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FURTHER INSTANCES OF PERIODICITY IN MAY'S HOST PARASITOID EQUATION

Abstract. May's host parasitoid equation is the difference equation

$$x_{n+1} = \frac{\alpha x_n^2}{(1+x_n)x_{n-1}}, \quad \alpha > 1. \tag{1}$$

We show that for each α there is a number k such that, whenever $n > k$, equation (1) has a one cycle periodic solution of period n. We also give some results on two cycle periodic solutions.

Key words: difference equation, periodicity.

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1. Introduction

May's host parasitoid equation (1) shows a lot of almost periodic cyclic behavior. In [1] it is shown that if $\alpha \in (0, 1]$ every solution goes to zero (hence our assumption that $\alpha > 1$), and that if $\alpha > 1$ then $\alpha - 1$ is the unique positive equilibrium, which is not asymptotically stable. In [2] it is shown that with $\alpha > 1$, if $x_{-1} = x_0 = 1$ then there are values of α giving one cycle periodic solutions of period n for all integers $n \geq 7$, and that there are no non-equilibrium periodic solutions of period less than 7. In this paper we adapt the results in [2] to get further periodicity results, including the results described in the abstract.

2. Getting periodic solutions

In [2] we fixed the initial conditions, setting $x_{-1} = x_0 = 1$, and thought of α as a variable. In the present context we still take $x_{-1} = x_0$ but we take the common value to be x. We assume $x > 0$ but think of x as a variable.

By taking $x_{-1} = x_0$ we get the same criteria for periodic solutions as in [2]:

Theorem 1. *Consider any difference equation which can be put in the form* $x_{n+1}x_{n-1} = f(x_n)$*, and take* $x_{-1} = x_0 = x$*. For* $k \geq 0$

(*i*) *we get a periodic solution of period* $2k + 2$ *if and only if* $x_k = x_{k+1}$*, and*

(*ii*) *we get a periodic solution of period* $2k + 3$ *if and only if* $x_k = x_{k+2}$.

Proof. The proof of Theorem 1 of [2] is easily adapted to cover this situation. ■

Note that this result applies to May's host parasitoid equation.

We can use (1) to compute a few values of x_i:

$$\begin{aligned} x_1 &= \tfrac{\alpha x}{1+x}, \\ x_2 &= \tfrac{\alpha^3 x}{(1+x)(1+x+\alpha x)}, \\ x_3 &= \tfrac{\alpha^6 x}{(1+x+\alpha x)((1+x)(1+x+\alpha x)+\alpha^3 x)}, \\ x_4 &= \tfrac{\alpha^{10} x(1+x)}{((1+x)(1+x+\alpha x)+\alpha^3 x)((1+x+\alpha x)((1+x)(1+x+\alpha x)+\alpha^3 x)+\alpha^6 x)}. \end{aligned}$$

Direct computation using theorem 1 shows that there are no solutions to equation (1) with $x_{-1} = x_0$ of prime period 2, 3, 4, 5, or 6.

With simplification, solutions of period 8 (where $x_3 = x_4$) correspond to solutions of

$$\begin{aligned} 0 &= x^3(\alpha^2 + 2\alpha + 1) + x^2(-\alpha^5 + \alpha^3 + \alpha^2 + 4\alpha + 3) + \\ &\quad + x(\alpha^6 - \alpha^5 - 2\alpha^4 + \alpha^3 + 2\alpha + 3) + (-\alpha^4 + 1). \end{aligned} \tag{2}$$

We anticipate that one solution to this equation will correspond to the equilibrium of (1), the value $x = \alpha - 1$. Thus $x - (\alpha - 1)$ will be a factor of the right hand side of (2); we factor it out and set the other factor equal to zero to get

$$\begin{aligned} 0 &= x^2(\alpha^2 + 2\alpha + 1) + x(-\alpha^5 + 2\alpha^3 + 2\alpha^2 + 3\alpha + 2) + \\ &\quad + (\alpha^3 + \alpha^2 + \alpha + 1). \end{aligned} \tag{3}$$

