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ON THE SOLUTIONS OF THE RECURSIVE SEQUENCE $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$

Abstract. In this paper we study the solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-(2k+1)}}{-a + x_{n-k}x_{n-(2k+1)}} \quad \text{for} \quad n = 0, 1, 2, ...$$

where a, $x_{-(2k+1)}$, $x_{-(2k)}$, $x_{-(2k-1)}$, ..., x_0 are the real numbers such that $x_0 x_{-(k+1)} \neq a, x_{-1}x_{-(k+2)} \neq a, x_{-2}x_{-(k+3)} \neq a, \ldots$, $x_{-k}x_{-(2k+1)} \neq a$ and k is a natural number.

Key words: recursive sequence, solution, periodicity.

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1. Introduction

Difference equations have played an important role in analysis of mathematical models of biology, physics and engineering. Many researchers have investigated the behavior of the solution of rational difference equations. For example see Refs. [1-15].

Aloqeili [12] studied the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_{n-1}x_n}$$

and gave the following formula

$$x_n = \begin{cases} x_0 \prod_{i=1}^{\frac{n}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i-1})x_{-1}x_0}{a^{2i}(1-a)-(1-a^{2i})x_{-1}x_0}, & n \;\; even, \\ x_{-1} \prod_{i=0}^{\frac{n+1}{2}} \frac{a^{2i-1}(1-a)-(1-a^{2i})x_{-1}x_0}{a^{2i+1}(1-a)-(1-a^{2i+1})x_{-1}x_0}, & n \;\; odd. \end{cases}$$

Andruch et al. [1] studied the asymtotic behavior of solutions of the difference equation

$$x_{n+1} = \frac{ax_{n-1}}{b + cx_n x_{n-1}}$$

and gave the following formula

$$x_n = \begin{cases} x_{-1}\dfrac{\prod_{i=0}^{\frac{n+1}{2}-1}\left[p^{2i}+x_0x_{-1}\sum_{k=0}^{2i-1}p^k\right]}{\prod_{i=0}^{\frac{n+1}{2}-1}\left[p^{2i+1}+x_0x_{-1}\sum_{k=0}^{2i}p^k\right]}, & n \ \ odd, \\ x_0\dfrac{\prod_{i=0}^{\frac{n}{2}-1}\left[p^{2i+1}+x_0x_{-1}\sum_{k=0}^{2i}p^k\right]}{\prod_{i=0}^{\frac{n}{2}-1}\left[p^{2i+2}+x_0x_{-1}\sum_{k=0}^{2i+1}p^k\right]}, & n \ \ even. \end{cases}$$

Cinar [3] investigated the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}.$$

Also, Cinar [4] investigated the positive solutions of the rational difference equation

$$x_{n+1} = \frac{ax_{n-1}}{-1+bx_nx_{n-1}}.$$

Yalçınkaya [10] investigated the global behaviour of the rational difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

El-Owaidy et al. [9] studied the dynamics of the recurcive sequence

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^p}.$$

Battaloğlu et al. [13] discussed the global asymptotic behavior and periodicity character of the following difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma x_{n-(k+1)}^p}$$

by generalizing the results due to El-Owaidy et al.

Hamza et al. [2] studied the asymptotic stability of the nonnegative equilibrium point of the difference equation

$$x_{n+1} = \frac{Ax_{n-1}}{B + C\prod_{i=l}^{k} x_{n-2i}}.$$

Gibbons et al. [6] investigated the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\alpha + x_n}.$$

Our aim in this paper is to investigate the solutions of the difference equation

$$(1) \qquad x_{n+1} = \frac{a x_{n-(2k+1)}}{-a + x_{n-k} x_{n-(2k+1)}} \quad \text{for } n = 0, 1, 2, \ldots$$

where

(2) $a, x_{-(2k+1)}, x_{-(2k)}, x_{-(2k-1)}, ..., x_0$ are the real numbers such that $x_0 x_{-(k+1)} \neq a, x_{-1} x_{-(k+2)} \neq a, x_{-2} x_{-(k+3)} \neq a, ..., x_{-k} x_{-(2k+1)} \neq a$

and k is a natural number.

