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A KOROVKIN TYPE APPROXIMATION THEOREM AND ITS APPLICATION

Abstract. In this paper, we investigate a Korovkin-type approximation theorem for sequences of positive linear operators on the space of all continuous real valued functions defined on $[a, b]$. We also obtain some approximation properties for sequences of positive linear operators constructed by means of the Bernstein operator.

Key words: positive linear operator, Korovkin-type theorem, Bernstein operator.

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1. Introduction

Let $\{L_n\}$ be a sequence of positive linear operators acting from $C[a, b]$ into $C[a, b]$, which is the space of all continuous real valued functions on $[a, b]$. In this case, Korovkin [8] first noticed necessary and sufficient conditions for the uniform convergence of $L_n f$ to a function f by using the test functions $e_i(t)$ defined by $e_i(t) = t^i$ $(i = 0, 1, 2)$. Later many researchers investigate these conditions for various operators defined on different spaces. In the present paper, we obtain following the Korovkin-type theorem and give results for the approximation properties of the generalized Bernstein operators $B_{n,\varphi}$ as an application. Note that throughout the paper, we always assume that $\varphi : [a, b] \to [a, b]$ be a bijection and a continuous function on $[a, b]$.

Theorem 1 (Korovkin-type theorem)**.** *Let $\{L_n\}$ be a sequence of positive linear operators acting from $C[a, b]$ into $C[a, b]$. Then, for all $f, \varphi \in C[a, b]$*

$$\lim_{n\to\infty} \|L_n(f) - f \circ \varphi\|_{C[a,b]} = 0 \tag{1}$$

if and only if

$$\lim_{n\to\infty} \|L_n(e_i) - e_i \circ \varphi\|_{C[a,b]} = 0, \quad (i = 0, 1, 2). \tag{2}$$

Proof. Since each $e_i \circ \varphi \in C[a,b]$, $(i = 0, 1, 2)$ the implication (1) $\Longrightarrow$ (2) is obvious. Suppose now that (2) holds. Since $f \circ \varphi$ is bounded on $[a,b]$, we can write

$$|f(\varphi(x))| \leq M.$$

Also, since f is continuous on $[a,b]$, for all $x \in [a,b]$, we write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|t - \varphi(x)| \leq \delta$, $|f(t) - f(\varphi(x))| < \varepsilon$. Hence, we get

$$|f(t) - f(\varphi(x))| < \varepsilon + 2M\frac{(t-\varphi(x))^2}{\delta^2}, \quad \forall t, x \in [a,b].$$

Since L_n is linear and positive, we obtain

$$\begin{aligned} |L_n(f;x) - f(\varphi(x))| &= |L_n(f(\cdot) - f(\varphi(x));x) \\ &\quad + f(\varphi(x))(L_n(e_0(\cdot);x) - e_0(\varphi(x)))| \\ &\leq L_n(|f(\cdot) - f(\varphi(x))|;x) \\ &\quad + M\,|L_n(e_0(\cdot);x) - e_0(\varphi(x))| \\ &\leq L_n(\varepsilon + 2M\frac{(\cdot - \varphi(x))^2}{\delta^2};x) \\ &\quad + M\,|L_n(e_0(\cdot);x) - e_0(\varphi(x))| \\ &\leq (\varepsilon + M + 2M\frac{k^2}{\delta^2})\,|L_n(e_0(t);x) - e_0(\varphi(x))| \\ &\quad + \frac{4M}{\delta^2}\,|L_n(e_1(t);x) - e_1(\varphi(x))| \\ &\quad + \frac{2M}{\delta^2}\,|L_n(e_2(t);x) - e_2(\varphi(x))| + \varepsilon \end{aligned}$$

where $k = \max_{a \leq x \leq b} \varphi(x)$. Then we get

$$\begin{aligned} \|L_n(f) - f \circ \varphi\|_{C[a,b]} &\leq \varepsilon + M' \Big\{ \|L_n(e_0(t);x) - e_0(\varphi(x))\|_{C[a,b]} \\ &\quad + \|L_n(e_1(t);x) - e_1(\varphi(x))\|_{C[a,b]} \\ &\quad + \|L_n(e_2(t);x) - e_2(\varphi(x))\|_{C[a,b]} \Big\} \end{aligned}$$

from (2), we obtain

$$\lim_{n\to\infty} \|L_n(f) - f \circ \varphi\|_{C[a,b]} = 0.$$

The proof is complete. ∎

Remark 1. If we take $\varphi(x) = x$ in Theorem 1, then we immediately get the classical result which was introduced by Korovkin [8].

