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Takashi Noiri and Valeriu Popa

# A GENERALIZATION OF $\omega$-CONTINUITY

Abstract. In this paper, by using $mg^*$-closed sets [23], we define and investigate the notion of $mg^*$-continuity which is a generalization of $\omega$-continuity [32] or $\hat{g}$-continuity [35].

Key words: $m$-structure, $g$-closed, $\omega$-closed, $mg^*$-closed, $\omega$-continuous, $mg^*$-continuous.

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## 1. Introduction

The concept of generalized closed (briefly $g$-closed) sets in topological spaces was introduced by Levine [15]. These sets also considered by Dunham [10] and Dunham and Levine [11]. In 1981, Munshi and Bassan [20] introduced the notion of generalized continuous (briefly $g$-continuous) functions. The notion of $g$-continuity is also studied in [5], [6], [7], [8] and other papers. The notions of $sg^*$-closed sets, $pg^*$-closed sets, $\alpha g^*$-closed sets, and $\beta g^*$-closed sets are introduced and studied in [21] by using semi-open sets, preopen sets, $\alpha$-open sets, and $\beta$-open sets, respectively. The notion of $sg^*$-closed sets is called $\omega$-closed [34], $\hat{g}$-closed [35], or semi-star generalized closed [30]. The notion of $\omega$-continuous (or $\hat{g}$-continuous) functions is introduced and studied in [35], [29], [33], and [34].

In [25] and [26], the present authors introduced and studied the notions of $m$-structures, $m$-spaces and $m$-continuity. In [23], the present authors introduced the notion of $mg^*$-closed sets in order to establish the unified theory of the notions: $sg^*$-closed sets, $pg^*$-closed sets, $\alpha g^*$-closed sets, and $\beta g^*$-closed sets and obtain the basic properties, characterizations and preserving properties.

In this paper, by using $mg^*$-closed sets and $m$-continuity, we introduce the notion of $mg^*$-continuous functions as a generalization of $\omega$-continuity or $\hat{g}$-continuity. We obtain some characterizations and properties of $mg^*$-continuity. In the last section, we introduce new generalizations of $\omega$-continuous functions.

## 2. Preliminaries

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. We recall some generalized open sets in topological spaces.

**Definition 1.** *Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be*
(1) *$\alpha$-open [22] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,*
(2) *semi-open [14] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,*
(3) *preopen [17] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,*
(4) *$\beta$-open [1] or semi-preopen [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.*

The family of all $\alpha$-open (resp. semi-open, preopen, $\beta$-open) sets in $(X, \tau)$ is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$).

**Definition 2.** *Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be $\alpha$-closed [18] (resp. semi-closed [9], preclosed [17], $\beta$-closed [1]) if the complement of $A$ is $\alpha$-open (resp. semi-open, preopen, $\beta$-open).*

**Definition 3.** *Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The intersection of all $\alpha$-closed (resp. semi-closed, preclosed, $\beta$-closed) sets of $X$ containing $A$ is called the* $\alpha$-closure *[18] (resp.* semi-closure *[9],* preclosure *[12],* $\beta$-closure *[2]) of $A$ and is denoted by* $\alpha\mathrm{Cl}(\mathrm{A})$ *(resp.* $\mathrm{sCl}(\mathrm{A})$, $\mathrm{pCl}(\mathrm{A})$, ${}_\beta\mathrm{Cl}(\mathrm{A})$*).*

**Definition 4.** *Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The union of all $\alpha$-open (resp. semi-open, preopen, $\beta$-open) sets of $X$ contained in $A$ is called the* $\alpha$-interior *[18] (resp.* semi-interior *[9],* preinterior *[12],* $\beta$-interior *[2]) of $A$ and is denoted by* $\alpha\mathrm{Int}(\mathrm{A})$ *(resp.* $\mathrm{sInt}(\mathrm{A})$, $\mathrm{pInt}(\mathrm{A})$, ${}_\beta\mathrm{Int}(\mathrm{A})$*).*

## 3. Minimal structures and $m$-continuity

**Definition 5.** *Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m_X$ of $\mathcal{P}(X)$ is called a* minimal structure *(briefly* $m$-structure*) on $X$ [25], [26] if $\emptyset \in m_X$ and $X \in m_X$.*

By $(X, m_X)$, we denote a nonempty set $X$ with an $m$-structure $m_X$ on $X$ and call it an $m$-space. Each member of $m_X$ is said to be *$m_X$-open* and the complement of an $m_X$-open set is said to be *$m_X$-closed*.

**Remark 1.** Let $(X, \tau)$ be a topological space. Then the family $\alpha(X)$ is a topology which is finer than $\tau$. The families $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, and $\beta(X)$ are all $m$-structures on $X$.

**Definition 6.** *Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$. For a subset $A$ of $X$, the* $m_X$-closure *of $A$ and the* $m_X$-interior *of $A$ are defined in [16] as follows:*
(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

**Remark 2.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$), then we have
(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_{\beta}\mathrm{Cl}(A)$),
(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_{\beta}\mathrm{Int}(A)$) .

