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Mohamed Akkouchi

WELL-POSEDNESS OF THE FIXED POINT PROBLEM FOR ϕ-MAX-CONTRACTIONS

Abstract. We study the well-posedness of the fixed point problem for self-mappings of a metric space which are ϕ-max-contractions (see [6]).

Key words: well-posedness, fixed point problem, fixed points, ϕ-max-contractions, metric spaces, orbitally complete spaces.

AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction

In 1974, Ćirić ([3]) has first introduced orbitally continuous mappings and orbitally complete metric spaces.

Definition 1. *Let T be a self-mapping on a metric space (X, d). If for any $x \in X$, every Cauchy sequence of the orbit $O_T(x) := \{x, Tx, T^2x, \ldots\}$ is convergent in X, then the metric space is said to be $T-$orbitally complete.*

Remark 1. Every complete metric space is $T-$orbitally complete for any T. An orbitally complete space may not be complete metric space (see [8], Example and [14], Example 1).

In [6], to generalize some results of Boyd and Wong [2], Ćirić [4], Massa [9], Sehgal [15] and Daneš [7], J. Daneš [6] introduced the notion of ϕ-max-contractions.

Let (X, d) be a metric space, $T : X \to X$ a self-mapping of X. For any arbitrary point x in X, the orbit of x under T is defined as the set $O_T(x) := \{x, Tx, T^2x, \ldots\}$.

Let $\mathcal{D}$ be the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

$(D1)$: ϕ is right continuous on $[0, \infty)$.

$(D2)$: ϕ is non-decreasing on $[0, \infty)$.

$(D3)$: $\phi(t) < t$ for all $t \in (0, \infty)$.

We recall the following definition from [6].

Definition 2. ([6]) *Let (X,d) be a metric space and $T : X \to X$ a mapping. For $x, y \in X$, we denote*

$$M(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \tag{1}$$

Let $\phi \in \mathcal{D}$. The mapping T is called a ϕ-max-contraction if the following inequality

$$d(Tx,Ty) \leq \phi(M(x,y)) \tag{2}$$

holds true for all $x, y \in X$.

Using this concept J. Daneš has proved some fixed point theorems in [6].

The aim of this paper is to study the well-posedness (see Definition 3 below) of the fixed point problem for the ϕ-max-contractions of orbitally complete metric spaces. More precisely we provide natural conditions on the functions ϕ which ensure the well-posedness of the fixed point problem for the associated ϕ-max-contractions.

The notion of well-posednes of a fixed point problem has evoked much interest to a several mathematicians, for examples, F.S. De Blasi and J. Myjak (see [1]), S. Reich and A. J. Zaslavski (see [12]), B.K. Lahiri and P. Das (see [8]) and V. Popa (see [10] and [11]).

Definition 3. *Let (X,d) be a metric space and $T : (X,d) \to (X,d)$ a mapping. The fixed point problem of T is said to be well posed if:*

(a) T has a unique fixed point z in X;

(b) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, z) = 0$.

2. Main result

For any arbitrary function $\phi : [0,\infty) \to [0,\infty)$ and for each real number $t \in [0,\infty)$, we set

$$J_\phi(t) := \{s \in [0,\infty) : s - \phi(s) \leq t\}. \tag{3}$$

In fact, for each non-negative number t, we have $J_\phi(t) = (Id - \phi)^{-1}([0,t])$.

We introduce the following definition.

Definition 4. *We denote $\mathcal{A}$ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*

$(A1)$: ϕ is right upper semi-continuous on $[0,\infty)$.

$(A2)$: ϕ is non-decreasing on $[0,\infty)$.

$(A3)$: $\phi(t) < t$ for all $t \in (0,\infty)$.

$(A4)$: (a) For each $t \in [0,\infty)$, the set $J_\phi(t)$ is bounded, and we have

(b) $\lim_{t\to 0} \sup\{s : s \in J_\phi(t)\} = 0$.

To simplify notations, if $\phi \in \mathcal{A}$, we set

$$\psi(t) := \sup\{s : s \in J_\phi(t)\},$$

for every $t \geq 0$.

We give examples of elements of the class $\mathcal{A}$.

Examples.
(1) $\phi(t) = qt$, for all $t \in [0, \infty)$, where $0 \leq q < 1$.
(2) $\phi(t) = \frac{t}{1+t}$, for all $t \in [0, \infty)$.

We recall the following elementary and classical result.

Lemma 1. *Let $f : [0, \infty) \to [0, \infty)$ be a function satisfying the conditions $(A1)$, $(A2)$ and $(A3)$, then f satisfies*

$$\lim_{n\to\infty} f^n(t) = 0 \quad \forall t \geq 0,$$

where $f^n : f \circ f \ldots \circ f$ n-times. (By definition $f^0 = Id$).

Before giving the main result, we need to recall the following lemma of [6].