Remark 2. The multivariate analogue of Theorem 1 is proved different method by Guessab and Schmeisser (See [5]).

2. Application of Korovkin-type theorem

Since several concrete operators on $C[a, b]$ are positive and linear, Korovkin's theorem plays fundamental role in his theory of approximation for example, the Bernstein operator

$$B_n(f; x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq x \leq 1 \tag{3}$$

is linear and positive on $[0, 1]$ for every $n > 0$. Here, inspired by the Bernstein operators, we introduce the following sequence of generalized Bernstein operator:

$$B_{n,\varphi}(f; x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} \varphi^k(x)(1-\varphi(x))^{n-k}, \quad 0 \leq \varphi(x) \leq 1 \tag{4}$$

where φ is a bijection and a continuous function on $[0, 1]$. We get the classical Bernstein operator, given in (3), by putting $\varphi(x) = x$ in (4).

Definition 1. *Let* $W^2 = \{g \in C[0, 1] : \ g', g'' \in C[0, 1]\}$. *For* $f \in C[0, 1]$ *and* $\delta > 0$, *the Peetre-K Functional is defined by*

$$K_f^2(\delta) = \inf_{g \in W^2} \left\{ \|f - g\|_{C[0,1]} + \delta \left\|g''\right\|_{C[0,1]} \right\} \tag{5}$$

where $\|g\|_{C[0,1]} = \sup_{0 \leq x \leq 1} |g(x)|$.

Definition 2. *For* $f \in C[0, 1]$ *and* δ *is a positive number. The continuity modulus of a function* f *is*

$$\omega_f(\delta) = \sup_{0<h\leq\delta} \ \sup_{x,x+h \in [0,1]} \{|f(x+h) - f(x)|\} \tag{6}$$

and the second order modulus of smoothness of a function f *is*

$$\omega_f^2(\delta) = \sup_{0<h\leq\delta} \ \sup_{x,x+2h \in [0,1]} \{|f(x+2h) - 2f(x+h) + f(x)|\} .$$

From [3, p. 177, Theorem 2.4], we have

$$K_f^2(\delta) \leq C\omega_f^2(\sqrt{\delta}) \tag{7}$$

where C *is an absolute positive constant.*

Theorem 2. *Let $\varphi \in C\,[0,1]$ be bijection function. For any $f \in C\,[0,1]$, the polynomials $B_{n,\varphi}f$ converge uniformly to $f \circ \varphi$ on $[0,1]$ as $n \to \infty$.*

Proof. Since $\sum\limits_{k=0}^{n} \binom{n}{k} \varphi^k(x)(1-\varphi(x))^{n-k} = 1$, from definition (4) of $B_{n,\varphi}(f;x)$, we have

$$B_{n,\varphi}(e_0(\cdot);x) = 1. \tag{8}$$

Now, consider the case where f is the identity function $f = e_1 : t \to t$; thus,

$$\begin{aligned} B_{n,\varphi}(e_1(.);x) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k}\varphi^k(x)(1-\varphi(x))^{n-k} \\ &= \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}\varphi^k(x)(1-\varphi(x))^{n-k} \\ &= \varphi(x)\sum_{k=0}^{n-1}\binom{n-1}{k}\varphi^k(x)(1-\varphi(x))^{n-k-1} \end{aligned} \tag{9}$$