**Lemma 1** (Maki et al. [16]). *Let $X$ be a nonempty set and $m_X$ a minimal structure on $X$. For subsets $A$ and $B$ of $X$, the following properties hold:*
(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,
(2) *If* $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

**Lemma 2** (Popa and Noiri [25]). *Let $X$ be a nonempty set with a minimal structure $m_X$ and $A$ a subset of $X$. Then $x \in \mathrm{mCl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m_X$ containing $x$.*

**Definition 7.** *A minimal structure $m_X$ on a nonempty set $X$ is said to have* property $\mathcal{B}$ *[16] if the union of any family of subsets belong to $m_X$ belongs to $m_X$.*

**Remark 3.** If $(X, \tau)$ is a topological space, then $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, and $\beta(X)$ have property $\mathcal{B}$.

**Lemma 3** (Popa and Noiri [28]). *Let $X$ be a nonempty set and $m_X$ an $m$-structure on $X$ satisfying property $\mathcal{B}$. For a subset $A$ of $X$, the following properties hold:*
(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) *$A$ is $m_X$-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is $m_X$-closed.*

**Definition 8.** *A function $f : (X, m_X) \to (Y, \sigma)$ is said to be $m$-continuous [26] if for each $x \in X$ and each open set $V$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$.*

**Theorem 1** (Popa and Noiri [26]). *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*
(1) *$f$ is $m$-continuous;*

(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every open set* $V$ *of* $Y$;
(3) $f^{-1}(F) = \mathrm{mCl}(f^{-1}(F))$ *for every closed set* $F$ *of* $Y$;
(4) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* $B$ *of* $Y$;
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset* $A$ *of* $X$;
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ *for every subset* $B$ *of* $Y$.

**Corollary 1** (Popa and Noiri [26]). *For a function* $f : (X, m_X) \to (Y, \sigma)$, *where* $m_X$ *has property* $\mathcal{B}$, *the following properties are equivalent:*
(1) $f$ *is m-continuous;*
(2) $f^{-1}(V)$ *is* $m_X$*-open in* $X$ *for every open set* $V$ *of* $Y$;
(3) $f^{-1}(F)$ *is* $m_X$*-closed in* $X$ *for every closed set* $F$ *of* $Y$.

**Definition 9.** *A function* $f : (X, m_X) \to (Y, \sigma)$ *is said to be* $m^*$*-continuous [19] if* $f^{-1}(V)$ *is* $m_X$*-open in* $X$ *for each open set* $V$ *of* $Y$.

**Remark 4.** (1) *If* $f : (X, m_X) \to (Y, \sigma)$ *is* $m^*$*-continuous, then it is m-continuous. By Example 3.4 of [19], an m-continuous function may not be* $m^*$*-continuous.*
(2) *Let* $m_X$ *have property* $\mathcal{B}$, *then it follows from Corollary 1 that* $f$ *is m-continuous if and only if* $f$ *is* $m^*$*-continuous.*

## 4. $m$-continuity and $mg^*$-continuity

**Definition 10.** *Let* $(X, \tau)$ *be a topological space. A subset* $A$ *of* $X$ *is said to be* $g$-closed *[15] (resp.* $sg^*$-closed, $pg^*$-closed, $\alpha g^*$-closed, $\beta g^*$-closed *[21]) if* $\mathrm{Cl}(A) \subset U$ *whenever* $A \subset U$ *and* $U$ *is open (resp. semi-open, preopen,* $\alpha$*-open,* $\beta$*-open) in* $X$.

**Remark 5.** (1) *An* $sg^*$*-closed set is called* $\omega$*-closed [34],* $\hat{g}$*-closed [35], or semi-star generalized closed [30].*
(2) *By the definitions, we obtain the following diagram:*

**DIAGRAM I**

$$\begin{array}{ccccc} g\text{-closed} \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & \\ & \Uparrow & & \Uparrow & \\ sg^*\text{-closed} \Leftarrow & \beta g^*\text{-closed} & \Leftarrow & \text{closed} & \end{array}$$

**Definition 11.** *A subset* $A$ *of a topological space* $(X, \tau)$ *is said to be* $g$-open *(resp.* $sg^*$-open *or* $\omega$-open, $pg^*$-open, $\alpha g^*$-open, $\beta g^*$-open *[21]) if* $X - A$ *is* $g$*-closed (resp.* $sg^*$*-closed,* $pg^*$*-closed,* $\alpha g^*$*-closed,* $\beta g^*$*-closed).*

The family of all $g$-open (resp. $sg^*$-open, $pg^*$-open, $\alpha g^*$-open, $\beta g^*$-open) sets of $X$ is denoted by $\mathrm{GO}(X)$ (resp. $\mathrm{SG^*O}(X)$ or $\omega(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$, $\beta\mathrm{G^*O}(X)$).

