F A S C I C U L I   M A T H E M A T I C I

Nr 45 2010

Baburao G. Pachpatte

APPROXIMATE SOLUTIONS OF A CERTAIN SECOND ORDER INTEGRODIFFERENTIAL EQUATION

Abstract. The main aim of this paper is to study the approximate solutions of a certain second order Volterra type integrodifferential equation with given initial values. A variant of a certain basic integral inequality with explicit estimate is used to establish the results.

Key words: approximate solutions, second order, integrodifferential equation, Volterra type, integral inequality, explicit estimate, closeness of solutions.

AMS Mathematics Subject Classification: 34K10, 35R10.

1. Introduction

Consider the initial value problem (IVP for short) for the second order Volterra type integrodifferential equation

$$x''(t) = f\left(t, x(t), x'(t), Hx(t)\right), \tag{1}$$

for $t \in R_+ = [0, \infty)$, with the given initial conditions

$$x(0) = x_0, \; x'(0) = \bar{x}_0, \tag{2}$$

where

$$Hx(t) := \int_0^t h\left(t, \sigma, x(\sigma), x'(\sigma)\right) d\sigma, \tag{3}$$

f, h are given functions, x is the unknown function to be found and $'$ denotes the derivative. We assume that $f \in C(R_+ \times R^n \times R^n \times R^n, R^n)$ and for $\sigma \leq t$; $h \in C\left(R_+^2 \times R^n \times R^n, R^n\right)$, where R^n denotes the n-dimensional Euclidean space with appropriate norm denoted by $|.|$. The problem of existence and some other fundamental properties of solutions of more general versions of IVP (1)-(2) are recently dealt with in [8, 9] for $t \in [0, b] \subset R_+$, see also [1-5, 10].

In general, the solutions to the IVP (1)-(2) cannot be found analytically and thus will need a new insight to handle the qualitative properties of its solutions. The method of approximations to the solutions is a very powerful tool which provides valuable information, without the need to know in advance the solutions explicitly of various dynamic equations. In the present paper, we apply the method of approximations to the solutions of IVP (1)-(2) and investigate new estimate on the difference between the two approximate solutions of equation (1) and convergence properties of solutions of approximate problems. The main tool employed in the analysis is based on the application of a variant of a certain integral inequality with explicit estimate due to the present author given in [7] (see also [6]).

2. Main results

Let $x_i(t) \in C(R_+, R^n)$ $(i = 1, 2)$ be functions such that $x_i''(t)$ exist for $t \in R_+$ and satisfy the inequalities

$$\left|x_i''(t) - f\left(t, x_i(t), x_i'(t), Hx_i(t)\right)\right| \le \varepsilon_i, \tag{4}$$

for given constants $\varepsilon_i \ge 0$, where it is assumed that the initial conditions

$$x_i(0) = x_i, \qquad x_i'(0) = \bar{x}_i, \tag{5}$$

are fulfilled. Then we call $x_i(t)$ the ε_i-approximate solutions with respect to the equation (1).

The following variant of the integral inequality established by the present author in [7, p. 152] is crucial in the proof of our main results.

Lemma. *Let $u, e, b \in C(R_+, R_+)$ and for $s \le t$; $a(t,s)$, $c(t,s) \in C(R_+^2, R_+)$. If $e(t)$ and $a(t,s)$ be nondecreasing in $t \in R_+$ and*

$$u(t) \le e(t) + \int_0^t a(t,s)\left[b(s)u(s) + \int_0^s c(s,\sigma)u(\sigma)\,d\sigma\right]ds, \tag{6}$$

for $t \in R_+$, then

$$u(t) \le e(t)\exp\left(\int_0^t a(t,s)\left[b(s) + \int_0^s c(s,\sigma)\,d\sigma\right]ds\right), \tag{7}$$

for $t \in R_+$.

Proof. First we assume that $e(t)$ is positive and fix $T \in R_+$. Then from (6) it is easy to observe that

$$\frac{u(t)}{e(t)} \le 1 + \int_0^t a(T,s)\left[b(s)\frac{u(s)}{e(s)} + \int_0^s c(s,\sigma)\frac{u(\sigma)}{e(\sigma)}d\sigma\right]ds, \tag{8}$$

on $0 \le t \le T$. Define $z(t)$ by the right hand side of (8). Then $z(0) = 1$, $\frac{u(t)}{e(t)} \le z(t)$, $z(t)$ is nondecreasing for $0 \le t \le T$ and

$$z'(t) = a(T,t)\left[b(t)\frac{u(t)}{e(t)} + \int_0^t c(t,\sigma)\frac{u(\sigma)}{e(\sigma)}d\sigma\right] \le a(T,t)\left[b(t) + \int_0^t c(t,\sigma)\,d\sigma\right]z(t). \tag{9}$$