so we obtain

$$B_{n,\varphi}(e_1(\cdot);x) = \varphi(x). \tag{10}$$

Next, take $f = e_2 : t \to t^2$ and from (9) find

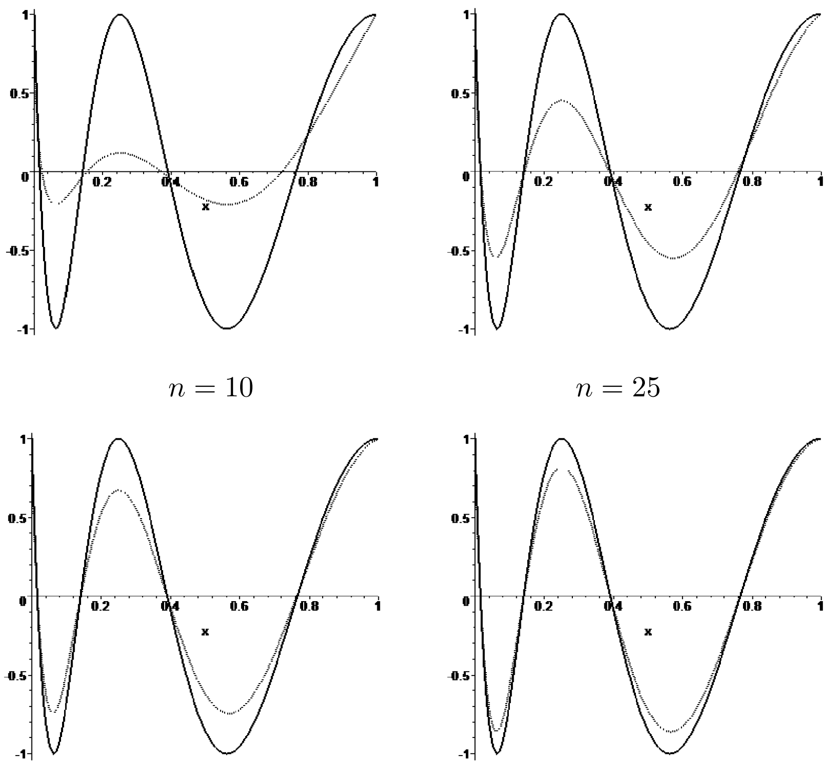
$$\begin{aligned} B_{n,\varphi}(e_2(\cdot);x) &= \varphi(x)\sum_{k=0}^{n-1}\frac{k+1}{n}\binom{n-1}{k}\varphi^k(x)(1-\varphi(x))^{n-k-1} \\ &= \frac{\varphi(x)}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}\varphi^k(x)(1-\varphi(x))^{n-k-1} \\ &\quad + \varphi(x)\sum_{k=0}^{n-1}\frac{k}{n}\binom{n-1}{k}\varphi^k(x)(1-\varphi(x))^{n-k-1} \\ &= \frac{\varphi(x)}{n} + (1-\frac{1}{n})\varphi(x)\sum_{k=1}^{n-1}\frac{(n-2)!}{(k-1)!(n-1-k)!} \\ &\qquad\qquad \times \varphi^k(x)(1-\varphi(x))^{n-k-1} \\ &= \frac{\varphi(x)}{n} + (1-\frac{1}{n})\varphi^2(x)\sum_{k=0}^{n-2}\binom{n-2}{k}\varphi^k(x)(1-\varphi(x))^{n-k-2} \\ &= \frac{\varphi(x)}{n} + (1-\frac{1}{n})\varphi^2(x) \end{aligned}$$

so we obtain

$$B_{n,\varphi}(e_2(\cdot);x) = \varphi^2(x) + \frac{\varphi(x) - \varphi^2(x)}{n}. \tag{11}$$

The proof of uniform convergence is then completed by applying the Korovkin-type Theorem 1. ■

Example 1. For $n = 10, 25, 50, 100$ and $\varphi(x) = \sqrt{x}$, the convergence of $B_{n,\varphi}(f;x)$(dot line) to $f(\varphi(x)) = \cos(4\pi\sqrt{x})$(solid line) will be illustrated in the following Figure 1.



$n = 10$ $n = 25$

$n = 50$ $n = 100$

Figure 1. *Convergence of* $B_{n,\varphi}(f;x)$ *to* $f(\varphi(x))$.

