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# OPERATIONS ON BITOPOLOGICAL SPACES

Abstract. In 1979, Kasahara [8], introduced the concept of operations on topological spaces. Further investigations of this concept are given in [7,15]. Operations on bitopological spaces were discussed in some manner in [ 1,10]. In this paper we use a different technique of that in [1,10] to study the concepts of operations on bitopological spaces by generalizing the results obtained in [8,15] to such spaces.

Key words: bitopological spaces, operation, filterbase, compact and $(\gamma, \beta)$-continuous.

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## 1. Introduction

The aim of this paper is to study the concept of operations on bitopological spaces. In Section 2, we introduce the concept of a family $\tau_{i\gamma}$ of all $\gamma_i$-open sets by using the operation $\gamma$ on a bitopological space $(X, \tau_1, \tau_2)$. Also we introduce the concepts of $\gamma_i$-closure and $\tau_{i\gamma}$-closure and discuss the relation between them. In Section 3, we introduce the notion of pairwise $\gamma - T_{\frac{1}{2}}$-spaces and we characterize them by the notion of $\gamma_i$-closed or $\gamma_i$-open sets. Section 4 is devoted to the study of $(\gamma, \beta)_i$-continuity of mappings. The concepts of $\gamma_i$-convergence and $\gamma_i$-accumulation of filters are discussed in Section 5. Finally, in section 6, we study $\gamma_i$-compactness of bitopological spaces

Throughout the paper, $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ ( or briefly, $X$ and $Y$ ) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let $A$ be a subset of $X$, by $i$-int$(A)$ and $i$-cl$(A)$ we denote respectively the interior and the closure of A with respect to $\tau_i$ (or $\sigma_i$) for $i = 1, 2$. By $id$ we mean the identity and nbd is the abbreviation of neighborhood. Also $X \backslash A = A^c$, is the complement of $A$ in $X$.

## 2. Operations

**Definition 1.** *Let $(X, \tau_1, \tau_2)$ be a bitopological space. An operation $\gamma$ on $\tau_1 \cup \tau_2$ is a mapping $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that $V \subset V^\gamma$ for each $V \in \tau_1 \cup \tau_2$, where $V^\gamma$ denotes the value of $\gamma$ at $V$. The operators $\gamma(V) = V$, $\gamma(V) = j - cl(V)$ and $\gamma(V) = i - int(j - cl(V))$ for $V \in \tau_i$ are operations on $\tau_1 \cup \tau_2$.*

**Definition 2.** *A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ will be called a $\gamma_i$-open set if for each $x \in A$ there exists a $\tau_i$-open set $U$ such that $x \in U$ and $U^\gamma \subset A$. $\tau_{i\gamma}$ will denote the set of all $\gamma_i$-open sets. Clearly we have $\tau_{i\gamma} \subset \tau_i$. A subset $B$ of $X$ is said to be $\gamma_i$-closed if $X \backslash B$ is $\gamma_i$-open in $X$.*

If $\gamma(U) = U$ (resp. $j - cl(U)$ and $j - cl(i - int(U))$ for each $U \in \tau_i$ then, the concept of $\gamma_i$-open sets coincides with the concept of $\tau_i$-open (resp. $ij - \theta$-open [2] and $ij - \delta$-open [2]) sets.

**Definition 3.** *An operation $\gamma$ on $\tau_1 \cup \tau_2$ in $(X, \tau_1, \tau_2)$ is said to be i-regular if for every pair $U$, $V$ of $\tau_i$-open nbds of each point $x \in X$, there exists a $\tau_i$-open nbd $W$ of $x$ such that$W^\gamma \subset U^\gamma \subset V^\gamma$.*

**Definition 4.** *Let $\gamma, \mu$ be two operations on $\tau_1 \cup \tau_2$ in $(X, \tau_1, \tau_2)$. Then $(X, \tau_1, \tau_2)$ is called $(\gamma, \mu)_i$-regular if for each $x \in X$ and each $\tau_i$-open nbd $V$ of $x$ there exists $\tau_i$-open set $U$ containing $x$ such that $U^\gamma \subset V^\mu$. If $\mu = id$ then the $(\gamma, \mu)_i$-regular space is called $\gamma_i$-regular. If $(X, \tau_1, \tau_2)$ is $(\gamma, \mu)_1$-regular and $(\gamma, \mu)_2$-regular then it is called pairwise $(\gamma, \mu)$-regular.*