The inequality (9) implies

$$z(t) \le \exp\left(\int_0^t a(T,s)\left[b(s) + \int_0^s c(s,\sigma)\,d\sigma\right]ds\right), \tag{10}$$

for $0 \le t \le T$. Since $T \in R_+$ is arbitrary, from (10) with T replaced by $t \in R_+$ and the fact that $\frac{u(t)}{e(t)} \le z(t)$, we get (7). If $e(t)$ is nonnegative, we carry out the above procedure with $e(t) + \varepsilon$ instead of $e(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (7). ■

Our main result given in the following theorem estimates the difference between the two approximate solutions of equation (1).

Theorem 1. *Suppose that the functions f, h in equation (1) satisfy the conditions*

$$|f(t,x,y,z) - f(t,\bar{x},\bar{y},\bar{z})| \le p(t)\left[|x-\bar{x}| + |y-\bar{y}|\right] + |z-\bar{z}|, \tag{11}$$

$$|h(t,\sigma,x,y) - h(t,\sigma,\bar{x},\bar{y})| \le q(t,\sigma)\left[|x-\bar{x}| + |y-\bar{y}|\right], \tag{12}$$

where $p \in C(R_+, R_+)$ and for $\sigma \le t; q(t,\sigma) \in C\left(R_+^2, R_+\right)$. Let $x_i(t)$ $(i = 1,2)$ be respectively ε_i-approximate solutions of equation (1) with (5) on R_+ such that

$$|x_1 - x_2| \le \delta, \quad |\bar{x}_1 - \bar{x}_2| \le \bar{\delta}, \tag{13}$$

where δ, $\bar{\delta}$ are nonnegative constants. Then

$$|x_1(t) - x_2(t)| + \left|x_1'(t) - x_2'(t)\right| \le m(t)\exp\left(\int_0^t (t-s+1)\left[p(s) + \int_0^s q(s,\sigma)\,d\sigma\right]ds\right), \tag{14}$$

for $t \in R_+$, *where*

$$m(t) = (\varepsilon_1 + \varepsilon_2)\left(\frac{t^2}{2} + t\right) + (t+1)\,\bar{\delta} + \delta. \tag{15}$$

Proof. Since $x_i(t)$ $(i = 1, 2)$ for $t \in R_+$ are respectively ε_i-approximate solutions of equation (1) with (5), we have (4). By taking $t = \tau$ in (4) and integrating both sides with respect to τ from 0 to t, we have

$$\begin{aligned} \varepsilon_i t &\geq \int_0^t \left| x_i''(\tau) - f\left(\tau, x_i(\tau), x_i'(\tau), Hx_i(\tau)\right) \right| d\tau \\ &\geq \left| \int_0^t \left\{ x_i''(\tau) - f\left(\tau, x_i(\tau), x_i'(\tau), Hx_i(\tau)\right) \right\} d\tau \right| \\ &= \left| \left\{ x_i'(t) - \bar{x}_i - \int_0^t f\left(\tau, x_i(\tau), x_i'(\tau), Hx_i(\tau)\right) d\tau \right\} \right|. \end{aligned} \tag{16}$$

By taking $t = s$ in (16) and integrating both sides with respect to s from 0 to t, we have

$$\begin{aligned} \varepsilon_i \frac{t^2}{2} &\geq \int_0^t \left| \left\{ x_i'(s) - \bar{x}_i - \int_0^s f\left(\tau, x_i(\tau), x_i'(\tau), Hx_i(\tau)\right) d\tau \right\} \right| ds \\ &\geq \left| \int_0^t \left\{ x_i'(s) - \bar{x}_i - \int_0^s f\left(\tau, x_i(\tau), x_i'(\tau), Hx_i(\tau)\right) d\tau \right\} ds \right| \\ &= \left| \left\{ x_i(t) - [x_i + \bar{x}_i t] - \int_0^t (t-s) f\left(s, x_i(s), x_i'(s), Hx_i(s)\right) ds \right\} \right|. \end{aligned} \tag{17}$$