Similar to the references in this paper, we define Eq.(1) with (2) and investigate the solutions of this difference equation.

Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then for every set of initial conditions x_{-k}, $x_{-(k+1)}$, $\ldots$, $x_0 \in I$, the difference equation

$$(3) \qquad x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right), \quad n = 0, 1, \ldots$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Periodicity). *A sequence $\{x_n\}_{n=-k}^{\infty}$ of Eq.(3) is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

2. Main results

Theorem 1. *Assume that (2) holds and let $\{x_n\}_{n=-(2k+1)}^{\infty}$ be a solution of Eq.(1). Then for $n = 0, 1, \ldots$ all solutions of Eq.(1) are*

$$(4) \qquad x_{2(k+1)n+i} = \begin{cases} \dfrac{a^{n+1} x_{-(2k+2-i)}}{(-a + x_{-(k+1-i)} x_{-(2k+2-i)})^{n+1}}, \\ \qquad\qquad i = 1, 2, \ldots, k+1, \\ \frac{1}{a^{n+1}} x_{-(2k+2-i)} \\ \quad \times \left(-a + x_{-(2k+2-i)} x_{-(3k+3-i)}\right)^{n+1}, \\ \qquad\qquad i = k+2, ..., 2k+2, \end{cases}$$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $(n-1)$. That is

$$(5) \qquad x_{2(k+1)n-(2k+2-i)} = \begin{cases} \dfrac{a^{n} x_{-(2k+2-i)}}{(-a + x_{-(k+1-i)} x_{-(2k+2-i)})^{n}}, \\ \qquad\qquad i = 1, 2, ..., k+1 \\ \frac{1}{a^{n}} x_{-(2k+2-i)} \\ \quad \times \left(-a + x_{-(2k+2-i)} x_{-(3k+3-i)}\right)^{n}, \\ \qquad\qquad i = k+2, \ldots, 2k+2 \end{cases}$$

Now, it follows from Eq.(1) and Eq.(2) that

$$\begin{aligned}
x_{2(k+1)n+1} &= \frac{ax_{2(k+1)n-(2k+1)}}{-a+x_{2(k+1)n-k}x_{2(k+1)n-(2k+1)}} \\
&= \frac{a\frac{a^n x_{-(2k+1)}}{(-a+x_{-k}x_{-(2k+1)})^n}}{-a+\frac{1}{a^n}x_{-k}(-a+x_{-k}x_{-(2k+1)})^n\frac{a^n x_{-(2k+1)}}{(-a+x_{-k}x_{-(2k+1)})^n}} \\
&= \frac{\frac{a^{n+1}x_{-(2k+1)}}{(-a+x_{-k}x_{-(2k+1)})^n}}{-a+x_{-k}x_{-(2k+1)}}.
\end{aligned}$$

Hence, we have

$$x_{2(k+1)n+1} = \frac{a^{n+1}x_{-(2k+1)}}{(-a+x_{-k}x_{-(2k+1)})^{n+1}}.$$

Similarly

$$\begin{aligned}
x_{2(k+1)n+k+2} &= \frac{ax_{(2k+1)n-k}}{-a+x_{(2k+1)n+1}x_{(2k+1)n-k}} \\
&= \frac{a\frac{1}{a^n}x_{-k}(-a+x_{-k}x_{-(2k+1)})^n}{-a+\frac{a^{n+1}x_{-(2k+1)}}{(-a+x_{-k}x_{-(2k+1)})^{n+1}}\cdot\frac{1}{a^n}x_{-k}(-a+x_{-k}x_{-(2k+1)})^n} \\
&= \frac{\frac{ax_{-k}(-a+x_{-k}x_{-(2k+1)})^n}{a^n}}{-a+\frac{ax_{-(2k+1)}x_{-k}}{(-a+x_{-k}x_{-(2k+1)})}} \\
&= \frac{\frac{ax_{-k}(-a+x_{-k}x_{-(2k+1)})^n}{a^n}}{\frac{a^2}{(-a+x_{-k}x_{-(2k+1)})}}.
\end{aligned}$$

Hence, we have

$$x_{2(k+1)n+k+2} = \frac{1}{a^{n+1}}x_{-k}(-a+x_{-k}x_{-(2k+1)})^{n+1}.$$

Similarly, the other cases can be obtained. Thus, the proof is completed. ■

Theorem 2. *Assume that* $x_0x_{-(k+1)} = x_{-1}x_{-(k+2)} = \ldots = x_{-k}x_{-(2k+1)} = 2a$. *Then every solution of Eq.(1) is periodic with period* $(2k+2)$.