If $(\gamma, \mu) = (j - cl, id)$ (resp. $(i - int^o j - cl, id)$ and $(j - cl, i - int^o j - cl)$ ) then pairwise $(\gamma, \mu)$-regularity is coincide with pairwise regularity [9] ( resp. pairwise semi regularity [18] and pairwise almost regularity [19]) ( where $(i - int^o j - cl)(V) = i - int(j - cl)V)))$. In the rest of this paper, instead of " $\gamma$ operation on $\tau_1 \cup \tau_2$ in $(X, \tau_1, \tau_2)$" we shall say $\gamma$ operation on $(X, \tau_1, \tau_2)$".

**Proposition 1.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. Then $(X, \tau_1, \tau_2)$ is a $\gamma_i$-regular space if and only if $\tau_i = \tau_{i\gamma}$ holds.*

**Example 1.** Let $X = \{a, b, c, d\}$, $\tau_1 = P(X)$, $\tau_2 = \{X, \{a\}, \{b\}, \{d\}, \{a, b\}, \{b, d\}, \{a, d\}, \{a, b, d\}\}$ and let $\gamma$ be an operation on $(X, \tau_1, \tau_2)$ defined by $\gamma(A) = j - cl(A)$ for $A \in \tau_i$. Then $\tau_{1\gamma} = \{X, \varphi, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. Clearly, $\tau_{1\gamma} \subsetneqq \tau_1$ We can see that $(X, \tau_1, \tau_2)$ is not $\gamma_1$-regular.

**Example 2.** Let $X = \{a, b, c, d\}$, $\tau_1 = \{X, \phi, \{b\}, \{c, d\}, \{b, c, d\}\}$ , $\tau_2 = \{X, \phi\{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$ and let $\gamma$ be an operation on $(X, \tau_1, \tau_2)$ defined by $\gamma(A) = j - cl(A)$ for $A \in \tau_i$. Then $\tau_1 = \tau_{1\gamma}$. We can see that $(X, \tau_1, \tau_2)$ is $\gamma_1$-regular.

**Definition 5.** *An operation $\gamma$ on $(X, \tau_1, \tau_2)$ is said to be i-open if for every $\tau_i$ -open nbd $U$ of each $x \in X$, there exists a $\gamma_i$-open set $S$ such that $x \in S$ and $S \subset U^\gamma$.*

By [15 Example 2.7] there is an operation $\gamma$ on a bitopological space $(X, \tau_1, \tau_2)$ which is $i$-regular but not $i$-open and an operation $\delta$ which is $i$-regular and $i$-open. Also, by [16 Example 2.8] there is an operation $\gamma$ which is $i$-open but not $i$-regular.

**Proposition 2.** *Let $\gamma$ be an i-regular operation on a bitopological space $(X, \tau_1, \tau_2)$*

*$(i)$ If $A$ and $B$ are $\gamma_i$-open sets of $X$, then $A \cap B$ is $\gamma_i$-open.*

*$(ii)$ $\tau_{i\gamma}$ is a topology on $X$ .*

**Remark 1.** If $\gamma$ is not $i$-regular, then the above proposition is not true in general. By [15 Example 2.8] there is an operation $\gamma$ on a bitopological space $(X, \tau_1, \tau_2)$ which is not $i$-regular and $\tau_{i\gamma}$ is not a topology on $X$.

**Definition 6.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. A point $x \in X$ is in the $\gamma_i$-closure of a set $A \subset X$ if $U^\gamma \cap A \neq \phi$ for each $\tau_i$-open nbd $U$ of $x$. The $\gamma_i$-closure of the set $A$ is denoted by $cl_{i\gamma}$. A subset $A$ of $X$ is said to be $\gamma_i$-closed (in the sense of$cl_{i\gamma}(A)$) if $cl_{i\gamma}(A) \subset A$.*

If $\gamma = id$(resp. $j - cl$ and $i - int^o j - cl$ ) then the $\gamma_i$-closure coincide with the $i$-closure (resp. $ij - \theta$-closure and $ij - \delta$-closure [5]).