**Definition 12.** *Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The intersection of all $g$-closed (resp. $sg^*$-closed, $pg^*$-closed, $\alpha g^*$-closed, $\beta g^*$-closed) sets of $X$ containing $A$ is called the* $g$-closure *[10] (resp.* $sg^*$-closure *or* $\omega$-closure *[33],* $pg^*$-closure, $\alpha g^*$-closure, $\beta g^*$-closure*) of $A$ and is denoted by $g\mathrm{Cl}(A)$ (resp. $sg^*\mathrm{Cl}(A)$ or $\omega\mathrm{Cl}(A)$, $pg^*\mathrm{Cl}(A)$, $\alpha g^*\mathrm{Cl}(A)$, $\beta g^*\mathrm{Cl}(A)$).*

**Definition 13.** *Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The union of all $g$-open (resp. $sg^*$-open, $pg^*$-open, $\alpha g^*$-open, $\beta g^*$-open) sets of $X$ contained in $A$ is called the* $g$-interior *[8] (resp.* $sg^*$-interior, $pg^*$-interior, $\alpha g^*$-interior, $\beta g^*$-interior*) of $A$ and is denoted by $g\mathrm{Int}(A)$ (resp. $sg^*\mathrm{Int}(A)$, $pg^*\mathrm{Int}(A)$, $\alpha g^*\mathrm{Int}(A)$, $\beta g^*\mathrm{Int}(A)$).*

**Remark 6.** Let $(X, \tau)$ be a topological space and $A$ a subset of $X$.

(1) Then, $\mathrm{GO}(X)$, $\mathrm{SG^*O}(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$ and $\beta\mathrm{G^*O}(X)$ are all $m$-structures on $X$. Hence, if we put $m_X = \mathrm{GO}(X)$ (resp. $\mathrm{SG^*O}(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$, $\beta\mathrm{G^*O}(X)$), then we have

(*i*) $\mathrm{mCl}(A) = g\mathrm{Cl}(A)$ (resp. $sg^*\mathrm{Cl}(A)$, $pg^*\mathrm{Cl}(A)$, $\alpha g^*\mathrm{Cl}(A)$, $\beta g^*\mathrm{Cl}(A)$),

(*ii*) $\mathrm{mInt}(A) = g\mathrm{Int}(A)$ (resp. $sg^*\mathrm{Int}(A)$), $pg^*\mathrm{Int}(A)$, $\alpha g^*\mathrm{Int}(A)$, $\beta g^*\mathrm{Int}(A)$).

(2) If $m_X = \mathrm{GO}(X)$, then by Lemma 1 we obtain Theorem 2.1 (5) and Theorem 2.8 (2), (3), (5), (6), (7) established in [8]. By Lemma 2, we obtain Theorem 2.1 (4) in [8].

(3) If $m_X = \mathrm{SG^*O}(X)$, then by Lemma 2 we obtain Proposition 3.16 of [33].

(4) The $m$-structures $\mathrm{GO}(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$, $\beta\mathrm{G^*O}(X)$ do not have property $\mathcal{B}$, in general. However, it is known in [32] that $\mathrm{SG^*O}(X)$ is a topology for $X$.

**Definition 14.** *Let $(X, \tau)$ be a topological space and $m_X$ an $m$-structure on $X$. A subset $A$ of $X$ is said to be $mg^*$-closed [23] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.*

**Remark 7.** Let $(X, \tau)$ be a topological space and $m_X$ an $m$-structure on $X$. We put $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$). Then, an $mg^*$-closed set is a $g$-closed (resp. $sg^*$-closed, $pg^*$-closed, $\alpha g^*$-closed, $\beta g^*$-closed ) set.

**Lemma 4** (Noiri and Popa [23]). *Let $(X, \tau)$ be a topological space and $m_X$ an $m$-structure on $X$. Let $\tau \subset m_X$. Then the following implications hold: closed $\Rightarrow$ $mg^*$-closed $\Rightarrow$ $g$-closed.*

**Remark 8.** Let $(X, \tau)$ be a topological space and $m_X$ an $m$-structure on $X$. Let $\tau \subset m_X$.

(1) By Lemma 4, we obtain Propositions 2.4 and 2.5 in [33], Theorem 1.33 (a) and (c) in [30], and Theorem 3.01 in [35].

(2) The implications in Lemma 4 are strict as seen from Examples 3.01 and 3.02 in [35] or Examples 1.3.5 and 1.3.6 in [30].

The complement of an $mg^*$-closed set is said to be *$mg^*$-open*. The family of all $mg^*$-open sets is denoted by $\mathrm{mG^*O}(X)$. Obviously, $\mathrm{mG^*O}(X)$ is an $m$-structure on $X$ and is called an *$mg^*$-structure* on $X$. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$), then $\mathrm{mG^*O}(X) = \mathrm{GO}(X)$ (resp. $\mathrm{SG^*O}(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$, $\beta\mathrm{G^*O}(X)$).

**Definition 15.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\omega$-continuous [32] or $\hat{g}$-continuous [35] if $f^{-1}(K)$ is $\omega$-closed in $X$ for every closed set $K$ of $Y$.*

**Remark 9.** (1) Let $(X, \tau)$ be a topological space and $m_X = \mathrm{SG^*O}(X) = \omega(X)$, then $f : (X, \tau) \to (Y, \sigma)$ is $\omega$-continuous if and only if $f : (X, \omega(X)) \to (Y, \sigma)$ is $m^*$-continuous.

(2) By Lemma 4, we have the following implications: continuity $\Rightarrow$ $\omega$-continuity $\Rightarrow$ $g$-continuity. The implications are strict as seen from Examples 3.8 and 3.9 of [33].

