F A S C I C U L I   M A T H E M A T I C I

Nr 45 2010

Haribhau L. Tidke and Machindra B. Dhakne

EXISTENCE AND UNIQUENESS OF SOLUTION OF DIFFERENTIAL EQUATION OF SECOND ORDER IN CONE METRIC SPACES

Abstract. In this paper we investigate the existence and uniqueness for fractional and non-fractional differential equations in cone metric spaces. The result is obtained by using the some extensions of Banach's contraction principle in complete cone metric space, fractional calculus and the theory of strongly continuous cosine family.

Key words: cone metric space, fixed point, cosine family, fractional calculus.

AMS Mathematics Subject Classification: 47G20, 34K05, 47H10, 34A12.

1. Introduction

The purpose of this paper is study the existence and uniqueness for fractional and non-fractional differential equations with classical condition in cone metric spaces.

In Section 3 we consider the consider the following fractional evolution equation of the form:

$$D^q x(t) = A(t)x(t), \qquad t \in [0, b], \tag{1}$$

$$x(0) = x_0, \tag{2}$$

where $0 < q < 1$, $A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space X, and x_0 is a given element of X. The operator D^q denotes the Capto fractional derivative of order q.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. Furthermore, the fractional differential and fractional Integrodifferential equations are now recognized as an excellent tool for the description of memory and hereditary properties of various materials and processes due to the existence of a "memory" term in a model. This memory term insures the history and its impact to the present and future. The mathematical modelling of systems and processes involving the fractional order occur in many areas: physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer, biotechnology, rheology, study of control system, game theory, programming languages etc. In recent years, the fractional differential and fractional integrodifferential equations have become a very active area of research and we refer the reader to the monographs [10, 17, 24] and the papers [2, 4, 5, 13, 20].

In Section 4 we study the existence and uniqueness of mild solution of abstract semilinear differential equation of second order the type:

$$y''(t) = Ay(t) + g\left(t, y(t)\right), \qquad 0 \leq t \leq b \tag{3}$$

$$y(0) = y_0, \quad y'(0) = y_1, \tag{4}$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t : t \in \mathbb{R})\}$ in a Banach space X, $g : [0, b] \times X \to X$ is appropriate function and y_0, y_1 are given elements of X.

The theory of differential equations with classical conditions has been extensively studied in the literature. Many authors have been studied the problems of existence, uniqueness, continuation and other properties of solutions of these type or special forms of the equations (3)–(4) are studied by different techniques, for example, see [6, 16, 18, 22, 23, 30, 31, 32] and the references given therein.

The objective of the present paper is to study the existence and uniqueness of solution of the system (1)–(2) and the system (3)–(4) under the conditions in respect of the cone metric space and fixed point theory. In particular for the system (3)–(4), neither the cosine family $\{C(t) : t \in \mathbb{R}\}$ nor the function g is needed to be compact in our result. Hence we extend and improve some results reported in [2, 5, 16, 22, 23, 26, 30, 32]. We are motivated by the work of P. Raja and S. M. Vaezpour in [26] and influenced by the work of K. Balchandran [5].

The paper is organized as follows: Section 2, we discuss the preliminaries. Section 3, we dealt with study of the fractional differential equation and in Section 4, we consider an abstract semilinear differential equation of second order. Finally in Section 5, we give examples to illustrate the application of our results.

2. Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader to [1, 3, 7, 9, 12, 14, 15, 19, 25, 27, 28, 29, 33] for the more details.

Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1. P is closed, nonempty and $P \neq \{0\}$;
2. $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \Rightarrow ax + by \in P$;
3. $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following we always suppose E is a real Banach space , P is a cone in E with $intP \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

(d_1) *$0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;*
(d_2) *$d(x, y) = d(y, x)$, for all $x, y \in X$;*
(d_3) *$d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.*

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of metric space.

The following example is a cone metric space, (see [11]).