**Definition 7.** *Let $A \subset (X, \tau_1, \tau_2)$. For the family $\tau_{i\gamma}$, we define a set $\tau_{i\gamma} - cl(A)$ as follows:*

$$\tau_{i\gamma} - cl(A) = \cap\{F : A \subset F, X \backslash F \in \tau_{i\gamma}\}$$

**Proposition 3.** *For a point $x \in X$, $x \in \tau_{i\gamma} - cl(A)$ iff $V \cap A \neq \phi$ for any $V \in \tau_{i\gamma}$ such that $x \in V$.*

**Remark 2.** It is easily shown that for any subset $A$ of $(X, \tau_1, \tau_2)$ $A \subset \tau_i - cl(A) \subset cl_{i\gamma}(A) \subset \tau_{i\gamma} - cl(A)$ .

**Remark 3.** Generally $\tau_{i\gamma} - cl(A) \nsubseteq cl_{i\gamma}(A)$ (see [15 Remark 3.5]).

**Theorem 1.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$, then*

*$(i)$ $cl_{i\gamma}(A)$ is $\tau_i$-closed in $(X, \tau_1, \tau_2)$.*

*$(ii)$ If $(X, \tau_1, \tau_2)$ is $\gamma_i$-regular, then $cl_{i\gamma}(A) = \tau_i - cl(A)$ holds.*

*$(iii)$ If $\gamma$ is i-open, then $cl_{i\gamma}(A) = \tau_{i\gamma} - cl(A)$ and $cl_{i\gamma}(cl_{i\gamma}(A)) = cl_{i\gamma}(A)$ hold and $cl_{i\gamma}(A)$ is $\gamma_i$-closed (in the sense of Definition 6).*

**Theorem 2.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$ and $A \subset X$, the following are equivalent*

(*a*) *$A$ is $\gamma_i$-open in $(X, \tau_1, \tau_2)$.*

(*b*) *$cl_{i\gamma}(X \backslash A) = X \backslash A$ (i.e., $X \backslash A$ is $\gamma_i$-closed in the sense of Definition 6).*

(*c*) *$\tau_{i\gamma} - cl(X \backslash A) = X \backslash A$.*

(*d*) *$X \backslash A$ is $\gamma_i$-closed (in the sense of Definition 7) in $(X, \tau_1, \tau_2)$*

**Remark 4.** By [15 Remark 3.9] there exists an operation $\gamma$ which is not $i$-open such that $cl_{i\gamma}(cl_{i\gamma}(A)) \neq cl_{i\gamma}(A)$.

**Lemma 1.** *If $\gamma$ is an i-regular operation on $(X, \tau_1, \tau_2)$, then $cl_{i\gamma}(A \cup B) = cl_{i\gamma}(A) \cup cl_{i\gamma}(B)$, for any subsets $A$ and $B$ of $X$.*

**Proof.** It is straight forward. ■

**Corollary 1.** *If $\gamma$ is an i-regular and i-open operation on $(X, \tau_1, \tau_2)$, then $cl_{i\gamma}$satisfies the Kuratowski closure axioms .*

## 3. Pairwise $\gamma - T_{\frac{1}{2}}$ spaces

Let $\gamma$ be an operation on $(X, \tau_1, \tau_2)$.

**Definition 8.** *A subset $A$ of $(X, \tau_1, \tau_2)$ is said to be $ij - \gamma - g$-closed if $cl_{j\gamma}(A) \subset U$ whenever $A \subset U$ and $U$ is $\gamma_i$-opene in $(X, \tau_1, \tau_2)$.*

Clearly, it follows from definition and Theorem 2, that every $\gamma_i$-closed set is $ij - \gamma - g$-closed. If $\gamma = id$, then the concept of $ij - \gamma - g$-closed coincide with the concept of $ij - g$-closed [6]. In case $\gamma(V) = j - cl(V)$ (resp. $\gamma(V) = i - int(j - cl(V))$) an $ij - \gamma - g$-closed set is called $ij - \theta - g - closed$ (resp. $ij - \delta - g$-closed).