Descartes' Rule of Signs tells us that if the x coefficient is greater than or equal to 0 there are no positive solutions for x, and that there is one positive value of α making this coefficient 0. This α value turns out to be 2, and we see there are no periodic solutions of prime period 8 to equation (1) if $\alpha \leq 2$. If $\alpha > 2$ solutions to (3) are given by the quadratic formula, and these are real only if "$\mathrm{b}^2 - 4\mathrm{ac}$"$\geq 0$. This is the inequality

$$\alpha^{10} - 4\alpha^8 - 4\alpha^7 - 2\alpha^6 + 4\alpha^4 + 8\alpha^3 + \alpha^2 \geq 0. \tag{4}$$

Descartes' Rule of Signs says the equation (4) has two or zero positive roots; in fact, it has two, one less than 2 and one between 2.4 and 2.45. With the restriction $\alpha > 2$ we get

Note 1. There is a number s between 2.4 and 2.45 such that equation (1) has a prime period 8 solution with $x_{-1} = x_0$ for $\alpha > s$ and not for $\alpha \leq s$. We get two solutions for x in these cases, corresponding to the values $x_{-1} = x_0$ and $x_3 = x_4$ above. For one of these values the sequence will start out decreasing with $x_0 > x_1$, and for the other it will start out increasing.

A similar but more difficult analysis shows

Note 2. There is a number t between 4.0 and 4.1 such that equation (1) has a prime period 7 solution with $x_{-1} = x_0$ if and only if $\alpha > t$. Again we get two values of x, one with the sequence starting out increasing and one with the sequence starting out decreasing.

In this case, after dividing out by the factor $x - (\alpha - 1)$, we want to solve the equation

$$(5) \quad \begin{aligned} 0 = \; & x^4(\alpha^3 + 3\alpha^2 + 3\alpha + 1) + x^3(2\alpha^5 + 5\alpha^4 + 6\alpha^3 + 9\alpha^2 + 10\alpha + 4) \\ & + x^2(-\alpha^8 + \alpha^7 + 4\alpha^6 + 5\alpha^5 + 9\alpha^4 + 10\alpha^3 + 10\alpha^2 + 12\alpha + 6) \\ & + x(\alpha^7 + 4\alpha^6 + 4\alpha^5 + 5\alpha^4 + 6\alpha^3 + 5\alpha^2 + 6\alpha + 4) \\ & + (\alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1). \end{aligned}$$

Note 3. We can rewrite equations (3) and (5) thinking of α as the variable and having coefficients involving x. Descartes' Rule of Signs again says that for fixed postitive x there is exactly one solution for α, and this solution will be greater than 1. Thus for fixed $x > 0$ there will be one and only one value of $\alpha > 1$ so that May's Host Parasitoid Equation has a periodic solution of period 7 or 8 with $x_{-1} = x_0 = x$.

3. Long decreasing sequences

As in [2], the key to establishing periodicity results for longer periods lies in showing that there are solution sequences which start with arbitrarily many decreasing terms. Specifically,

Theorem 2. *With May's host parasitoid equation (1), for a fixed value of $\alpha > 1$ and for any natural number N, there is a number M such that for any value $x > M$, if $x_{-1} = x_0 = x$, then $x_0 > x_1 > \ldots > x_N$.*

Proof. Rewriting (1) we get

$$(6) \qquad \frac{x_{n+1}}{x_n} = \frac{\alpha}{(1 + x_n)} \frac{x_n}{x_{n-1}}.$$

Using (6) recursively we get

$$\frac{x_{n+1}}{x_n} = \frac{\alpha}{(1+x_n)} \frac{\alpha}{(1+x_{n-1})} \cdots \frac{\alpha}{(1+x_0)} \frac{x_0}{x_{-1}},$$

or

$$x_{n+1} = \frac{\alpha}{(1+x_n)} \frac{\alpha}{(1+x_{n-1})} \cdots \frac{\alpha}{(1+x_0)} \frac{x_0}{x_{-1}} x_n.$$