Let (X, d) be a metric space and $T : X \to X$ a mapping. For x in X and n and integer, let

$$O_T(x, n) := \{x, Tx, \ldots, T^n x\}.$$

Then we have

$$O_T(x) = \cup_{n\geq 1} O_T(x, n).$$

Lemma 2 ([6])**.** *Let (X, d) be a metric space. Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying the condition $(D2)$. Suppose that $T : X \to X$ is a ϕ-max-contraction. i.e., T satisfies the inequality*

$$d(Tx, Ty) \leq \phi(\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\})$$

for all $x, y \in X$.

Let x be an arbitrary $x \in X$. Then

(i) for any non-negative integers n and m, we have

$$\operatorname{diam}(O_T(T^m x, n)) \leq \phi^m(\operatorname{diam}(O_T(x, n + m))). \tag{4}$$

(ii) For any non-negative integer m, we have

$$\operatorname{diam}(O_T(T^m x)) \leq \phi^m(\operatorname{diam}(O_T(x))), \tag{5}$$

provided that $\operatorname{diam}(O_T(x))$ *is finite.*

The main result of this paper reads as follows.

Theorem 1. *Let (X,d) be a metric space and $T : X \to X$ be a self-mapping satisfying the inequality*

$$d(Tx,Ty) \leq \phi\left(\max\{d(x,y), d(Tx,x), d(Ty,y), d(Tx,y), d(Ty,x)\}\right) \tag{6}$$

for all $x, y \in X$, where ϕ is a given element in $\mathcal{A}$.

Suppose that (X,d) is $T-$orbitally complete. Then, we have:

(i) T has a unique fixed point z in X.

(ii) The fixed point problem of T is well-posed.

(iii) T is continuous at its unique fixed point.

Proof. (i) Let x_0 be an arbitrary point in X. We consider the Picard sequence associated to x_0. That is the sequence $\{x_n\}$ defined by $x_{n+1} := T^n x_0 = T(x_0)$, for every non negative integer n.

We start by showing that the Picard sequence $\{x_n\}$ is a Cauchy sequence. For each non negative integer n, we consider the set $O_T(x_0, n) := \{T^j x_0 : 0 \leq j \leq n\}$. We observe that

$$\operatorname{diam}(O_T(x_0)) = \sup\{\operatorname{diam}(O_T(x_0,n)) : n \geq 0\},$$

where for any subset A of X, we denote $\operatorname{diam}(A)$ to mean the diameter of A.

By (i) of Lemma 2, we know that

$$\operatorname{diam}(O_T(T^m x_0, n)) \leq \phi^m\left(\operatorname{diam}(O_T(x_0, n+m))\right) \tag{7}$$

holds true for any positive integers n and m.

Let $n \geq 1$. For all integers i, j such that $1 \leq i, j \leq n$, by (6), we have

$$\begin{aligned} d(T^i x, T^j x) &= d(T(T^{i-1}x), T(T^{j-1}x)) \\ &\leq \phi\big(\max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), \\ &\qquad d(T^{j-1}x, T^j x), d(T^i x, T^{j-1}x), d(T^j x, T^{i-1}x)\}\big) \\ &\leq \phi\left(\operatorname{diam}(O_T(x_0,n))\right). \end{aligned} \tag{8}$$

From (8), we deduce that there exists k such that $1 \leq k \leq n$ and

$$\operatorname{diam}(O_T(x_0,n)) = d(x_0, T^k x_0).$$

Then

$$\begin{aligned} \operatorname{diam}(O_T(x_0,n)) &= d(x_0, T^k x_0) \leq d(x_0, Tx_0) + d(Tx_0, T^k x_0) \\ &\leq d(x_0, Tx_0) + \operatorname{diam}(O_T(Tx_0, n-1)). \end{aligned}$$

Taking into account the inequality (i) of Lemma 2, we obtain that

$$\operatorname{diam}(O_T(x_0,n)) \leq d(x_0, Tx_0) + \phi\left(\operatorname{diam}(O_T(x_0,n))\right).$$

By virtue of the properties (a) and (b) of the assumption (A4), the previous inequality implies that

$$\operatorname{diam}\left(O_T(x_0,n)\right) \leq \psi\left(d(x_0,Tx_0)\right), \quad \forall n \geq 1. \tag{9}$$

From (9), we deduce that $\operatorname{diam}\left(O_T(x_0)\right)$ is finite and that

$$\operatorname{diam}\left(O_T(x_0)\right) \leq \psi\left(d(x_0,Tx_0)\right).$$

By using (ii) of Lemma 2, we obtain that

$$\operatorname{diam}\left(O_T(T^m x_0)\right) \leq \phi^m\left(\psi\left(d(x_0,Tx_0)\right)\right) \tag{10}$$

holds true for all positive integer m. In particular, (10) implies

$$d(T^p x_0, T^p x_0) \leq \phi^m\left(\psi\left(d(x_0,Tx_0)\right)\right), \quad \text{for all integers } \; p,q \geq m. \tag{11}$$