**Definition 9.** *A bitopological space $(X, \tau_1, \tau_2)$ is called an $ij - \gamma - T_{\frac{1}{2}}$ space if every $ij - \gamma - g$-closed set in $X$ is $\gamma_j$-closed. In case $\gamma(V) = V$, the concept of $ij - \gamma - T_{\frac{1}{2}}$ coincide with the concept of $ij - T_{\frac{1}{2}}$ [6]. In case $\gamma(V) = j - cl(V)$ (resp. $\gamma(V) = i - int(j - cl(V))$) the concept of $ij - \gamma - T_{\frac{1}{2}}$ is called $ij - \theta - T_{\frac{1}{2}}$ (resp. $ij - \delta - T_{\frac{1}{2}}$).*

**Proposition 4.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$ and $A \subset X$, then*

(*i*) *If $\tau_{i\gamma} - cl(\{x\}) \cap A \neq \phi$ for every $x \in cl_{j\gamma}(A)$, then $A$ is $ij - \gamma - g$-closed in $(X, \tau_1, \tau_2)$.*

(*ii*) *If $\gamma$ is i-regular, then the converse of (i) is true .*

If $\gamma = id$ in Proposition 4, then we have the following

**Corollary 2** (6, Proposition 2.13($ii$)). *$A$ is $ij-g$-closed if and only if $\tau_i - cl\{x\} \cap A \neq \phi$ holds for each $x \in cl_j(A)$.*

**Proposition 5.** *If $\tau_{i\gamma} - cl(\{x\}) \cap A \neq \phi$ for every $x \in cl_{j\gamma}(A)$, then $cl_{j\gamma}(A) - A$ does not contain nonempty $\gamma_i$-closed subsets .*

**Remark 5.** Let $\gamma$ be an $i$-regular operation. If $A$ is $ij-\gamma-g$-closed set then $cl_{j\gamma}(A) - A$ does not contain nonempty $\gamma_i$-closed subsets. In fact this is obtained by Propositions 4($ii$) and 5.

**Corollary 3** (6, Proposition 2.13($i$)). *If $A$ is $ij-g$-closed then $\tau_j - cl(A) - A$ contains no non-empty $\tau_i$-closed set.*

**Corollary 4.** *If $A$ is $ij-\theta-g$-closed then $ji - cl_\theta(A) - A$ contains no nonempty $ij-\theta$-closed set.*

**Corollary 5.** *If $A$ is $ij-\delta-g$-closed then $ji - cl_\delta(A) - A$ contains no nonempty $ij-\delta$-closed set.*

**Proposition 6.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. For each $x \in X$, $\{x\}$ is $\gamma_i$-closed or $\{x\}^c$ is $ij-\gamma-g$-closed in $(X, \tau_1, \tau_2)$.*

**Corollary 6.** *In a bitopological space $(X, \tau_1, \tau_2)$, for each $x \in X$, $\{x\}$ is $\tau_i$-closed or $\{x\}^c$ is $ij-g$-closed.*

**Corollary 7.** *In a bitopological space $(X, \tau_1, \tau_2)$, for each $x \in X$, $\{x\}$ is $ij-\theta$-closed or $\{x\}^c$ is $ij-\theta-g$-closed.*

**Corollary 8.** *In a bitopological space $(X, \tau_1, \tau_2)$, for each $x \in X$, $\{x\}$ is $ij-\delta$-closed or $\{x\}^c$ is $ij-\delta-g$-closed.*

For a $i$-regular operation $\gamma$, we give a necessary and sufficient condition for a space to be $ij-\gamma-T_{\frac{1}{2}}$.

