (3) and improves the similar result from Theorem 1 of [29].

Finally, let us observe that the method presented in this paper could be applied to differential equations of higher order.

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