From (16), (17) and using the elementary inequalities

$$|v - z| \leq |v| + |z|, \ \ |v| - |z| \leq |v - z|, \tag{18}$$

we observe that

$$\begin{aligned} (\varepsilon_1 + \varepsilon_2)\, t \geq & \left| \left\{ x_1'(t) - \bar{x}_1 - \int_0^t f\left(s, x_1(s), x_1'(s), Hx_1(s)\right) ds \right\} \right| \\ & + \left| \left\{ x_2'(t) - \bar{x}_2 - \int_0^t f\left(s, x_2(s), x_2'(s), Hx_2(s)\right) ds \right\} \right| \end{aligned} \tag{19}$$

$$\geq \left|\left\{x_1'(t)-\bar{x}_1-\int_0^t f\left(s,x_1(s),x_1'(s),Hx_1(s)\right)ds\right\}\right.$$
$$\left.-\left\{x_2'(t)-\bar{x}_2-\int_0^t f\left(s,x_2(s),x_2'(s),Hx_2(s)\right)ds\right\}\right|$$
$$\geq \left|x_1'(t)-x_2'(t)\right|-\left|\bar{x}_1-\bar{x}_2\right|$$
$$-\int_0^t \left|f\left(s,x_1(s),x_1'(s),Hx_1(s)\right)\right.$$
$$\left.-\ f\left(s,x_2(s),x_2'(s),Hx_2(s)\right)\right|ds,$$

and

$$(20)\ (\varepsilon_1+\varepsilon_2)\frac{t^2}{2}$$
$$\geq \left|\left\{x_1(t)-[x_1+\bar{x}_1t]-\int_0^t (t-s)f\left(s,x_1(s),x_1'(s),Hx_1(s)\right)ds\right\}\right|$$
$$+\left|\left\{x_2(t)-[x_2+\bar{x}_2t]-\int_0^t (t-s)f\left(s,x_2(s),x_2'(s),Hx_2(s)\right)ds\right\}\right|$$
$$\geq \left|\left\{x_1(t)-[x_1+\bar{x}_1t]-\int_0^t (t-s)f\left(s,x_1(s),x_1'(s),Hx_1(s)\right)ds\right\}\right.$$
$$\left.-\left\{x_2(t)-[x_2+\bar{x}_2t]-\int_0^t (t-s)f\left(s,x_2(s),x_2'(s),Hx_2(s)\right)ds\right\}\right|$$
$$\geq \left|x_1(t)-x_2(t)\right|-\left|[x_1+\bar{x}_1t]-[x_2+\bar{x}_2t]\right|$$
$$-\int_0^t \left|(t-s)f\left(s,x_1(s),x_1'(s),Hx_1(s)\right)\right.$$
$$\left.-\ (t-s)f\left(s,x_2(s),x_2'(s),Hx_2(s)\right)\right|ds.$$

Let $u(t)=\left|x_1(t)-x_2(t)\right|+\left|x_1'(t)-x_2'(t)\right|, t\in R_+$. From (19) and (20) and using the hypotheses, we observe that

$$(21)\qquad u(t)\leq(\varepsilon_1+\varepsilon_2)\frac{t^2}{2}+\left|(x_1-x_2)+(\bar{x}_1-\bar{x}_2)t\right|$$
$$+\int_0^t (t-s)\left[p(s)u(s)+\int_0^s q(s,\sigma)u(\sigma)d\sigma\right]ds$$

$$+ \ (\varepsilon_1 + \varepsilon_2) t + |\bar{x}_1 - \bar{x}_2| + \int_0^t \left[p(s) u(s) + \int_0^s q(s, \sigma) u(\sigma) d\sigma \right] ds$$

$$\leq (\varepsilon_1 + \varepsilon_2) \left(\frac{t^2}{2} + t \right) + \{ |x_1 - x_2| + |\bar{x}_1 - \bar{x}_2| t + |\bar{x}_1 - \bar{x}_2| \}$$

$$+ \int_0^t (t - s + 1) \left[p(s) u(s) + \int_0^s q(s, \sigma) u(\sigma) d\sigma \right] ds$$

$$\leq m(t) + \int_0^t (t - s + 1) \left[p(s) u(s) + \int_0^s q(s, \sigma) u(\sigma) d\sigma \right] ds,$$

where $m(t)$ is given by (15). Clearly $m(t)$ is nondecreasing for $t \in R_+$. Now a suitable application of Lemma to (21) yields (14). ■