Proof. From assumption and Theorem 1, we have

$$\begin{aligned}
x_{2(k+1)n+1} &= x_{-(2k+1)}, \\
x_{2(k+1)n+2} &= x_{-(2k)}, \\
&\vdots
\end{aligned}$$

$$\begin{aligned} x_{2(k+1)n+k+1} &= x_{-(k+1)}, \\ x_{2(k+1)n+k+2} &= x_{-k}, \\ x_{2(k+1)n+k+3} &= x_{-(k-1)}, \\ &\vdots \\ x_{2(k+1)n+2(k+1)} &= x_0. \end{aligned}$$

It is obvious that every solution of Eq.(1) is periodic with period $(2k+2)$. ■

Corollary 1. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.*(1)*. Assume that*

$$a, x_{-(2k+1)}, x_{-(2k)}, x_{-(2k-1)}, ..., x_0 > 0$$

and

$$x_0 x_{-(k+1)} > a, x_{-1} x_{-(k+2)} > a, ..., x_{-k} x_{-(2k+1)} > a.$$

Then all solutions of Eq.(1) are positive.

Proof. From the Eq.(2) all solutions of Eq.(1) are positive. ■

Corollary 2. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.(1). Assume that*

$$a > 0, x_{-(2k+1)}, x_{-(2k)}, x_{-(2k-1)}, ..., x_0 < 0$$

and

$$x_0 x_{-(k+1)} > a, x_{-1} x_{-(k+2)} > a, ..., x_{-k} x_{-(2k+1)} > a.$$

Then all solutions of Eq.(1) are negative.

Proof. From the Eq.(2) all solutions of Eq.(1) are negative. ■

Corollary 3. *Let* $\{x_n\}_{n=-(2k+1)}^{\infty}$ *be a solution of Eq.(1). Assume that*

$$a = 1, x_{-(2k+1)}, x_{-(2k)}, ..., x_0 > 0$$

and

$$x_0 x_{-(k+1)} > 2, x_{-1} x_{-(k+2)} > 2, ..., x_{-k} x_{-(2k+1)} > 2.$$

Then

$$\lim_{n\to\infty} x_{2(k+1)n+i} = 0 \quad (i = 1, 2, ..., k+1).$$

$$\lim_{n\to\infty} x_{2(k+1)n+i} = \infty \quad (i = k+2, k+3, ..., 2k+2).$$

Proof. Let

$$x_{-(2k+1)}, x_{-(2k)}, x_{-(2k-1)}, ..., x_0 > 0$$

and

$$x_0 x_{-(k+1)} > 2, x_{-1} x_{-(k+2)} > 2, ..., x_{-k} x_{-(2k+1)} > 2.$$

Then

$$x_0 x_{-(k+1)} - 1 > 1, x_{-1} x_{-(k+2)} - 1 > 1, ..., x_{-k} x_{-(2k+1)} - 1 > 1.$$

From the Eq.(2), we have

$$\begin{aligned}
\lim_{n\to\infty} x_{2(k+1)n+1} &= \lim_{n\to\infty} \frac{x_{-(2k+1)}}{(-1 + x_{-k} x_{-(2k+1)})^{n+1}} = 0, \\
\lim_{n\to\infty} x_{2(k+1)n+2} &= \lim_{n\to\infty} \frac{x_{-(2k)}}{(-1 + x_{-(k-1)} x_{-(2k)})^{n+1}} = 0, \\
&\cdots \\
\lim_{n\to\infty} x_{2(k+1)n+k+1} &= \lim_{n\to\infty} \frac{x_{-(k+1)}}{(-1 + x_0 x_{-(k+1)})^{n+1}} = 0, \\
\lim_{n\to\infty} x_{2(k+1)n+k+2} &= \lim_{n\to\infty} x_{-k}(-1 + x_{-k} x_{-(2k+1)})^{n+1} = \infty, \\
\lim_{n\to\infty} x_{2(k+1)n+k+3} &= \lim_{n\to\infty} x_{-(k-1)}(-1 + x_{-(k-1)} x_{-(2k)})^{n+1} = \infty, \\
&\cdots \\
\lim_{n\to\infty} x_{2(k+1)n+2(k+1)} &= \lim_{n\to\infty} x_0(-1 + x_0 x_{-(k+1)})^{n+1} = \infty
\end{aligned}$$