Now we are assuming that $x_{-1} = x_0 = x$. Also, since $x > 0$ it follows that $x_k > 0$ for all k, so $\frac{1}{1+x_k} < 1$. Thus $x_{n+1} < \frac{\alpha^{n+1}}{1+x} x_n$. If we choose $x > \alpha^{N+1} - 1 = M$, the theorem follows. ■

Even though we get arbitrarily long decreasing sequences, it follows from [1] that every such sequence eventually stops decreasing–that is, there will be some k with $x_{k+1} > x_k$.

4. Periodic solutions

We now prove a theorem which will have as a corollary the existence of more periodic solutions in May's host parasitoid equation.

Theorem 3. *Let the difference equation $x_{n+1}x_{n-1} = f(x_n)$ involve a parameter c. Suppose $f(x)$ is continuous in c for values of c in the interval I, and that no value of c in I gives an equilibrium solution to the difference equation. Suppose there are numbers a and b in I so that when $c = a$ we get $x_{-1} = x_0 > x_1 > x_2 > \ldots > x_{m-1}$, $x_{m-1} < x_m$, and that when $c = b$ we get $x_{-1} = x_0 > x_1 > x_2 > \ldots > x_{m+k}$. Then there are values of c giving periodic solutions of periods $2m, 2m+1, \ldots, 2m+2k-1$.*

Proof. Case 1: Even periods. Let $n = 2l$ be in the list of periods above, so $m \leq l < m+k$. Without loss of generality suppose $b > a$. Let $X = \{d|$ for all c in $[d, b]$, when c is used in $f(x)$, $x_{-1} = x_0 \geq x_1 \geq \ldots \geq x_{l-1} \geq x_l\}$. Then X is non-empty as b is in X. Also X is bounded below by a. Thus X has a greatest lower bound L, which will be in I. We claim that when $c = L$, $x_{-1} = x_0 > x_1 > \ldots > x_{l-1} = x_l$, so we get a periodic cycle of period $2l$ by theorem 1. Since the values of x_k are continuous in the parameter, by the definition of X we conclude that for $c = L$ we get

$$(*) \qquad x_{-1} = x_0 \geq x_1 \geq \ldots \geq x_{l-1} \geq x_l.$$

If we have $x_0 = x_1$ we have an equilibrium solution, but $L \in I$ and no value of c in I gives an equilibrium. Thus $x_0 > x_1$. If we have $x_k = x_{k+1}$ for $1 \leq k < l-1$, then $x_{k+2} = x_{k-1}$ because of the form of the difference equation, and hence $x_{k+2} > x_{k+1}$, contradicting $(*)$. Thus we have $x_0 > x_1 > \ldots > x_{l-1}$. If also $x_{l-1} > x_l$, by the continuity of the x_k's we can find an interval $[L-\epsilon, L]$ in X, contradicting the statement that L is the greatest lower bound of X. Thus $x_{l-1} = x_l$ and we get a periodic solution of period $2l$ by Theorem 1.

The case of odd periods is similarly proved using the set $Y = \{d|$ for all c in $[d, b]$, when c is used in $f(x)$, $x_{-1} = x_0 \geq x_1 \geq \ldots \geq x_{l-1} \geq x_l, x_{l+1} \geq x_{l-1}\}$. ■

Corollary 1. *Let $\alpha > 1$ be fixed. There is a positive number N so that for all natural numbers $n > N$, May's host parasitoid equation (1) has a single cycle solution of period n.*