Example 1. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x, y) = (|x - y|, a|x - y|)$, where $a \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2. *Let X be a an ordered space. A function $\Phi : X \to X$ is said to be a comparison function if for every $x, y \in X$, $x \leq y$ implies that $\Phi(x) \leq \Phi(y)$, $\Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$ for all $x \in X$.*

Example 2. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$. It is easy to check that $\Phi : E \to E$, with $\Phi(x, y) = (cx, cy)$, for some $c \in (0, 1)$ is a comparison function. Also if Φ_1, Φ_2 are two comparison functions over $\mathbb{R}$, then $\Phi(x, y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

Let X be a Banach space with norm $\| \cdot \|$. Let $B = C([0, b], X)$ be the Banach space of all continuous functions from $[0, b]$ into X endowed with supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in [0, b]\}.$$

Let $P = \{(x, y) : x, y \geq 0\} \subset E = \mathbb{R}^2$, and define $d(f, g) = (\|f - g\|_B, a\|f - g\|_B)$, for every $f, g \in B$ and $a \geq 0$. Then it is easily seen that (B, d) is a cone metric space.

We need the following theorem for further discussion:

Theorem 1 ([26]). *Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that*

$$d(f(x), f(y)) \leq \Phi(d(x, y)),$$

for every $x, y \in X$. Then f has a unique fixed point.

3. Fractional differential equation

We need some basic definitions and properties of fractional calculus which are used in this section.

Definition 3. *A real function $f(t)$, $t > 0$, is said to be an element of the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p g(t)$, where $g(t) \in C[0, \infty)$, and it is said to be an element of the space C^n_μ whenever $f^{(n)} \in C_\mu$, for all $n \geq 1$.*

Definition 4. *A function $f \in C_\mu$, $\mu \geq -1$ is said to be Riemann-Liouville fractional integrable of order $\alpha \in \mathbb{R}^+$ if*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds < \infty,$$

where Γ is the Euler gamma function and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Definition 5. *The fractional derivative in the Capto sense is defined as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{n-\alpha}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds$$

for $n - 1 < \alpha \leq n$, $n \geq 1$, $t > 0$ and $f \in C^n{}_{-1}$. If $0 < \alpha \leq 1$, then

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds,$$

where $f'(s) = \frac{df(s)}{ds}$ and f is an abstract function with values in X.

The properties of the above operators and the common symbols can be found in [17] and the general theory of fractional differential equations can be found [21].

Remark 1. The problem (1)–(2) is equivalent to the integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)x(s)ds.$$

By a solution of the problem (1)–(2), we mean a function x such that the following conditions are satisfied:
(a) $x \in B$ and $x \in D(A(t))$;
(b) $D^q x(t)$ is continuous on $[0, b]$, where $0 < q < 1$;
(c) x satisfies equation (1) with the initial condition (2).

We list the following hypotheses for our convenience:

(H_1) $A(t)$ is a bounded linear operator on X for each $t \in [0, b]$, the function $t \to A(t)$ is continuous in the uniform operator topology and there exist a constant K such that

$$K = \max_{t\in[0,b]} \|A(t)\|.$$

(H_2) There exists a comparison function $\Phi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\|x(t) - y(t)\|, a\|x(t) - y(t)\|) \leq \Phi_1(d(x, y)),$$

for every $t \in [0, b]$ and $x, y \in B$.

(H_3)

$$\frac{Kb^{q+1}}{\Gamma(q+1)} \leq 1.$$

Theorem 2. *Assume that hypotheses* $(H_1) - (H_3)$ *hold. Then the evolution equations* (1)–(2) *has a unique solution* x *on* $[0, b]$.

Proof. The operator $F : B \to B$ is defined by

$$Fx(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} A(s)x(s)ds, \qquad t \in [0, b]. \tag{5}$$

By using the hypotheses $(H_1) - (H_3)$, we have

$$\begin{aligned}
&(\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|) \\
&\leq (\frac{Kb^q}{\Gamma(q+1)} \int_0^t \|x(s) - y(s)\|ds, \\
&\qquad\qquad a\frac{Kb^q}{\Gamma(q+1)} \int_0^t \|x(s) - y(s)\|ds) \\
&\leq \frac{Kb^q}{\Gamma(q+1)} \int_0^t (\|x(s) - y(s)\|, a\|x(s) - y(s)\|)ds \\
&\leq \Phi_1(\|x-y\|_B, a\|x-y\|_B)\frac{Kb^{q+1}}{\Gamma(q+1)} \\
&\leq \Phi_1(\|x-y\|_B, a\|x-y\|_B),
\end{aligned} \tag{6}$$

for all $x, y \in B$. This implies that $d(Fx, Fy) \leq \Phi_1(d(x, y))$, for all $x, y \in B$. The Theorem 1 can be applied to guarantee the mild solution of the semilinear differential equations (1)–(2). ■