Example 2. For $n = 10, 25, 50, 100$ and $\varphi(x) = x^5$, the convergence of $B_n(f \circ \varphi; x)$(dash dot line) and $B_{n,\varphi}(f;x)$(dot line) to $f(\varphi(x)) = \sin(2\pi x^5)$(solid line) will be illustrated in the following Figure 2.

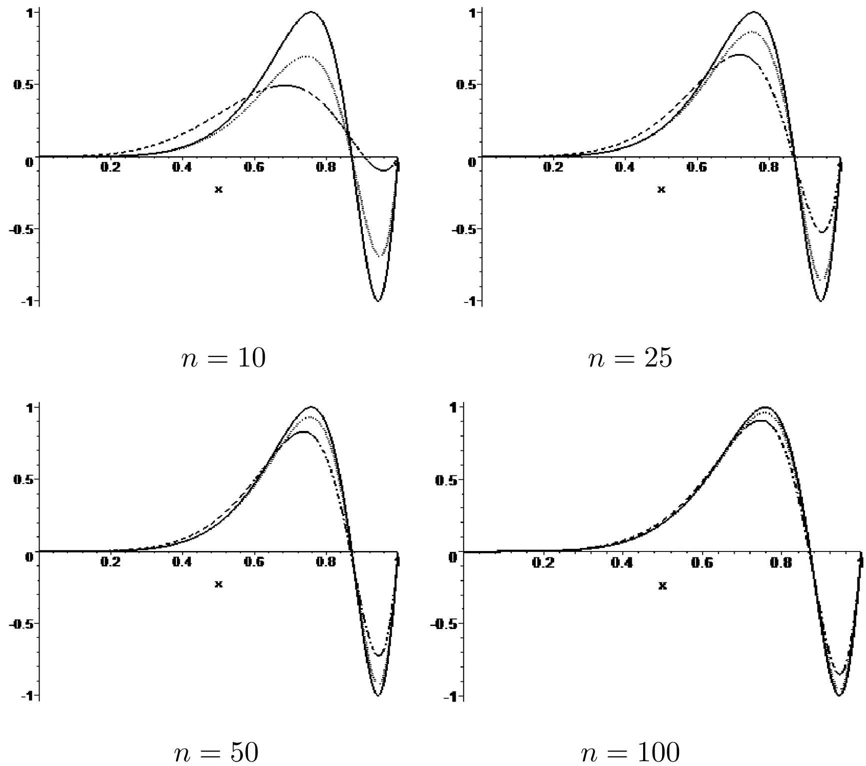


Figure 2. *Comparation of Bernstein and generalized Bernstein operators for* $n = 10, 25, 50, 100$.

From figure 2, we can see that the generalized Bernstein operator approximation is better than the classical Bernstein operator.

Theorem 3. *Let* $\varphi \in C\,[0,1]$ *be bijection function. For any* $f \in C[0,1]$,

$$|B_{n,\varphi}(f;x) - f(\varphi(x))| \leq \left(\sqrt{C_\varphi} + 1\right)\omega_f(\frac{1}{\sqrt{n}})$$

where $C_\varphi = \max_{0\leq x\leq 1}\left\{\varphi(x) - \varphi^2(x)\right\}$ *and* $\omega_f(\cdot)$ *is the modulus of continuity as given in (6).*

Proof. Using the relation $\sum\limits_{k=0}^{n} \binom{n}{k}\varphi^k(x)(1-\varphi(x))^{n-k} = 1$ we can express the difference between $B_{n,\varphi}(f;x)$ and $f(\varphi(x))$ as follow:

$$B_{n,\varphi}(f;x) - f(\varphi(x)) = \sum_{k=0}^{n}\left\{f(\frac{k}{n}) - f(\varphi(x))\right\}\binom{n}{k}\varphi^k(x)(1-\varphi(x))^{n-k}$$

and so

$$|B_{n,\varphi}(f;x)-f(\varphi(x))| \leq \sum_{k=0}^{n} \left| f(\frac{k}{n})-f(\varphi(x)) \right| \binom{n}{k} \varphi^k(x)(1-\varphi(x))^{n-k}.$$