Remark 1. *We note that the estimate obtained in (14) yields not only the bound on the difference between the two approximate solutions of equation (1) but also the bound on the difference between their derivatives. If $x_1(t)$ is a solution of equation (1) with $x_1(0) = x_1$, $x_1'(0) = \bar{x}_1$, then we have $\varepsilon_1 = 0$ and from (14), we see that $x_2(t) \to x_1(t)$ as $\varepsilon_2 \to 0$ and $\delta \to 0, \bar{\delta} \to 0$. Moreover, if we put (i) $\varepsilon_1 = \varepsilon_2 = 0$ and $x_1 = x_2$, $\bar{x}_1 = \bar{x}_2$ in (14), then the uniqueness of solutions of equation (1) is established and (ii) $\varepsilon_1 = \varepsilon_2 = 0$ in (14), then we get the bound which shows the dependency of solutions of equation (1) on given initial values.*

Consider the IVP (1)-(2) together with

$$y''(t) = g\left(t, y(t), y'(t), Hy(t)\right), \tag{22}$$

$$y(0) = y_0, \qquad y'(0) = \bar{y}_0, \tag{23}$$

for $t \in R_+$, where H is given by (3) and $g \in C(R_+ \times R^n \times R^n \times R^n, R^n)$.

The following theorem concerns the closeness of solutions of IVP (1)-(2) and IVP (22)-(23).

Theorem 2. *Suppose that the functions f, h in equation (1) satisfy the conditions (11), (12) and there exist nonnegative constants $\bar{\varepsilon}$, δ_0, $\bar{\delta}_0$ such that*

$$|f(t, u, v, w) - g(t, u, v, w)| \leq \bar{\varepsilon}, \tag{24}$$

$$|x_0 - y_0| \leq \delta_0, \qquad |\bar{x}_0 - \bar{y}_0| \leq \bar{\delta}_0, \tag{25}$$

where f, x_0, $\bar{x}_0$ and g, y_0, $\bar{y}_0$ are as in IVP (1)-(2) and IVP (22)-(23). Let $x(t)$ and $y(t)$ be respectively, solutions of IVP (1)-(2) and IVP (22)-(23) on R_+. Then

$$|x(t)-y(t)|+\left|x'(t)-y'(t)\right| \leq n(t)\exp\left(\int_0^t (t-s+1)\left[p(s)+\int_0^s q(s,\sigma)\,d\sigma\right]ds\right), \tag{26}$$

for $t \in R_+$, where

$$n(t)=\bar{\varepsilon}\left(\frac{t^2}{2}+t\right)+(t+1)\,\bar{\delta}_0+\delta_0. \tag{27}$$

Proof. Let $v(t)=|x(t)-y(t)|+|x'(t)-y'(t)|$, $t \in R_+$. Using the facts that $x(t)$ and $y(t)$ are the solutions of IVP (1)-(2) and IVP (22)-(23) and hypotheses, we observe that

$$\begin{aligned}(28)\ v(t) \leq\ & \Bigg|\left\{[x_0+\bar{x}_0 t]+\int_0^t (t-s)\,f\left(s,x(s),x'(s),Hx(s)\right)ds\right\} \\ & -\left\{[y_0+\bar{y}_0 t]+\int_0^t (t-s)\,g\left(s,y(s),y'(s),Hy(s)\right)ds\right\}\Bigg| \\ & +\Bigg|\left\{\bar{x}_0+\int_0^t f\left(s,x(s),x'(s),Hx(s)\right)ds\right\} \\ & -\left\{\bar{y}_0+\int_0^t g\left(s,y(s),y'(s),Hy(s)\right)ds\right\}\Bigg|\end{aligned}$$

$$\begin{aligned}&\leq |x_0-y_0|+|\bar{x}_0-\bar{y}_0|\,t \\ &+\int_0^t (t-s)\left|f\left(s,x(s),x'(s),Hx(s)\right)-f\left(s,y(s),y'(s),Hy(s)\right)\right|ds \\ &+\int_0^t (t-s)\left|f\left(s,y(s),y'(s),Hy(s)\right)-g\left(s,y(s),y'(s),Hy(s)\right)\right|ds \\ &+|\bar{x}_0-\bar{y}_0| \\ &+\int_0^t \left|f\left(s,x(s),x'(s),Hx(s)\right)-f\left(s,y(s),y'(s),Hy(s)\right)\right|ds\end{aligned}$$

$$+\int_0^t \left|f\left(s,y\left(s\right),y'\left(s\right),Hy\left(s\right)\right)-g\left(s,y\left(s\right),y'\left(s\right),Hy\left(s\right)\right)\right|ds$$