4. Second order differential equation

In many cases it is advantageous to treat second abstract differential equations directly rather than to convert first order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family. Next, we only mention a few results and notations needed to establish our results. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

(a) $C(0) = I$ (I is the identity operator);

(b) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in X$;

(c) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in X, then $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family, is defined by

$$S(t)x = \int_0^t C(s)x ds, \qquad x \in X, \quad t \in \mathbb{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}, \qquad x \in D(A),$$

where $D(A) = \{x \in X : C(.)x \in C^2(\mathbb{R}, X)\}$. Moreover, $M \geq 1$ and N are positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in [0, b]$.

Definition 6. *Let $g \in L^1(0, b; X)$. The function $y \in B$ given by*

$$y(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t - s)g(s, y(s))ds, \quad t \in [0, b]$$

is called the mild solution of the initial value problem (3)–(4).

We list the following hypotheses for our convenience:

(H_4) $g : [0, b] \times X \to X$ and there exists a continuous function $p_1 : [0, b] \to \mathbb{R}^+$ and a comparison function $\Phi_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\|g(t, x) - g(t, y)\|, a\|g(t, x) - g(t, y)\|) \leq p_1(t)\Phi_2(d(x, y)),$$

for all $t \in [0, b]$ and $x, y \in X$.

(H_5)

$$N\int_0^b p_1(t)dt = 1.$$

Theorem 3. *Assume that hypotheses* $(H_4)-(H_5)$ *hold. Then the abstract problem* (3)–(4) *has a unique solution* x *on* $[0,b]$.

Proof. We want to prove that the operator $G : B \to B$ is defined by

$$Gy(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)g(s,y(s))ds, \quad t \in [0,b] \tag{7}$$

has unique fixed point. This fixed point is a solution of equations (3)–(4). For every $x, y \in B$, we have

$$\begin{aligned}
&(\|Gx(t) - Gy(t)\|, a\|Gx(t) - Gy(t)\|) \\
&= (\|\int_0^t S(t-s)[g(s,x(s)) - g(s,y(s))]ds\|, \\
&\qquad a\|\int_0^t S(t-s)[g(s,x(s)) - g(s,y(s))]ds\|) \\
&\le (\int_0^t \|S(t-s)\|\|[g(s,x(s)) - g(s,y(s))]\|ds, \\
&\qquad a\int_0^t \|S(t-s)\|\|[g(s,x(s)) - g(s,y(s))]\|ds) \\
&\le (\int_0^t N\|[g(s,x(s)) - g(s,y(s))]\|ds, \\
&\qquad a\int_0^t N\|\|[g(s,x(s)) - g(s,y(s))]\|ds) \\
&\le N\int_0^t p_1(s)\Phi_2(\|x(s) - y(s)\|, a\|x(s) - y(s)\|)ds \\
&\le \Phi_2(\|x-y\|_B, a\|x-y\|_B)N\int_0^b p_1(s)ds \\
&= \Phi_2(\|x-y\|_B, a\|x-y\|_B).
\end{aligned} \tag{8}$$

Hence $d(Gx, Gy) \le \Phi_2(d(x,y))$, for all $x, y \in B$. The conclusion follows now from Theorem 1. ■

5. Applications

In this section we give examples to illustrate the usefulness of our results. Let us consider first example of fractional initial value problem:

$$D^r x(t) = \frac{t}{7}x(t), \qquad t \in [0,1], \quad 0 < r < 1, \tag{9}$$

$$(10) \qquad x(0) = x_0.$$

Consider a metric $d(x,y) = (\|x-y\|_B, a\|x-y\|_B)$ on $C([0,1],\mathbb{R})$ for $a \geq 0$. Then clearly $C([0,1],\mathbb{R})$ is a complete cone metric space. Here $A(t) = \frac{t}{7}$, $t \in [0,1]$. Clearly, $K = \frac{1}{7}$ and if

$$\frac{1}{7} \leq \Gamma(r+1),$$

then all conditions of Theorem 2 are satisfied, the problem (9)–(10) has a unique solution $x \in C([0,1],\mathbb{R})$ on $[0,1]$.