Letting $y=\frac{k}{n}$ and $|y-\varphi(x)|=\mu\delta$, we have $|f(y)-f(\varphi(x))| \leq \omega_f(\mu\delta) \leq (1+\mu)\omega_f(\delta)$. Thus,

$$\left| f(\frac{k}{n})-f(\varphi(x)) \right| \leq \left(1+\frac{\left|\frac{k}{n}-\varphi(x)\right|}{\delta} \right) \omega_f(\delta)$$

and hence

$$\begin{aligned} |B_{n,\varphi}(f;x)-f(\varphi(x))| \ &\leq \ \omega_f(\delta) \sum_{k=0}^{n} \left(1+\frac{\left|\frac{k}{n}-\varphi(x)\right|}{\delta} \right) \\ &\qquad \times \binom{n}{k} \varphi^k(x)(1-\varphi(x))^{n-k} \\ &\leq \ \omega_f(\delta) \left\{ 1+\frac{1}{\delta} \left[\sum_{k=0}^{n} \left(\frac{k}{n}-\varphi(x)\right)^2 \binom{n}{k} \varphi^k(x)(1-\varphi(x))^{n-k} \right]^{1/2} \right\} \end{aligned}$$

where we have invoked the Cauchy-Schwartz inequality. Expanding the squared term and making use of (8), (10) and (11), we obtain

$$\begin{aligned} |B_{n,\varphi}(f;x)-f(\varphi(x))| \ &\leq \ \omega_f(\delta) \left\{ 1+\frac{1}{\delta} \left[\frac{\varphi(x)-\varphi^2(x)}{n} \right]^{1/2} \right\} \\ &\leq \ \omega_f(\delta) \left\{ 1+\frac{1}{\delta} \left(\frac{\max_{0\leq x\leq 1} \{\varphi(x)-\varphi^2(x)\}}{n} \right)^{1/2} \right\}. \end{aligned} \tag{12}$$

Therefore, by choosing $\delta=\frac{1}{\sqrt{n}}$ in (12), we get the desired result. ■

Remark 3. It is observed that from above theorem that if $\varphi \in C^1[0,1]$ and $\varphi'(x) \neq 0$ for $x \in (0,1)$, we get $C_\varphi=\frac{1}{4}$. For example, If we take $\varphi(x)=x$, we obtain the rate of convergence for the classical Bernstein polynomials.

Theorem 4. *Let $f \in C[0,1]$, then we have*

$$|B_{n,\varphi}(f;x)-f(\varphi(x))| \leq 2C\omega_f^2 \left(\frac{1}{2}\sqrt{\frac{\varphi(x)-\varphi^2(x)}{n}} \right)$$

where C is an absolute positive constant.

Proof. Let $g \in W^2$. From Taylor's expansion we write

$$g(t) = g(\varphi(x)) + g'(\varphi(x))(t - \varphi(x)) + \int\limits_{\varphi(x)}^{t} (t-u)g''(u)du$$

we get

$$\begin{aligned} B_{n,\varphi}(g;x) &= g(\varphi(x)) + g'(\varphi(x))B_{n,\varphi}(\cdot - \varphi(x);x) \\ &\quad + B_{n,\varphi}\left(\int\limits_{0}^{t-\varphi(x)} (\cdot - \varphi(x) - u)g''(u)du;x \right) \end{aligned}$$

and from (10) we obtain

$$|B_{n,\varphi}(g;x) - g(\varphi(x))| \leq \frac{1}{2} \left\|g''\right\|_{C[0,1]} B_{n,\varphi}((. - \varphi(x))^2;x). \tag{13}$$

On the other hand, from (8), (10) and (11) we have

$$B_{n,\varphi}((\cdot - \varphi(x))^2;x) = \frac{\varphi(x) - \varphi^2(x)}{n}.$$

Then, by (13), we get

$$|B_{n,\varphi}(g;x) - g(\varphi(x))| \leq \frac{1}{2}\frac{\varphi(x) - \varphi^2(x)}{n} \left\|g''\right\|_{C[0,1]}.$$