**Proposition 7.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$*

($i$) *If $(X, \tau_1, \tau_2)$ is $ij-\gamma-T_{\frac{1}{2}}$, then for each $x \in X$, $\{x\}$ is $\gamma_i$-closed or $\gamma_j$-open.*

($ii$) *If $\gamma$ is $i$-regular then the converse of ($i$) is true.*

**Corollary 9** (6, Theorem 2.15). *A bitopological space $(X, \tau_1, \tau_2)$ is $ij-T_{\frac{1}{2}}$ if and only if $\{x\}$ is $\tau_i$-closed or $\tau_j$-open for each $x \in X$.*

**Corollary 10.** *A bitopological space $(X, \tau_1, \tau_2)$ is $ij-\theta-T_{\frac{1}{2}}$ if and only if $\{x\}$ is $ij-\theta$-closed or $ji-\theta$-open for each $x \in X$.*

**Corollary 11.** *A bitopological space $(X, \tau_1, \tau_2)$ is $ij-\delta-T_{\frac{1}{2}}$ if and only if $\{x\}$ is $ij-\delta$-closed or $ji-\delta$-open for each $x \in X$.*

## 4. Pairwise $(\gamma, \beta)$-continuous mappings

Throughout this section let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces and let $\gamma$ and $\beta$ be operations on $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$, respectively.

**Definition 10.** *A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\gamma, \beta)_i$-continuous if for each point $x$ of $X$ and each $\sigma_i$-open set $V$ containing $f(x)$, there exists a $\tau_i$-open set $U$ such that $x \in U$ and $f(U^\gamma) \subset V^\beta$. If $f$ is $(\gamma, \beta)_i$-continuous for $i = 1, 2$, then it is called pairwise $(\gamma, \beta)$-continuous.*

If $(\gamma, \beta) = (id, id)$ (resp. $(id, j - cl)$, $(j - cl, id)$, $(j - cl, j - cl)$, $(id, i - int^o j - cl)$, $(i - int^o j - cl, id)$, $(i - int^o j - cl, j - cl)$, $(j - cl, i - int^o j - cl)$ and $(i - int^o j - cl, i - int^o j - cl)$) then pairwise $(\gamma, \beta)$-continuity coincides with pairwise continuity [9] (resp. pairwise weak continuity [4], pairwise strong $\theta$-continuity [2], pairwise $\theta$-continuity [3], pairwise almost continuity [4], pairwise super continuity [12], pairwise weak $\theta$-continuity [11], pairwise almost strong $\theta$-continuity [2] and pairwise $\delta$-continuity [2]).

**Remark 6.** *When $\gamma = id$, $(\gamma, \beta)_i$-continuity of $f$ may be written $\beta_i$-continuity.*

**Proposition 8.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $(\gamma, \beta)_i$-continuous mapping, then*

(*i*) *$f(cl_{i\gamma}(A)) \subset cl_{i\beta}(f(A))$ for every $A \subset X$.*

(*ii*) *for any $\beta_i$-closed set $B$ of $Y$, $f^{-1}(B)$ is $\gamma_i$-closed in $X$, i.e., for any $U \in \sigma_{i\beta}$, $f^{-1}(U) \in \tau_{i\gamma}$.*

**Remark 7.** In Proposition 8, if $(Y, \sigma_1, \sigma_2)$ is $\beta_i$-regular, then $(\gamma, \beta)_i$-continuity of $f$, (i) and (ii) are equivalent .

**Remark 8.** In Proposition 8, if $\beta$ is an $i$-open operation, then $(i)$ and $(ii)$ are equivalent.

**Proposition 9.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an i-continuous mapping of a $\gamma_i$-regular space $X$ into $Y$, then $f$ is $(\gamma, id)_i$-continuous ($f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is i-continuous if $f : (X, \tau_i) \to (Y, \sigma_i)$ is continuous).*

**Definition 11.** *A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(\gamma, \beta)_i$-closed if for any $\gamma_i$-closed set $A$ of $(X, \tau_1, \tau_2)$, $f(A)$ is a $\beta_i$-closed set in $(Y, \sigma_1, \sigma_2)$.*

If $f$ is $(id, \beta)_i$-closed, then $f(F)$ is a $\beta_i$-closed set for any $\tau_i$-closed set $F$ of $(X, \tau_1, \tau_2)$. Clearly if $f$ is bijective and $f^{-1} : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $(\beta, id)_i$-continuous, then $f$ is $(id, \beta)_i$-closed.