$$\leq \delta_0+\bar{\delta}_0 t+\int_0^t (t-s)\left[p\left(s\right)v\left(s\right)+\int_0^s q\left(s,\sigma\right)v\left(\sigma\right)d\sigma\right]ds$$

$$+\int_0^t (t-s)\,\bar{\varepsilon}ds+\bar{\delta}_0+\int_0^t \left[p\left(s\right)v\left(s\right)+\int_0^s q\left(s,\sigma\right)v\left(\sigma\right)d\sigma\right]ds$$

$$+\int_0^t \bar{\varepsilon}ds=n\left(t\right)+\int_0^t (t-s+1)\left[p\left(s\right)v\left(s\right)+\int_0^s q\left(s,\sigma\right)v\left(\sigma\right)d\sigma\right]ds,$$

for $t\in R_+$, where $n(t)$ is given by (27). Clearly $n(t)$ is nondecreasing for $t\in R_+$. Now an application of Lemma to (28) yields (26). ∎

Remark 2. The result given in Theorem 2 relates the solutions of IVP (1)-(2) and of IVP (22)-(23) in the sense that if f is close to g, x_0 is close to y_0 and $\bar{x}_0$ is close to $\bar{y}_0$, then the solutions of IVP (1)-(2) and of IVP (22)-(23) are also close to each other.

Next, consider the IVP (1)-(2) together with

$$y''\left(t\right)=f_k\left(t,y\left(t\right),y'\left(t\right),Hy\left(t\right)\right), \tag{29}$$

$$y\left(0\right)=\alpha_k,\qquad y'\left(0\right)=\bar{\alpha}_k, \tag{30}$$

for $k=1,2,...$, where H is given by (3), α_k, $\bar{\alpha}_k$ are sequences in R^n and $f_k\in C\left(R_+\times R^n\times R^n\times R^n,R^n\right)$.

As an immediate consequence of Theorem 2, we have the following corollary.

Corollary 1. *Suppose that the functions f, h in equation (1) satisfy the conditions (11), (12) and there exist nonnegative constants ε_k, δ_k, $\bar{\delta}_k$ $(k=1,2,...)$ such that*

$$\left|f\left(t,u,v,w\right)-f_k\left(t,u,v,w\right)\right|\leq\varepsilon_k, \tag{31}$$

$$\left|x_0-\alpha_k\right|\leq\delta_k,\qquad\left|\bar{x}_0-\bar{\alpha}_k\right|\leq\bar{\delta}_k, \tag{32}$$

with $\varepsilon_k\to 0$ and $\delta_k\to 0$, $\bar{\delta}_k\to 0$ as $k\to\infty$, where f, x_0, $\bar{x}_0$ and f_k, α_k, $\bar{\alpha}_k$ are as in (1), (2) and (29), (30). If $y_k(t)$ $(k=1,2,...)$ and $x(t)$ are respectively the solutions of IVPs (29)-(30) and IVP (1)-(2) on R_+, then as $k\to\infty$, $y_k\left(t\right)\to x\left(t\right)$ on R_+.

Proof. For $k = 1, 2, ...$, the conditions of Theorem 2 hold. As an application of Theorem 2 yields

$$(33)\qquad |y_k(t) - x(t)| + |y_k'(t) - x'(t)| \\ \le n_k(t) \exp\left(\int_0^t (t-s+1)\left[p(s) + \int_0^s q(s,\sigma)\, d\sigma\right] ds\right),$$

for $t \in R_+$ and $k = 1, 2, ...$, where

$$(34)\qquad n_k(t) = \bar{\varepsilon}_k \left(\frac{t^2}{2} + t\right) + (t+1)\bar{\delta}_k + \delta_k.$$

■

The required result follows from (33).

Remark 3. The result obtained in Corollary provide sufficient conditions that ensures solutions of IVPs (29)-(30) will converge to the solutions of IVP (1)-(2). For some other qualitative properties of solutions of more general versions of IVP (1)-(2), we refer the interested readers to [8-10].

A slight variant of Theorem 2 is given in the following theorem.