**Definition 16.** *Let $(X, \tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be*

(1) *$mg^*$-continuous if $f : (X, \mathrm{mG^*O}(X)) \to (Y, \sigma)$ is $m$-continuous, equivalently if for each $x \in X$ and each open set $V$ containing $f(x)$ there exists an $mg^*$-open set $U$ containing $x$ such that $f(U) \subset V$,*

(2) *$m^*g^*$-continuous if $f : (X, \mathrm{mG^*O}(X)) \to (Y, \sigma)$ is $m^*$-continuous, equivalently if $f^{-1}(K)$ is $mg^*$-closed in $X$ for each closed set $K$ of $Y$.*

By DIAGRAM I and Definition 16, we obtain the following diagram:

**DIAGRAM II**

$$\begin{array}{ccccc} g\text{-continuity} & \Leftarrow & \alpha g^*\text{-continuity} & \Leftarrow & pg^*\text{-continuity} \\ & & \Uparrow & & \Uparrow \\ & & sg^*\text{-continuity} \Leftarrow & \beta g^*\text{-continuity} & \Leftarrow \text{continuity} \end{array}$$

**Definition 17.** *Let $X$ be a nonempty set and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$. For a subset $A$ of $X$, the $mg^*$-closure of $A$ and the $mg^*$-interior of $A$ are defined as follows:*

(1) $mg^*\mathrm{Cl}(A) = \cap\{F : A \subset F, X - F \in \mathrm{mG^*O}(X)\}$,

(2) $mg^*\mathrm{Int}(A) = \cup\{U : U \subset A, U \in \mathrm{mG^*O}(X)\}$.

By Theorem 1, we obtain the following theorem and corollary.

**Theorem 2.** *Let $(X, \tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) $f$ *is* $mg^*$*-continuous;*
(2) $f^{-1}(V) = mg^*\mathrm{Int}(f^{-1}(V))$ *for every open set* $V$ *of* $Y$*;*
(3) $f^{-1}(F) = mg^*\mathrm{Cl}(f^{-1}(F))$ *for every closed set* $F$ *of* $Y$*;*
(4) $mg^*\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset* $B$ *of* $Y$*;*
(5) $f(mg^*\mathrm{Cl}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset* $A$ *of* $X$*;*
(6) $f^{-1}(\mathrm{Int}(B)) \subset mg^*\mathrm{Int}(f^{-1}(B))$ *for every subset* $B$ *of* $Y$*.*

**Corollary 2.** *Let* $(X, \tau)$ *be a topological space and* $\mathrm{mG^*O}(X)$ *an* $mg^*$*-structure with property* $\mathcal{B}$ *on* $X$*. Then, for a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) $f$ *is* $m^*g^*$*-continuous;*
(2) $f^{-1}(V)$ *is* $mg^*$*-open in* $X$ *for every open set* $V$ *of* $Y$*;*
(3) $f^{-1}(F)$ *is* $mg^*$*-closed in* $X$ *for every closed set* $F$ *of* $Y$*.*

**Remark 10.** If $\mathrm{mG^*O}(X) = \mathrm{SG^*O}(X)$, then by Theorem 2 and Corollary 2 we obtain the result established in Theorem 3.17 of [33].

Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. A point $x \in X$ is called a $\theta$-cluster point of $A$ if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\theta$-cluster points of $A$ is called the *$\theta$-closure* of $A$ and is denoted by $\mathrm{Cl}_\theta(A)$ [36]. If $A = \mathrm{Cl}_\theta(A)$, then $A$ is said to be *$\theta$-closed.* The complement of a $\theta$-closed set is said to be *$\theta$-open.*

**Theorem 3.** *Let* $(Y, \sigma)$ *be a regular space. For a function* $f : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) $f$ *is m-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}(f^{-1}(\mathrm{Cl}_\theta(B)))$ *for every subset* $B$ *of* $Y$*;*
(3) $f^{-1}(K) = \mathrm{mCl}(f^{-1}(K))$ *for every* $\theta$*-closed set* $K$ *of* $Y$*;*
(4) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every* $\theta$*-open set* $V$ *of* $Y$*.*

**Proof.** It is known in [36] that $\mathrm{Cl}_\theta(B)$ is closed in $Y$ for every subset $B$ of $Y$. Since $(Y, \sigma)$ is regular, every open (resp. closed) set of $Y$ is $\theta$-open (resp. $\theta$-closed). Therefore, the proof follows easily from Theorem 1. ■

**Corollary 3.** *Let* $(Y, \sigma)$ *be a regular space and* $m_X$ *an m-structure with property* $\mathcal{B}$*. For a function* $f : (X, m_X) \to (Y, \sigma)$*, the following properties are equivalent:*
(1) $f$ *is m-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B))$ *is m-closed for every subset* $B$ *of* $Y$*;*
(3) $f^{-1}(K)$ *is m-closed in* $X$ *for every* $\theta$*-closed set* $K$ *of* $Y$*;*
(4) $f^{-1}(V)$ *is m-open in* $X$ *for every* $\theta$*-open set* $V$ *of* $Y$*.*