**Proposition 10.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $(\gamma, \beta)_i$-continuous and $(id, \beta)_i$-closed mapping, then for an $ij - \gamma - g$-closed set $A$ of $X$, $f(A)$ is $ij - \beta - g$-closed in $Y$.*

**Theorem 3.** *Suppose that $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise $(\gamma, \beta)$-continuous and $(id, \beta)_i$-closed injection. If $Y$ is $ij - \beta - T\frac{1}{2}$, then $X$ is $ij - \gamma - T\frac{1}{2}$.*

**Proposition 11.** *If $\gamma$ is an $i$-regular operation on $(X, \tau_1, \tau_2)$, then $(X, \tau_1, \tau_2)$ is $ij - \gamma - T\frac{1}{2}$ if and only if $(X, \tau_{1\gamma}, \tau_{2\gamma})$ is $ij - T\frac{1}{2}$.*

**Corollary 12.** *A bitopological space $(X, \tau_1, \tau_2)$ is $ij - \theta - T\frac{1}{2}$ if and only if $(X, \tau_{1\theta}, \tau_{2\theta})$ is $ij - T\frac{1}{2}$, where $\tau_{i\theta}$ is the topology on $X$ has the $ij - \theta$-open sets as a base.*

**Corollary 13.** *A bitopological space $(X, \tau_1, \tau_2)$ is $ij - \delta - T\frac{1}{2}$ if and only if $(X, \tau_{1\delta}, \tau_{2\delta})$ is $ij - T\frac{1}{2}$, where $\tau_{i\delta}$ is the topology on $X$ has the $ij - \delta$-open sets as a base.*

## 5. $\gamma_i$-convergence and $\gamma_i$-accumulation

**Definition 12.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. A filterbase $\Phi$ in $X$ is said to be $\gamma_i$-converges to a point $x \in X$, if for every $\tau_i$-open nbd $V$ of $x$, there exists an $F \in \Phi$ such that $F \subset V^\gamma$. Further, we say that a filterbase $\Phi$ in $X$ $\gamma_i$-accumulates to $x \in X$ if $F \cap V^\gamma \neq \phi$ for every $F \in \Phi$ and for every $\tau_i$-open nbd $V$ of $x$.*

The convergence and the accumulation in the usual sense are identical with the $\gamma_i$-convergence and $\gamma_i$-accumulation for the identity operation, respectively. If $\gamma = j - cl$ (resp. $i - int^o j - cl$) for $U \in \tau_i$, then $\gamma_i$-convergence and $\gamma_i$-accumulation coincide with $ij - \theta$-convergence and $ij - \theta$-accumulation [17] (resp. $ij - \delta$-convergence and $ij - \delta$-accumulation [14]. It is easy to see that $f : X \to Y$ is $\gamma_i$-continuous if and only if for every $x \in X$ and every filterbase $\Phi$ in $X$ $i$-converging to $x$, the filterbase $f(\Phi)$ $\gamma_i$-converges to $f(x)$.

**Proposition 12.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$, the following statements hold:*

1. *If a filterbase $\Phi$ in $X$ is $\gamma_i$-converges to $x \in X$, then $\Phi$ $\gamma_i$-accumulates to $x$.*
2. *If a filterbase $\Phi$ in $X$ is contained in a filterbase which $\gamma_i$-accumulates to $x \in X$, then $\Phi$ $\gamma_i$-accumulates to $x$.*
3. *If a maximal filterbase in $X$ $\gamma_i$-accumulates to $x \in X$ then it is $\gamma_i$-converges to $x$.*

**Proposition 13.** *Let $\gamma$ be an i-regular operation on a bitopological space $(X, \tau_1, \tau_2)$. If a filterbase $\Phi$ in $X$ $\gamma_i$-accumulates to $x \in X$, then there exists a filterbase $\Psi$ in $X$ such that $\Phi \subset \Psi$ and $\Psi$ $\gamma_i$-converges to $x$.*

**Proposition 14.** *Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces and $\gamma$ an operation on $Y$. If $f : X \to Y$ is $\gamma_i$-continuous, then $G \in \sigma_i$ and $G^\gamma = G$ imply $f^{-1}(G) \in \tau_i$. The converse is true if $G^{\gamma\gamma} = G^\gamma \in \sigma_i$ for all $G \in \sigma_i$.*