On the other hand, from definition (4) and (8) we have

$$|B_{n,\varphi}(f;x)| \leq \|f\|_{C[0,1]} B_{n,\varphi}(1;x) = \|f\|_{C[0,1]}. \tag{14}$$

Now (13) and (14) imply

$$\begin{aligned} |B_{n,\varphi}(f;x) - f(\varphi(x))| &\leq |B_{n,\varphi}(f-g;x) - (f-g)(\varphi(x))| \\ &\quad + |B_{n,\varphi}(g;x) - g(\varphi(x))| \\ &\leq 2\left[\|f-g\|_{C[0,1]} + \frac{1}{4}\frac{\varphi(x) - \varphi^2(x)}{n} \left\|g''\right\|_{C[0,1]} \right]. \end{aligned}$$

Hence taking infimum on the right hand side over all $g \in W^2$ we get

$$|B_{n,\varphi}(f;x) - f(\varphi(x))| \leq 2K_f^2(\frac{1}{4}\frac{\varphi(x) - \varphi^2(x)}{n})$$

and by (7), we get

$$|B_{n,\varphi}(f;x) - f(\varphi(x))| \leq 2C\omega_f^2 \left(\frac{1}{2}\sqrt{\frac{\varphi(x) - \varphi^2(x)}{n}} \right).$$

This completes the proof of the theorem. ∎

The following inverse theorem can be proved for the generalized Bernstein operator (4). As in the previous theorems, $\varphi(x)$ is a bijection function on $[0,1]$.

Lemma 1 (from [2] p.696)**.** *With $h,\delta\in(0,1]$, if $\omega_f(h)\le K_1\{\delta^\gamma+(h/\delta)\times\omega_f(\delta)\}$ for some $K_1>0$ and $0<\gamma<1$, then there exists a constant $K_2>0$ such that $\omega_f(h)\le K_2h^\gamma$.*

Theorem 5. *For $f,\varphi'\in C[0,1]$ and $\omega_f(\delta)\le\omega_{f\circ\varphi}(\delta)$,*

$$|B_{n,\varphi}(f;x)-f(\varphi(x))|=O\left(n^{-\gamma}\right),\quad 0<\gamma<1 \tag{15}$$

implies the composite function $f\circ\varphi\in\mathrm{Lip}(\gamma,C[0,1])$.

Proof. For $0<x<1$, taking the derivative of (4) with respect to x:

$$\begin{aligned}B'_{n,\varphi}(f;x)&=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}\frac{d}{dx}\left[\varphi^k(x)(1-\varphi(x))^{n-k}\right]\\&=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}\left[k\varphi^{k-1}(x)\varphi'(x)-(n-k)\varphi'(x)(1-\varphi(x))^{n-k-1}\right]\\&=\varphi'(x)\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}\left[k\varphi^{k-1}(x)-(n-k)(1-\varphi(x))^{n-k-1}\right].\end{aligned}$$

Using properties of binomial coefficients, we write

$$\begin{aligned}B'_{n,\varphi}(f;x)&=n\varphi'(x)\left\{\sum_{k=1}^{n}f(\frac{k}{n})\binom{n-1}{k-1}\varphi^{k-1}(x)(1-\varphi(x))^{n-k}\right.\\&\qquad\left.-\sum_{k=0}^{n-1}f(\frac{k}{n})\binom{n-1}{k}\varphi^{k}(x)(1-\varphi(x))^{n-k-1}\right\}\\&=n\varphi'(x)\sum_{k=0}^{n-1}\left\{f(\frac{k+1}{n})-f(\frac{k}{n})\right\}\binom{n-1}{k}\varphi^{k}(x)(1-\varphi(x))^{n-k-1}.\end{aligned}$$