**Proof.** The proof follows from Lemma 3 and Theorem 3. ■

**Theorem 4.** *Let* $(Y, \sigma)$ *be a regular space and* $\mathrm{mG^*O}(X)$ *an* $mg^*$*-structure on* $X$*. For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) $f$ *is* $mg^*$*-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B)) = mg^*\mathrm{Cl}(f^{-1}(\mathrm{Cl}_\theta(B)))$ *for every subset* $B$ *of* $Y$*;*
(3) $f^{-1}(K) = mg^*\mathrm{Cl}(f^{-1}(K))$ *for every* $\theta$*-closed set* $K$ *of* $Y$*;*
(4) $f^{-1}(V) = mg^*\mathrm{mInt}(f^{-1}(V))$ *for every* $\theta$*-open set* $V$ *of* $Y$*.*

**Proof.** The proof follows from Definition 16 and Theorem 3. ■

**Corollary 4.** *Let* $(Y, \sigma)$ *be a regular space and* $\mathrm{mG^*O}(X)$ *an* $mg^*$*-structure with property* $\mathcal{B}$ *on* $X$*. For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) $f$ *is* $mg^*$*-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B))$ *is* $mg^*$*-closed in* $X$ *for every subset* $B$ *of* $Y$*;*
(3) $f^{-1}(K)$ *is* $mg^*$*-closed in* $X$ *for every* $\theta$*-closed set* $K$ *of* $Y$*;*
(4) $f^{-1}(V)$ *is* $mg^*$*-open in* $X$ *for every* $\theta$*-open set* $V$ *of* $Y$*.*

**Proof.** The proof follows from Theorem 4 and Lemma 3. ■

**Corollary 5.** *Let* $(Y, \sigma)$ *be a regular space. For a function* $f : (X, \tau) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) $f$ *is* $\omega$*-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B))$ *is* $\omega$*-closed in* $X$ *for every subset* $B$ *of* $Y$*;*
(3) $f^{-1}(K)$ *is* $\omega$*-closed in* $X$ *for every* $\theta$*-closed set* $K$ *of* $Y$*;*
(4) $f^{-1}(V)$ *is* $\omega$*-open in* $X$ *for every* $\theta$*-open set* $V$ *of* $Y$*.*

**Proof.** The proof follows from Corollary 4 because the family of $\omega$-open sets is a topology for $X$ and hence it has property $\mathcal{B}$. ■

## 5. Some properties of $mg^*$-continuity

In this section, we investigate the relationships between $mg^*$-continuity and $mg^*$-compactness, $mg^*$-connectedness and strongly $mg^*$-closed graphs.

**Definition 18.** *An* $m$*-space* $(X, m_X)$ *is said to be* $m$*-*$T_2$ *[26] if for any distinct points* $x, y$ *of* $X$*, there exist* $U, V \in m_X$ *such that* $x \in U, y \in V$*, and* $U \cap V = \emptyset$*.*

**Remark 11.** (1) Let $(X, \tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$, then $(X, \tau)$ is said to be $mg^*$-$T_2$ if the $m$-space $(X, \mathrm{mG^*}\ O(X))$ is $m$-$T_2$.
(2) If $\mathrm{mG^*O}(X) = \mathrm{SG^*O}(X) = \omega(X)$, then $(X, \tau)$ is said to be $\omega$-$T_2$ [29].

**Lemma 5** (Popa and Noiri [26]). *If $f:(X,m_X)\to(Y,\sigma)$ is an $m$-continuous injection and $(Y,\sigma)$ is a $T_2$-space, then $(X,m_X)$ is $m$-$T_2$.*

**Theorem 5.** *If $f:(X,\tau)\to(Y,\sigma)$ is an $m^*g^*$-continuous injection and $(Y,\sigma)$ is a $T_2$-space, then $(X,\tau)$ is $mg^*$-$T_2$.*

**Proof.** The proof follows from Remark 4 and Lemma 5. ■

**Corollary 6.** *If $f:(X,\tau)\to(Y,\sigma)$ is an $\omega$-continuous injection and $(Y,\sigma)$ is a $T_2$-space, then $(X,\tau)$ is $\omega$-$T_2$.*

**Definition 19.** *An $m$-space $(X,m_X)$ is said to be $m$-*compact *[26] if every cover of $X$ by sets of $m_X$ has a finite subcover.*

A subset $K$ of an $m$-space $(X,m_X)$ is said to be *$m$-compact* [26] if every cover of $K$ by subsets of $m_X$ has a finite subcover.

**Remark 12.** Let $(X,\tau)$ be a topological space and $\mathrm{mG^*O}(X))$ an $mg^*$-structure on $X$.

(1) $(X,\tau)$ is said to be *$mg^*$-compact* if $(X,\mathrm{mG^*O}(X))$ is $m$-compact.

(2) If $\mathrm{mG^*O}(X)=\mathrm{SG^*O}(X)=\omega(X)$, then $(X,\tau)$ is said to be $\omega$-compact.