Proof. Our parameter c in the theorem is the value of $x_{-1} = x_0$. Pick an $x = a > \alpha - 1$. By Ladas et al. ([1]) the sequence we get is not (eventually) monotone, and since $x_1 = \frac{\alpha x}{1+x} < x = x_0$, this sequence must look like $x_{-1} = x_0 > x_1 \geq x_2 \geq \ldots \geq x_k$, $x_k < x_{k+1}$ for some k. By theorem 2, for some larger value of x, say $x = b$, we have $x_{-1} = x_0 > x_1 > \ldots > x_k > \ldots > x_n$, where n can be chosen as large as we like. With $I = [a, b]$ the hypotheses of Theorem 3 are satisfied and we can thus get one cycle periodic solutions of all periods greater than $2k + 1$. ■

Corollary 2. *For all values of α greater than the number s in Note 1, there are periodic soluitions of prime period n for all values of $n \geq 8$, and for all values of α greater than the number t in Note 2, there are periodic solutions of prime period n for all values of $n \geq 7$.*

5. Two cycle periodic solutions

The continuity of the functions x_i can be further exploited to get two cycle periodic solutions:

Theorem 4. (A) *Consider α to be fixed. If there is a value of x $(x = E)$ with*

$$E = x_{-1} = x_0 > x_1 > \ldots > x_k = x_{k+1} \quad (\textit{one cycle, period 2k+2})$$

then there is a value of x giving a two cycle periodic solution of period $4k+5$.

(B) *With α considered fixed, if there is a value of x $(x = F)$ with*

$$F = x_{-1} = x_0 > \ldots > x_k > x_{k+1}, \;\; x_k = x_{k+2}, \;\; (\textit{one cycle, period } 2k+3)$$

then there is a value of x giving a two cycle periodic solution of period $4k+7$.

Proof. (A) By Theorem 3 there is also a solution $(x = E_1 > E)$ with

$$E_1 = x_{-1} = x_0 > x_1 > \ldots > x_k > x_{k+1}, \quad x_{k+2} = x_k,$$
$$\text{(one cycle, period } 2k+3).$$

Let

$$S = \{x \big| x \geq E, \text{ and for all } y \in [E,x], \text{ if } y = x_{-1} = x_0, \\ \text{then } x_0 > x_1 > \ldots > x_k \geq x_{k+1} \\ \text{and } x_{2k+1} \geq x_{2k} > x_{2k-1} > \ldots > x_{k+1} \text{ and } x_{2k+2} \leq x_{2k}\}.$$

Then $E \in S$ so S is non-empty, and $E_1 \notin S$ so S is bounded above. Again the least upper bound of S gives a solution with $x_{2k+2} = x_{2k}$, that is, a solution of period $4k+5$. The symmetry of the solution before and after $x_{2k} = x_{2k+2}$ shows that this solution consists of two cycles.

(B) is proved similarly. ■

Likewise we get

Theorem 5. *Consider α to be variable, $\alpha > 1$, and take $x_{-1} = x_0 = 1$ (as in [2]). Then for any odd integer n greater than or equal to 19 there is a value of α giving a two cycle period n solution.*

Proof. The proof is adapted from proofs in [2] in a manner similar to the proof of Theorem 4. The requirement $n \geq 19$ is added to avoid getting the equilibrium solution. ■

Adapting the proof of theorem 4 to get periodic solutions of period k with n cycles, where $k/n \geq 7$, seems a formidable task at best. It would seem a different, as yet untried, approach will be needed to prove this result.

6. Problems

1. (Ladas) For $\alpha > 1$ and $x_{-1}, x_0 > 0$, show every solution to May's host parasitoid equation is bounded.
2. Show that no two cycles in a given solution to equation (1) differ in length by more than one element. (This could be used to solve problem 3).
3. Show that for some values of α equation (1) has periodic solutions of period k with n cycles whenever $k/n \geq 7$.

Also see [2].

References

[1] LADAS G., TOVBIS A., TZANETOPOULOS G., On May's host parasitoid model, *Journal of Difference Equations and Applications,* 2(1996), 195-204.

[2] SIZER W.S., Periodicity in May's host parasitoid equation, *Advances in Discrete Dynamical Systems, S. Elaydi et al., eds, Tokyo: Mathematical Society of Japan,* (2009), 333-337.

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