The proof is completed. ∎

Corollary 4. *Let $\{x_n\}_{n=-(2k+1)}^{\infty}$ be a solution of Eq.(1). Assume that*

$$a = 1, x_{-(2k+1)}, x_{-(2k)}, x_{-(2k-1)}, ..., x_0 < 0$$

and

$$x_0 x_{-(k+1)} > 2, x_{-1} x_{-(k+2)} > 2, ..., x_{-k} x_{-(2k+1)} > 2.$$

Then

$$\begin{aligned}
\lim_{n\to\infty} x_{2(k+1)n+i} &= 0 \quad (i = 1, 2, ..., k+1). \\
\lim_{n\to\infty} x_{2(k+1)n+i} &= -\infty \quad (i = k+2, k+3, ..., 2k+2).
\end{aligned}$$

Proof. The proof is similar to Corollary 3. ∎

3. Numerical results

Example 1. Let $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, $n = 0, 1, , ..., 7$ and $k = 1$, $a = 1$, $x_{-3} = 0.2$, $x_{-2} = 1$, $x_{-1} = 10$, $x_0 = 2$. Then we have the following results from Theorem 2:

n	x_n	n	x_n
1	$0,2$	5	$0,2$
2	1	6	1
3	10	7	10
4	2	8	2

Example 2. Let $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, $n = 0, 1, ..., 99$ and $k = 2$, $a = 3$, $x_{-5} = 5$, $x_{-4} = 4$, $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 1$, $x_0 = 2$. Then we have the following results from Corollary 1:

n	x_n	n	x_n
1	$1,25$	50	$78732,00057$
2	12	67	$2,980232242.10^{-7}$
6	$0,666666666$	88	$3,221225448.10^{9}$
10	$47,99999994$	100	$5,153960670.10^{10}$

Example 3. Let $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, $n = 0, 1, ..., 99$ and $k = 2$, $a = 2$, $x_{-5} = -1$, $x_{-4} = -2.5$, $x_{-3} = -1.5$, $x_{-2} = -3$, $x_{-1} = -1$, $x_0 = -2$. Then we have the following results from Corollary 2:

n	x_n	n	x_n
1	-2	50	$-6,55360.10^{5}$
2	-10	51	-768
10	$-0,75$	98	$-4,294967234.10^{10}$
11	$-0,0625$	100	$-0,00002288818376$

Example 4. Let $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, $n = 0, 1, ..., 99$ and $k = 1$, $a = 1$, $x_{-3} = 10$, $x_{-2} = 11$, $x_{-1} = 0.3$, $x_0 = 0.2$. Then we have the following results from Corollary 3:

n	x_n	n	x_n
1	5	51	$2457,6$
13	$0,625$	71	$78643,2$
25	$0,078125$	87	$1,258291210.10^{6}$
37	$0,009765625$	99	$1,006632974.10^{7}$

Example 5. Let $x_{n+1} = \frac{ax_{n-(2k+1)}}{-a+x_{n-k}x_{n-(2k+1)}}$, $n = 0, 1, ..., 99$ and $k = 1$, $a = 1$, $x_{-3} = -15$, $x_{-2} = 1 - 12$, $x_{-1} = -0.4$, $x_0 = -0.3$. Then we have the following results from Corollary 4:

n	x_n	n	x_n
1	-3	51	$-4,88281250.10^{8}$
13	$-0,024$	71	$-1,525878906.10^{12}$
25	$-0,000192$	87	$-9,536743160.10^{14}$
37	$-0,000001536$	99	$-1,192092895.10^{17}$

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