**Lemma 6** (Popa and Noiri [26]). *If a function $f:(X,m_X)\to(Y,\sigma)$ is $m$-continuous and $K$ is an $m$-compact set of $X$, then $f(K)$ is compact.*

**Theorem 6.** *If $f:(X,\tau)\to(Y,\sigma)$ is an $m^*g^*$-continuous function and $K$ is an $mg^*$-compact set of $X$, then $f(K)$ is compact.*

**Proof.** The proof follows from Definition 19, Remark 4 and Lemma 6. ■

**Corollary 7.** *If $f:(X,\tau)\to(Y,\sigma)$ is an $\omega$-continuous function and $K$ is an $\omega$-compact set of $X$, then $f(K)$ is compact.*

**Definition 20.** *An $m$-space $(X,m_X)$ is said to be $m$-*connected *[26] if $X$ cannot be written as the union of two nonempty disjoint $m$-open sets.*

**Remark 13.** Let $(X,\tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$.

(1) $(X,\tau)$ is said to be *$mg^*$-connected* if $(X,\mathrm{mG^*O}(X))$ is $m$-connected.

(2) If $\mathrm{mG^*O}(X)=\mathrm{SG^*O}(X)=\omega(X)$, then $(X,\tau)$ is said to be $\omega$-connected [32].

**Lemma 7.** *If $f:(X,m_X)\to(Y,\sigma)$ is an $m^*$-continuous surjection and $(X,m_X)$ is $m$-connected, then $(Y,\sigma)$ is connected.*

**Proof.** Assume that $(Y,\sigma)$ is not connected. Then there exist nonempty open sets $V_1$ and $V_2$ such that $V_1\cap V_2=\emptyset$ and $V_1\cup V_2=Y$. Hence we have $f^{-1}(V_1)\cap f^{-1}(V_2)=\emptyset$ and $f^{-1}(V_1)\cup f^{-1}(V_2)=X$. Since $f$ is an

$m^*$-continuous surjection, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty $m$-open sets. Therefore, $(X, m_X)$ is not $m$-connected. This is a contradiction and hence $Y$ is connected. ■

**Theorem 7.** *Let $(X,\tau)$ be a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure on $X$. If $f : (X,\tau) \to (Y,\sigma)$ is an $m^*g^*$-continuous surjection and $(X,\tau)$ is $mg^*$-connected, then $(Y,\sigma)$ is connected.*

**Proof.** The proof follows from Definition 20, Remark 13 and Lemma 7. ■

**Corollary 8.** *If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega$-continuous surjection and $(X,\tau)$ is $\omega$-connected, then $(Y,\sigma)$ is connected.*

**Definition 21.** *A function $f : (X, m_X) \to (Y,\sigma)$ is said to have a* strongly $m$-closed graph (*resp.* $m$-closed graph) *[26] if for each $(x,y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing $x$ and an open set $V$ of $Y$ containing $y$ such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).*

**Remark 14.** Let $(X,\tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$.

(1) A function $f : (X,\tau) \to (Y,\sigma)$ is said to have a *strongly $mg^*$-closed graph* (resp. *$mg^*$-closed graph*) if for each $(x,y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in \mathrm{mG^*O}(X)$ containing $x$ and an open set $V$ of $Y$ containing $y$ such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

(2) If $\mathrm{mG^*O}(X) = \mathrm{GO}(X)$ (resp. $\mathrm{SG^*O}(X)$, $\mathrm{PG^*O}(X)$, $\alpha\mathrm{G^*O}(X)$, $\beta\mathrm{G^*O}(X)$) and $f$ has a strongly $mg^*$-closed graph, then $f$ has a strongly $g$-closed graph (resp. strongly $sg^*$-closed graph or strongly $\omega$-closed graph, strongly $pg^*$-closed graph, strongly $\alpha g^*$-closed graph, strongly $\beta g^*$-closed graph). For $mg^*$-closed graphs, we define similarly.

**Lemma 8** (Popa and Noiri [26])**.** *A function $f : (X, m_X) \to (Y,\sigma)$ is $m$-continuous and $(Y,\sigma)$ is a Hausdorff space, then $f$ has a strongly $m$-closed graph.*

**Theorem 8.** *Let $(X,\tau)$ be a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure on $X$. If a function $f : (X,\tau) \to (Y,\sigma)$ is $mg^*$-continuous and $(Y,\sigma)$ is a Hausdorff space, then $f$ has a strongly $mg^*$-closed graph.*

**Proof.** The proof follows from Definition 21, Remark 14 and Lemma 8. ■

**Corollary 9.** *If a function $f : (X,\tau) \to (Y,\sigma)$ is an $\omega$-continuous function and $(Y,\sigma)$ is a Hausdorff space, then $f$ has a strongly $\omega$-closed graph.*

**Lemma 9** (Popa and Noiri [26]). *Let $(X, m_X)$ be an m-space and $(Y, \sigma)$ a topological space. If $f : (X, m_X) \to (Y, \sigma)$ is a surjective function with a strongly m-closed graph, then Y is Hausdorff.*

**Theorem 9.** *Let $(X, \tau)$ be a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure on X. If $f : (X, \tau) \to (Y, \sigma)$ is a surjective function with a strongly $mg^*$-closed graph, then Y is Hausdorff.*

**Proof.** The proof follows from Definition 21 and Lemma 9. ■

**Corollary 10.** *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective function with a strongly $\omega$-closed graph, then Y is Hausdorff.*