Upon taking absolute values of both sides and using the modulus of continuity, we obtain

$$\begin{aligned}\left|B'_{n,\varphi}(f;x)\right|&\le n\left|\varphi'(x)\right|\sum_{k=0}^{n-1}\left|f(\frac{k+1}{n})-f(\frac{k}{n})\right|\binom{n-1}{k}\\&\qquad\times\varphi^{k}(x)(1-\varphi(x))^{n-k-1}\\&\le n\left|\varphi'(x)\right|\sum_{k=0}^{n-1}\omega_f\left(\frac{1}{n}\right)\binom{n-1}{k}\varphi^{k}(x)(1-\varphi(x))^{n-k-1}\end{aligned}$$

$$
\begin{aligned}
&\leq n\left|\varphi'(x)\right|\sum_{k=0}^{n-1}\left\{1+\frac{1}{\delta}\frac{1}{n}\right\}\omega_f(\delta)\binom{n-1}{k}\varphi^k(x)(1-\varphi(x))^{n-k-1}\\
&\leq \left|\varphi'(x)\right|\omega_f(\delta)\left\{n+\frac{1}{\delta}\right\}.
\end{aligned}
$$

Since $\varphi' \in C[0,1]$, we have $\lambda = \max_{0\leq x\leq 1}|\varphi'(x)|$. For any fixed pair of points x, y in $[0,1]$, one obtains

$$
\begin{aligned}
\left|\int_y^x \left|B'_{n,\varphi}(f;u)\right|\,du\right| &\leq \omega_f(\delta)\left\{n+\frac{1}{\delta}\right\}\left|\int_y^x \left|\varphi'(u)\right|du\right| \\
&\leq \lambda\omega_f(\delta)\left\{n+\frac{1}{\delta}\right\}\left|\int_y^x du\right| \\
&= \lambda\left\{n+\frac{1}{\delta}\right\}\omega_f(\delta)|x-y|.
\end{aligned} \tag{16}
$$

Additionally, we have the following equality

$$
\int_y^x B'_{n,\varphi}(f;u)\,du = B_{n,\varphi}(f;x) - B_{n,\varphi}(f;y)
$$

therefore, any fixed pair of points x, y in $[0,1]$, using (15) and (16), we have

$$
\begin{aligned}
|f(\varphi(x)) - f(\varphi(y))| &= \Bigg|f(\varphi(x)) - f(\varphi(y)) - B_{n,\varphi}(f;x) + B_{n,\varphi}(f;y) \\
&\qquad + \int_y^x B'_{n,\varphi}(f;u)\,du\Bigg| \\
&\leq |f(\varphi(x)) - B_{n,\varphi}(f;x)| + |B_{n,\varphi}(f;y) - f(\varphi(y))| \\
&\qquad + \left|\int_y^x \left|B'_{n,\varphi}(f;u)\right|\,du\right| \\
&\leq 2K\left[\frac{1}{n}\right]^\gamma + \lambda\left\{n+\frac{1}{\delta}\right\}\omega_f(\delta)|x-y|.
\end{aligned}
$$

or, introducing $\delta_n = \dfrac{1}{n}$,

$$
|f(\varphi(x)) - f(\varphi(y))| \leq 2K\delta_n^\gamma + \lambda\left\{\frac{1}{\delta_n}+\frac{1}{\delta}\right\}\omega_f(\delta)|x-y|.
$$

The sequence δ_n decreases to zero as $n \to \infty$. For a fixed n, pick $\delta \in (0,1]$ such that $\delta_n \leq \delta < \delta_{n-1} \leq 2\delta_n$, consequently we have

$$
\begin{aligned}
|f(\varphi(x)) - f(\varphi(y))| &\leq 2K\delta^\gamma + 3\lambda\frac{|x-y|}{\delta}\omega_f(\delta) \\
&\leq K_1\left\{\delta^\gamma + \frac{|x-y|}{\delta}\omega_{f\circ\varphi}(\delta)\right\}
\end{aligned}
$$

where $K_1 = \max\{2K,\ 3\lambda\}$. Taking the maximum over all arbitrary pairs x, y in $[0,1]$ with $|x-y| = h \leq 1$, the last inequality gives

$$\omega_{f\circ\varphi}(h) \leq K_1 \left\{ \delta^{\gamma} + \frac{h}{\delta}\omega_{f\circ\varphi}(\delta) \right\}$$

where $0 < h,\ \delta \leq 1$. Lemma 12 then tells us that $\omega_{f\circ\varphi}(h) \leq K_2 h^{\gamma}$ for some constant K_2. This completes the proof. ■

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