By Lemma 1, we have

$$\lim_{m\to\infty} \phi^m(s) = 0, \quad \forall s \in [0,\infty).$$

We conclude from (11), that the Picard sequence $\{T^n x_0\}$ is a Cauchy sequence. Since (X,d) is a $T-$orbitally complete metric space, there is some z in X such that

$$\lim_{n\to\infty} x_n = z. \tag{12}$$

Now we show that z is a fixed point of T. By using (6), we have

$$\begin{aligned} d(Tz,x_{n+1}) &= d(Tz,Tx_n) \\ &\leq \phi\left(\max\{d(z,x_n), d(Tz,z), d(x_{n+1},x_n),\right. \\ &\qquad \left. d(Tz,x_n), d(x_{n+1},z)\}\right). \end{aligned} \tag{13}$$

By making $n \to \infty$ and using right upper-semi-continuity of the function ϕ, we obtain from (13) that

$$\begin{aligned} d(Tz,z) &\leq \phi\left(\max\{0, d(Tz,z), 0, d(Tz,z), 0\}\right) \\ &= \phi\left(d(Tz,z)\right), \end{aligned}$$

from which, with the help of the assumption (A3), we deduce that $d(Tz,z) = 0$, or equivalently, that z is a fixed point of T.

To complete the proof of the assertion (i), we need to prove the uniqueness of z. Let us suppose that u and v are two different fixed points of T. From (6), we have

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq \phi\left(\max\{d(u,v), d(Tu,u), d(Tv,v), d(Tu,v), d(Tv,u)\}\right) \\ &= \phi(d(u,v)), \end{aligned}$$

from which, with the help of the assumption $(A3)$, we deduce that that $d(u,v) = 0$, or equivalently, that $u = v$, which is a contradiction. We conclude that z is the unique fixed point of T. Thus we have proved the assertion (i).

(ii) We show the well-posedness. Let $\{y_n\}$ be any arbitrary sequence of points in X such that

$$\lim_{n\to\infty} d(Ty_n, y_n) = 0. \tag{14}$$

We have to prove that the sequence $\{y_n\}$ converges to the unique fixed point z of T.

By using (6), for every nonnegative integer n, we have

$$\begin{aligned} d(y_n,z) &\le d(y_n,Ty_n) + d(Ty_n,Tz) \\ &\le d(y_n,Ty_n) + \phi\left(\max\{d(y_n,z), d(Ty_n,y_n),\right. \\ &\qquad\qquad \left. d(Tz,z), d(Ty_n,z), d(Tz,y_n)\}\right) \\ &\le d(y_n,Ty_n) + \phi\left(\max\{d(y_n,z), d(Ty_n,y_n), 0,\right. \\ &\qquad\qquad \left. d(Ty_n,y_n) + d(y_n,z), d(z,y_n)\}\right) \\ &= d(y_n,Ty_n) + \phi\left((d(Ty_n,y_n) + d(y_n,z))\right). \end{aligned} \tag{15}$$

From (15), we get

$$d(y_n,z) + d(y_n,Ty_n) \le 2d(y_n,Ty_n) + \phi\left((d(Ty_n,y_n) + d(y_n,z))\right). \tag{16}$$

By using the conditions (a) and (b) of the assumption (A4), we deduce that

$$d(y_n,z) + d(y_n,Ty_n) \le \psi\left(2(d(Ty_n,y_n))\right),$$

which implies that $\lim_{n\to\infty} d(y_n,z) = 0$. This proves that the fixed point problem of T is well-posed. Thus we have established the assertion (ii).

(iii) It remains to show that T is continuous at z. To this end, let $\{w_n\}$ be any arbitrary sequence in X such that $w_n \to z = Tz$ (i.e., $\{w_n\}$ converges to z). Then from (6), we have

$$\begin{aligned} d(Tw_n,z) &= d(Tw_n,Tz) \\ &\le \phi\left(\max\{d(w_n,z), d(Tw_n,w_n), d(Tz,z),\right. \\ &\qquad\qquad \left. d(Tw_n,z), d(Tz,w_n)\}\right) \\ &\le \phi\left(\max\{d(w_n,z), d(Tw_n,z) + d(z,w_n), 0,\right. \\ &\qquad\qquad \left. (Tw_n,z), d(z,w_n)\}\right) \\ &= \phi\left(d(Tw_n,z) + d(z,w_n)\right) \end{aligned} \tag{17}$$

From (17), we obtain that

$$d(Tw_n,z) + d(z,w_n) \le d(z,w_n) + \phi\left(d(Tw_n,z) + d(z,w_n)\right). \tag{18}$$

From (18) and the assumption (b) of (A4), we deduce that

$$d(Tw_n, z) + d(z, w_n) \leq \psi(d(z, w_n)) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{19}$$

From (19) we deduce that the sequence $\{T(w_n)\}$ converges to $z = Tz$. Hence, T is continuous at its unique fixed point z. Thus the assertion (iii) is proved and this ends the proof. ■

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Mohamed Akkouchi
Université Cadi Ayyad
Faculté des Sciences-Semlalia
Département de Mathématiques
Av. Prince My Abdellah, BP. 2390
Marrakech, Maroc (Morocco)
e-mail: akkouchimo@yahoo.fr

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