Theorem 3. *Suppose that*

$$(35)\qquad |f(t,u,v,w) - g(t,\bar{u},\bar{v},\bar{w})| \le \bar{p}(t)\left[|u-\bar{u}| + |v-\bar{v}|\right] + |w-\bar{w}|,$$

where $\bar{p} \in C(R_+, R_+)$ and the conditions (12) and (25) hold. Let $x(t)$ and $y(t)$ be respectively, solutions of IVP (1)-(2) and IVP (22)-(23) on R_+. Then

$$(36)\qquad |x(t) - y(t)| + |x'(t) - y'(t)| \\ \le r(t) \exp\left(\int_0^t (t-s+1)\left[\bar{p}(s) + \int_0^s q(s,\sigma)\, d\sigma\right] ds\right),$$

for $t \in R_+$, where

$$(37)\qquad r(t) = (t+1)\bar{\delta}_0 + \delta_0.$$

Proof. Define $v(t)$ as in the proof of Theorem 2. Using the facts that $x(t)$ and $y(t)$ are respectively, solutions of IVP (1)-(2) and IVP (22)-(23) and hypotheses, we observe that

$$(38)\quad v(t) \le |[x_0 + \bar{x}_0 t] - [y_0 + \bar{y}_0 t]| \\ + \int_0^t (t-s)\left|f(s, x(s), x'(s), Hx(s)) - g(s, y(s), y'(s), Hy(s))\right| ds$$

$$+ |\bar{x}_0 - \bar{y}_0|$$

$$+ \int_0^t \left| f\left(s, x(s), x'(s), Hx(s)\right) - g\left(s, y(s), y'(s), Hy(s)\right)\right| ds$$

$$\leq |x_0 - y_0| + t|\bar{x}_0 - \bar{y}_0| + \int_0^t (t-s)\left[\bar{p}(s)v(s) + \int_0^s q(s,\sigma)v(\sigma)\,d\sigma\right] ds$$

$$+ |\bar{x}_0 - \bar{y}_0| + \int_0^t \left[\bar{p}(s)v(s) + \int_0^s q(s,\sigma)v(\sigma)\,d\sigma\right] ds$$

$$\leq r(t) + \int_0^t (t-s+1)\left[\bar{p}(s)v(s) + \int_0^s q(s,\sigma)v(\sigma)\,d\sigma\right] ds,$$

for $t \in R_+$, where $r(t)$ is given by (37). Clearly $r(t)$ is nondecreasing for $t \in R_+$. Now an application of Lemma to (38) yields (36). ■

Remark 4. An important feature of our approach here is that it is elementary and can be extended to obtain similar results as given above for more general equation of the form

$$x^{(n)}(t) = f\left(t, x(t), x'(t), ..., x^{(n-1)}(t), Gx(t)\right), \tag{39}$$

with the prescribed initial values

$$x^{(k)}(0) = x_k, \ (k = 0, 1, 2, ..., n-1), \tag{40}$$

for $t \in R_+$, where $n \geq 2$ is a given integar and

$$Gx(t) = \int_0^t g\left(t, \sigma, x(\sigma), x'(\sigma), ..., x^{(n-1)}(\sigma)\right) d\sigma, \tag{41}$$

under some suitable conditions on the functions involved in IVP (39)-(40). We note that the idea of this paper can be extended to the general vergions of IVP (1)-(2) and IVP (39)-(40), recently studied by the present author in [8-10]. We shall not pursue the detailed treatment here.

References

[1] BELLMAN R., *Stability Theory of Differential Equations,* McGraw-Hill, New York, 1953.

[2] Brunner H., *Collocation methods for Volterra integral and related functional differential equations,* Cambridge University Press, Cambridge, 2004.

[3] Coddington E.A., Levinson N., *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1955.

[4] Hartman P., *Ordinary Differential Equations,* Wiley, New York, 1964.

[5] Morchalo J., Construction of upper and lower functions by approximate integration of an integrodifferential equation of higher order, *Fasc. Math.,* 9(1975), 97-108.

[6] Pachpatte B.G., *Inequalities for Differential and Integral Equations,* Academic Press, New York, 1998.

[7] Pachpatte B.G., *Integral and Finite Difference Inequalities and Applications,* North-Holland Mathematics Studies, Vol. 205, Elsevier Science B.V., Amsterdam, 2006.

[8] Pachpatte B.G., On a certain second order integrodifferential equation, *RGMIA Research Report Collection,* 10(3)(2007), 463-476.

[9] Pachpatte B.G., On higher order Volterra-Fredholm integrodifferential equation, *Fasc. Math.,* 37(2007), 35-48.

[10] Pachpatte B.G., On iterated Volterra integrodifferential equation of higher order, *Fasc. Math.,* 39(2008), 47-61.

Baburao G. Pachpatte
57 Shri Niketan Colony, Near Abhinay Talkies
Aurangabad 431 001 (Maharashtra), India
e-mail: bgpachpatte@gmail.com

Received on 05.09.2009 and, in revised form, on 16.01.2010.