**Lemma 10** (Popa and Noiri [26]). *Let $(X, m_X)$ be an m-space, where $m_X$ has property $\mathcal{B}$. If $f : (X, m_X) \to (Y, \sigma)$ is an m-continuous injection with an m-closed graph, then X is m-$T_2$.*

**Theorem 10.** *Let $(X, \tau)$ be a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure satisfying property $\mathcal{B}$. If $f : (X, \tau) \to (Y, \sigma)$ is an $mg^*$-continuous injection with an $mg^*$-closed graph, then X is $mg^*$-$T_2$.*

**Proof.** The proof follows from Definition 21, Remark 14 and Lemma 10. ■

**Corollary 11.** *If $f : (X, \tau) \to (Y, \sigma)$ is an injective $\omega$-continuous function with an $\omega$-closed graph, then Y is $\omega$-$T_2$.*

**Definition 22.** *Let $(X, m_X)$ be an m-space and A a subset of X. The* $m_X$-frontier *of A,* $\mathrm{mFr}(A)$*, [26] is defined by* $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A)$.

If $(X, \tau)$ is a topological space and $\mathrm{mG^*O}(X)$ is an $mg^*$-structure, then $mg^*\mathrm{Fr}(A) = mg^*\mathrm{Cl}(A) \cap mg^*\mathrm{Cl}(X - A) = mg^*\mathrm{Cl}(A) - mg^*\mathrm{Int}(A)$. If $\mathrm{mG^*O}(X) = \mathrm{GO}(X)$, then we obtain the $g$-frontier in Definition 4 of [8].

**Theorem 11.** *The set of all points of X at which a function $f : (X, m_X) \to (Y, \sigma)$ is not m-continuous is identical with the union of the m-frontiers of the inverse images of open sets containing f(x).*

**Proof.** Suppose that $f$ is not $m$-continuous at $x \in X$. There exists an open set $V$ of $Y$ containing $f(x)$ such that $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every $m$-open set $U$ containing $x$. By Lemma 2, we have $x \in \mathrm{mCl}(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V)$ and hence $x \in \mathrm{mFr}(f^{-1}(V))$.

Conversely, suppose that $f$ is $m$-continuous at $x \in X$. Then, for any open set $V$ of $Y$ containing $f(x)$, there exists $U \in m_X$ containing $x$ such that $f(U) \subset V$; hence $U \subset f^{-1}(V)$. Therefore, we have $x \in U \subset \mathrm{mInt}(f^{-1}(V))$. This contradicts to the fact that $x \in \mathrm{mFr}(f^{-1}(V))$. ■

**Theorem 12.** *Let $(X, \tau)$ is a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure. Then, the set of all points of $X$ at which a function $f : (X, \tau) \to (Y, \sigma)$ is not $mg^*$-continuous is identical with the union of the $mg^*$-frontiers of the inverse images of open sets containing $f(x)$.*

**Proof.** The proof follows from Definition 22 and Theorem 11. ■

**Corollary 12.** *The set of all points at $x \in X$ which a function $f : (X, \tau) \to (Y, \sigma)$ is not $\omega$-continuous is identical with the union of the $\omega$-frontiers of the inverse images of open sets containing f(x).*

**Proof.** Since $\omega(X)$ is a topology for $X$, $m$-continuity coincides with $m^*$-continuity and hence the result follows from Theorem 12. ■

For a function $f : (X, m_X) \to (Y, \sigma)$, we define $D_m(f)$ as follows:

$$D_m(f) = \{x \in X : f \text{ is not } m\text{-continuous at } x\}.$$

**Lemma 11** (Popa and Noiri [27]). *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D_m(f) &= \bigcup_{G\in\sigma}\{f^{-1}(G) - \mathrm{mInt}(f^{-1}(G))\} \\ &= \bigcup_{B\in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) - \mathrm{mInt}(f^{-1}(B))\} \\ &= \bigcup_{B\in \mathcal{P}(Y)} \{\mathrm{mCl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in \mathcal{P}(X)} \{\mathrm{mCl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} \\ &= \bigcup_{F\in \mathcal{F}} \{\mathrm{mCl}(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

*where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.*

Let $(X, \tau)$ be a topological space and $\mathrm{mG^*O}(X)$ an $mg^*$-structure on $X$. For a function $f : (X, \tau) \to (Y, \sigma)$, we denote by $D_{mg^*}(f)$ the set of all points of $X$ at which the function $f$ is not $mg^*$-continuous.

**Theorem 13.** *Let $(X, \tau)$ be a topological space and* $\mathrm{mG^*O}(X)$ *an $mg^*$-structure on $X$. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D_{mg^*}(f) &= \bigcup_{G\in\sigma}\{f^{-1}(G) - mg^*\mathrm{Int}(f^{-1}(G))\} \\ &= \bigcup_{B\in \mathcal{P}(Y)} \{f^{-1}(\mathrm{Int}(B)) - mg^*\mathrm{Int}(f^{-1}(B))\} \\ &= \bigcup_{B\in \mathcal{P}(Y)} \{mg^*\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in \mathcal{P}(X)} \{mg^*\mathrm{Cl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} \\ &= \bigcup_{F\in \mathcal{F}} \{mg^*\mathrm{Cl}(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

*where $\mathcal{F}$ is the family of closed sets of $(Y, \sigma)$.*

**Proof.** The proof follows from Lemma 11. ■

Let $f : (X, \tau) \to (Y, \sigma)$ be a function. By $D_\omega(f)$, we denote the set of all points $x \in X$ at which $f$ is not $\omega$-continuous. Then by Theorem 13 we obtain the following corollary.