**Definition 13.** *Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be two bitopological spaces and $\gamma$ an operation on $Y$. We say that the graph $G(f)$ of $f : X \to Y$ is $\gamma_i$-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $\tau_i$-open and $\sigma_i$-open sets $U$ and $V$ containing $x$ and $y$, respectively, such that$(U \times V^\gamma) \cap G(f) = \phi$. Obviously, a mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ has a $\gamma_i$-closed graph if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $\tau_i$-open and $\sigma_i$-open sets $U$ and $V$ containing $x$ and $y$, respectively, such that$f(U) \cap V^\gamma = \phi$.*

If $\gamma = id$ (resp. $j{-}cl$ and $i{-}int^o j{-}cl$) in the definition of a $\gamma_i$-closed graph of a function then we obtain the definition of $\tau_i$-closed (resp. $ij - \theta$-closed and $ij - \delta$-closed) graph of the function.

**Theorem 4.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping and $\gamma$ an operation on $Y$. Then for the following statements (1)-(3), we have (1) $\Rightarrow$(2) and (2)$\Rightarrow$(3). If $\gamma$ is i-regular then (1) - (3) are equivalent.*

(1) *$f$ has a $\gamma_i$-closed graph.*
(2) *If there exists a filterbase $\Phi$ in $X$ i-converging to $x \in X$ such that $f(\Phi)$ $\gamma_i$-accumulates to $y \in Y$, then $f(x) = y$.*
(3) *If there exists a filterbase $\Phi$ in $X$ i-converging to $x \in X$ such that $f(\Phi)$ $\gamma_i$-converges to $y \in Y$, then $f(x) = y$.*

**Definition 14.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping and $\gamma$ an operation on $Y$. Then $f$ is called $\gamma_i$-subcontinuous if for every i-convergent filterbase $\Phi$ in $X$ (i.e. $\Phi$ i-converges to a point in $X$), the filterbase $f(\Phi)$ $\gamma_i$-accumulates to some point in $Y$.*

By (1) of Proposition 13, a $\gamma_i$-continuous mapping is $\gamma_i$-subcontinuous . Conversely we have the following

**Proposition 15.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $\gamma_i$-subcontinuous mapping, where $\gamma$ is an operation on $Y$. If $f$ has a $\gamma_i$-closed graph, then $f$ is $\gamma_i$-continuous .*

## 6. $\gamma_i$-compactness

**Definition 15.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. A subset $K \subset X$ is said to be $\gamma_i$-compact if for every $\tau_i$-open cover $C$ of $K$ there exists a finite subfamily $\{G_1, G_1, ...G_n\}$ of $C$ such that $K \subset \cup_{r=1}^n G_r^\gamma$.*

If $\gamma = id$ (resp. $j - cl$ and $i - int^o j - cl$) then $\gamma_i$-compactness is coincides with $\tau_i$-compactness (resp. $ij$-almost-compactness [13] and $ij$-near-compactness [2]).

**Theorem 5.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. A subset $K \subset X$ is $\gamma_i$-compact if and only if $K$ is $\gamma_i^*$-compact in$(K, \tau_1/K, \tau_2/K)$, where $\tau_i/K = \{G \cap K : G \in \tau_i$ and $\gamma^*$ is defined by $(G \cap K)^{\gamma^*} = G^\gamma \cap K$ for all $G \in \tau_i$ .*

**Proposition 16.** *If $(X, \tau_i)$ is compact, then $(X, \tau_1, \tau_2)$ is $\gamma_i$-compact for every operation $\gamma$ on $X$. If $(X, \tau_1, \tau_2)$ is $\gamma_i$-compact for some operation $\gamma$ on $X$ such that $(X, \tau_1, \tau_2)$ is $\gamma_i$-regular, then $(X, \tau_i)$ is compact.*