Now, we consider the following partial differential equation of the form:

$$(11) \qquad \frac{\partial^2}{\partial t^2} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \frac{2}{19+e^t} z(t,x(t)), \quad t \in [0,1], \; x \in [0,\pi]$$

$$(12) \qquad z(t,0) = z(t,\pi) = 0, \qquad t \in [0,1],$$

$$(13) \qquad z(0,x) = z_0(x), \qquad x \in [0,\pi],$$

$$(14) \qquad \frac{\partial z(t,x)}{\partial t}|_{t=0} = z_1(x), \qquad x \in [0,\pi].$$

Let

$$\begin{aligned} y(t)x &= z(t,x), \qquad t \in [0,1], \; x \in [0,\pi], \\ g(t,y)(x) &= \frac{2}{19+e^t} z(t,x(t)), \qquad x \in [0,\pi]. \end{aligned}$$

Let us take $X = L^2([0,\pi])$. We define the operator $A : D(A) \subset X \to X$ by $Aw = w_{uu}$, where $D(A) = \{w(\cdot) \in X : w, \; w^{'}$ are absolutely contnuous, $w(0) = w(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathbb{R}\}$ on X. Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized characteristics vectors $w_n(u) := \sqrt{\frac{2}{\pi}} \sin(nu)$, $n = 1,2,3,\cdots$, and the following conditions hold:

1) $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.
2) If $w \in D(A)$, then $Aw = -\sum_{n=1}^{\infty} n^2 < w, w_n > w_n$.
3) For $w \in X$, $C(t)w = \sum_{n=1}^{\infty} \cos(nt) < w, w_n > w_n$. Moreover, from these expression, it follows that $S(t)w = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} < w, w_n > w_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in [0,1]$.

4) If H denotes the group of translations on X defined by $H(t)x(u) = \tilde{x}(u+t)$, where $\tilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}\Big(H(t) + H(-t)\Big)$. If $G : X \to X$ is defined by $Gx = x'$, $D(G) = \{x \in X : x' \in X\}$, then it follows that $A = G^2$ (see [8]), where G is the infinitesimal generator of the group H.

With these choices of functions, the equations (11)–(14) can be formulated as an abstract semilinear differential equations (3)–(4). Since all the conditions of Theorem 3 are satisfied, the problem (11)–(14) has solution z on $[0,1] \times [0,\pi]$.

References

[1] Abbas M., Jungck G., Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.,* 341(1)(2008), 416-420.

[2] Abd El-Salam Sh.A., El-Sayed A.M.A., On the stability of some fractional order non-autonomous systems, *Electronic Journal of Qualitative Theory of Differential Equations,* 6(2007), 1-14.

[3] Azam A., Arshad M., Common fixed points of generalized contractive maps in cone metric spaces, *Bull. Iranian Math. Soc.,* (2009) (in press).

[4] Bashir A., Some existence results for boundary value problems of fractional semilinear evolution equations, *Electronic Journal of Qualitative Theory of Differential Equations,* 28(2009), 1-7.

[5] Balchandran K., Park J.Y., Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, *Nonlinear Analysis,* 71(2009), 4471-4475.

[6] Banas J., Solutions of a functional integral equation in $BC(\mathbb{R}_+)$, *International Mathematical Forum,* 1(24)(2006), 1181-1194.

[7] Choudhury B.S., Metiya N., Fixed points of weak contractions in cone metric spaces, *Nonlinear Analysis,* (2009) doi:10.1016/j.na.2009.08.040.

[8] Fattorini H.O., *Second Order Linear Differential Equations in Banach Spaces,* North-Holland Mathematics Studies, 108, North-Holland, Amsterdam, 1985.

[9] Haghi R.H., Rezapour Sh., Fixed points of multifunctions on regular cone metric spaces, *Expo. Math.,* (2009) doi: 10.1016 /j.exmath.2009.04.001.

[10] Hilfer R., *Application of Fractional Calculus in Physics,* World Scientific, Singapore, 2000.

[11] Huang L.G., Zhang X., Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.,* 332(2)(2007), 1468-1476.

[12] Ilic D., Rakocevic V., Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.,* 341(2)(2008), 876-882.