**Corollary 13.** *For a function $f : (X,\tau) \to (Y,\sigma)$, the following properties hold:*

$$\begin{aligned} D_\omega(f) &= \bigcup_{G\in\sigma}\{f^{-1}(G) - \omega\mathrm{Int}(f^{-1}(G))\} \\ &= \bigcup_{B\in\,\mathcal{P}(Y)}\{f^{-1}(\mathrm{Int}(B)) - \omega\mathrm{Int}(f^{-1}(B))\} \\ &= \bigcup_{B\in\,\mathcal{P}(Y)}\{\omega\mathrm{Cl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} \\ &= \bigcup_{A\in\,\mathcal{P}(X)}\{\omega\mathrm{Cl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} \\ &= \bigcup_{F\in\,\mathcal{F}}\{\omega\mathrm{Cl}(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

*where $\mathcal{F}$ is the family of closed sets of $(Y,\sigma)$.*

## 6. Other generalizations of $\omega$-continuity

First, we recall the $\delta$-closure of a subset in a topological space. Let $(X,\tau)$ be a topological space and $A$ a subset of $X$. A point $x \in X$ is called a $\delta$-cluster point of $A$ if $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$ for every open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the *$\delta$-closure* of $A$ and is denoted by $\mathrm{Cl}_\delta(A)$ [36].

**Definition 23.** *A subset of a topological space $(X,\tau)$ is said to be*

(1) *$\delta$-preopen [31] (resp. $\theta$-preopen [24]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),*

(2) *$\delta$-$\beta$-open [13] (resp. $\theta$-$\beta$-open [24]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$),*

(3) *$b$-open [4] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$.*

By $\delta\mathrm{PO}(X)$ (resp. $\delta\beta(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$, $\mathrm{BO}(X)$), we denote the collection of all $\delta$-preopen (resp. $\delta$-$\beta$-open, $\theta$-preopen, $\theta$-$\beta$-open, $b$-open) sets of a topological space $(X,\tau)$. These five collections are $m$-structures with property $\mathcal{B}$. In [24], the following diagram is known:

**DIAGRAM III**

$$\begin{array}{ccccccc} \alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ \text{semi-open} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open} & \Rightarrow & \theta\text{-}\beta\text{-open} \end{array}$$

For the new collections of subsets of a topological space $(X,\tau)$, we can define many new variations of $g$-closed sets. For example, in case $m_X = \delta\mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$, or $\mathrm{BO}(X)$ we can define new types of $g$-closed sets as follows:

**Definition 24.** *A subset $A$ of a topological space $(X,\tau)$ is said to be $\delta pg^*$-*closed (*resp. $\theta pg^*$-*closed, $\delta\beta g^*$-*closed, $\theta\beta g^*$-*closed, $bg^*$-closed) if Cl(A) $\subset U$ whenever $A \subset U$ and $U$ is $\delta$-preopen (resp. $\theta$-preopen, $\delta$-$\beta$-open, $\theta$-$\beta$-open, $b$-open) in $(X,\tau)$.*

By DIAGRAM I and Definition 24, we have the following diagram:

**DIAGRAM IV**

$$\begin{array}{ccccccccccc}
g\text{-closed} & \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & \Leftarrow & \delta pg^*\text{-closed} & \Leftarrow & \theta pg^*\text{-closed} & & \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & sg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed} & \Leftarrow & \delta\beta g^*\text{-closed} & \Leftarrow & \theta\beta g^*\text{-closed} & \Leftarrow & \text{closed}
\end{array}$$

**Definition 25.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be* $\delta pg^*$-continuous (*resp.* $\theta pg^*$-continuous, $\delta\beta g^*$-continuous, $\theta\beta g^*$-continuous, *$bg^*$-continuous*) *if $f^{-1}(K)$ is $\delta pg^*$-closed* (*resp. $\theta pg^*$-closed, $\delta\beta g^*$-closed, $\theta\beta g^*$-closed, $bg^*$-closed*) *in $X$ for each closed set $K$ of $Y$.*

Finally, we have to state the following remark:

**Remark 15.** The families $\delta\mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$, and $\mathrm{BO}(X)$ have property $\mathcal{B}$ and we can apply the results established in Sections 4 and 5 to the functions in Definition 25.

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TAKASHI NOIRI
2949-1 SHIOKITA-CHO, HINAGU
YATSUSHIRO-SHI, KUMAMOTO-KEN,
869-5142, JAPAN
*e-mail:* t.noiri@nifty.com

VALERIU POPA
DEPARTMENT OF MATHEMATICS
UNIV. VASILE ALECSANDRI OF BACĂU
600 114 BACĂU, ROMANIA
*e-mail:* vpopa@ub.ro

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