**Proof.** It will suffices to prove the second assertion. Let $C$ be a $\tau_i$-open cover of $X$. Since $(X, \tau_1, \tau_2)$ is $\gamma_i$-regular, the set $D$ of all $V \in \tau_i$ such that $V^\gamma \subset G$ for some $G \subset C$ is a $\tau_i$-open cover of $X$. Hence $X = \cup_r^n V_r^\gamma$ for some $V_1, V_2, ..., V_n \in D$. For $r \in \{1, 2, ..., n\}$, there exists $G_r \in C$ such that$V_r^\gamma \subset G$. Therefore we have $X = \cup_r^n G_r^\gamma$ as desired. ■

**Corollary 14.** *If $(X, \tau_1, \tau_2)$ is an $ij$-semi regular space, then $(X, \tau_1, \tau_2)$ is $ij$-nearly compact if and only if $(X, \tau_i)$ is compact.*

**Theorem 6.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$. If $X$ is $\gamma_i$-compact, then every cover of $X$ by $\gamma_i$-open sets has a finite subcover. If $\gamma$ is i-open then the converse is true .*

**Proposition 17.** *Let $\gamma$ be an i-open operation on a bitopological space $(X, \tau_1, \tau_2)$. If $X$ is $\gamma_i$-compact, then every $\gamma_i$-closed set of $X$ is $\gamma_i$-compact*

**Proposition 18.** *Let $\gamma$ be an i-open operation on a bitopological space $(X, \tau_1, \tau_2)$. Then $X$ is $\gamma_i$-compact if and only if every family of $\gamma_i$-closed subsets of $X$ having the finite intersection property , has a non empty intersection.*

**Theorem 7.** *Let $\gamma$ be an operation on a bitopological space $(X, \tau_1, \tau_2)$, the following statements are equivalent:*

(1) *$(X, \tau_1, \tau_2)$ is $\gamma_i$-compact*
(2) *Each filterbase in $X$ $\gamma_i$-accumulates to some point of $X$.*
(3) *Each maximal filterbase in $X$ $\gamma_i$-converges to some point of $X$.*

**Corollary 15.** *For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent*

(1) *$(X, \tau_1, \tau_2)$ is $ij$-almost compact.*
(2) *Each filterbase in $X$ $ij - \theta$-accumulates to some point of $X$.*
(3) *Each maximal filterbase in $X$ $ij - \theta$-converges to some point of $X$.*

**Corollary 16.** *For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent*

(1) $(X, \tau_1, \tau_2)$ *is $ij$-nearly compact.*

(2) *Each filterbase in $X$ $ij-\delta$-accumulates to some point of $X$.*

(3) *Each maximal filterbase in $X$ $ij-\delta$-converges to some point of $X$.*

**Theorem 8.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping and $\gamma$ an operation on $(Y, \sigma_1, \sigma_2)$. If $(X, \tau_i)$ is compact and $f$ is $\gamma_i$-continuous, then $f(X)$ is $\gamma_i$-compact in $(Y, \sigma_1, \sigma_2)$.*

**Theorem 9.** *If a bitopological space $(Y, \sigma_1, \sigma_2)$ is $\gamma_i$-compact for some operation $\gamma$ on $Y$, then every mapping $f$ of any bitopological space $(X, \tau_1, \tau_2)$ into $Y$ is $\gamma_i$-subcontinuous.*

**Proof.** Let $\Phi$ be an $i$-convergent filterbase in $X$. Then by Theorem 6, the filterbase $f(\Phi)$ $\gamma_i$-accumulates to some point of $Y$. Thus $f$ is $\gamma_i$-subcontinuous. ■

**Theorem 10.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping with $\gamma_i$-closed graph for some operation $\gamma$ on $(Y, \sigma_1, \sigma_2)$. If $(Y, \sigma_1, \sigma_2)$ is $\gamma_i$-compact, then $f$ is $\gamma_i$-continuous.*

**Corollary 17.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping with $ij-\theta$-closed graph. If $(Y, \sigma_1, \sigma_2)$ is $ij$-almost compact, then $f$ is $ij$-weakly continuous.*

**Corollary 18.** *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a mapping with $ij-\delta$-closed graph. If $(Y, \sigma_1, \sigma_2)$ is $ij$-nearly compact, then $f$ is $ij$-almost continuous.*

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