[13] Jaradat O.K., Al-Omari A., Momani S., Existence of the mild solution for fractional semilinear initial value problems, *Nonlinear Analysis,* 69(2008), 3153-3159.

[14] JANKOVIC S., KADELBURG Z., RADONEVIC R., RHOADES B.E., Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces, *Fixed Point Theory and Applications,* (2009), 16 pages, Article ID 761086, doi: 10.1155 /2009/761086.

[15] KADELBURG Z., RADONEVIC S., ROSIC B., Strict contractive conditions and common fixed point theorems in cone metric spaces, *Fixed Point Theory and Applications,* (2009), 14 pages, Article ID 173838, doi: 10.1155 /2009/173838.

[16] KAROUI A., On the existence of continuous solutions of nonlinear integral equations, *Applied Mathematics Letters,* 18(2005), 299-305.

[17] KILBAS A.A., SRIVASTAVA H.M., TRUJILLO J.J., *Theory and Applications of Fractional Differential Equations,* North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.

[18] KISYŃSKI J., On second order Cauchy's problem in a Banach space, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronm. Phys.,* 18(1970), 371-374.

[19] KWONG M.K., On Krasnoselskii's cone fixed point theorems, *Fixed Point Theory and Applications,* Volume 2008, Article ID 164537, 18pages.

[20] KOSMATOV N., Solutions to a class of nonlinear differential equations of fractional order, *Electronic Journal of Qualitative Theory of Differential Equations,* 20(2009), 1-10.

[21] MILLER K.S., ROSS B., *An Introduction to the Fractional Calculus and Fractional Differential Equations,* A Wiley-Interscience Publication, John Wiley and Sons, New York, NY, USA, 1993.

[22] PACHPATTE B.G., Applications of the Leray-Schauder Alternative to some Volterra integral and integrodifferential equations, *Indian J. Pure Appl. Math.,* 26(12)(1995), 1161-1168.

[23] PAZY A., Semigroups of Linear Operators and applications to Partial Differential Equations, *Springer-Verlag,* New York, 1983.

[24] PODLUBNY I., Fractional Differential Equations, *Vol. 198 of Mathematics in Science and Engineering, San Diego,* Academic Press, Calif, USA, 1999.

[25] RADONEVIC S., Common fixed points under contractive conditions in cone metric spaces, *Computer and Math. with Appl.,* (2009) doi:10.1016/j.camwa.2009.07.035.

[26] RAJA P., VAEZPOUR S.M., Some extensions of Banach's contraction principle in complete cone metric spaces, *Fixed Point Theory and Applications,* Volume 2008, Article ID 768294, 11 pages.

[27] REZAPOUR SH., HAMLBARANI R., Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.,* 345(2008), 719-724.

[28] REZAPOUR SH., HAGHI R.H., Fixed point of multifunctions on cone metric spaces, *Numer. Funct. Anal. and Opt.,* 30(7-8)(2009), 825-832.

[29] REZAPOUR SH., HAGHI R.H., Two results about fixed point of multifunctions, *Bull. Iranian Math. Soc.,* (2009)(in press).

[30] TIDKE H.L., DHAKNE M.B., On abstract nonlinear differential equations of second order, *Advances in Differential Equations and Control Processes,* 3(1)(2009), 33-39.

[31] TRAVIS C.C., WEBB G.F., Compactness, regularity, and uniform conti-

nuity properties of strongly continuous cosine families, *Houston J. Math.,* 3(4)(1977), 555-567.

[32] TRAVIS C.C., WEBB G.F., Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hungaricae,* 32(1978), 76-96.

[33] WLODARCZYK K., PLEBANIAK R., OBCZYNSKI C., Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasiasyptotic contractions in cone uniform spaces, *Computer and Math. with Appl.,* (2009) doi:10.1016/j.camwa.2009.07.035.

HARIBHAU L. TIDKE
DEPARTMENT OF MATHEMATICS
MAHARASHTRA UNIVERSITY
JALGAON-425 001, INDIA
e-mail: tharibhau@gmail.com

MACHINDRA B. DHAKNE
DEPARTMENT OF MATHEMATICS
DR. B. A. MARATHWADA UNIVERSITY
AURANGABAD-431 004, INDIA
e-mail: mbdhakne@yahoo.com

Received on 10.10.2009 and, in revised form, on 21.11.2009.