F A S C I C U L I M A T H E M A T I C I

Nr 46 2011

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A SHORT NOTE ON A MULTIPLIER OF THE NÖRLUND MEANS AND CONVEX MAPS OF THE UNIT DISC

Abstract. The problem of a multiplier $\omega(n)$ of the Nörlund means and convex maps of the unit disc is being considered again, but with a multiplier $\omega(n)$ of different form then that appeared by Ziad S. Ali in [1]. With this we would have brought up again in a rather unified approach the results of G. Pólya, and I.J. Schonberg in [7], T. Basgoze, J.L. Frank and F.R. Keogh in [3], and Ziad S. Ali in [2]. The new multiplier $w(n)$ gives rise to interesting applications related to function expansion by the Chebychev polynomials of the first and the second kind, an application to bionomial trigonometric formulas and an application to the subordination principle.

Key words: Nörlund, Cesaro, de La Vallee Poussin means, convex maps, combinatorial identities, Fourier series.

AMS Mathematics Subject Classification: 40G05, 40C15, 30C45, 05A19, 42C10.

1. Introduction

Let $\sum_{k=0}^{\infty} u_k$ be a given series, and let $\{S_n\}_0^\infty$ denote the sequence of its partial sums. Let $\{q_n\}_0^\infty$ be a sequence of real numbers with $q_0 > 0$, and $q_n \geq 0$ for all $n > 0$, and let $Q_n = \sum_{k=0}^{n} q_k$. By G. H. Hardy [5] the sequence-to-sequence transformation

$$T_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k$$

is called the Nörlund means of $\{S_n\}_0^\infty$, and is denoted by (N, q_n).

The (N, q_n) is regular if and only if $q_n = o(Q_n)$ as $n \to \infty$. By P. L. Duren [4] a function f analytic in a domain D is said to be simple, schlicht, or univalent if f is one-to-one mapping of D onto another domain. A domain

E of the complex plane is said to be convex if and only if the line segment joining any two points of E lies entirely in E. A function f which is analytic, univalent in the unit disc $D = \{z : z < 1\}$, and is normalized by $f(0) = f'(0) - 1 = 0$ is said to belong to the class S. Now $f \in S$ is said to belong to the class K if and only if it is a conformal mapping of the unit disc $D = \{z : z < 1\}$ onto a convex domain. An analytic function g is said to be subordinate to an analytic function f (written $g \prec f$) if

$$g(z) = f\big(\omega(z)\big), \qquad |z| < 1$$

for some analytic function ω with $|\omega(z)| \leq |z|$. It is known by the Koebe-One-Quarter theorem that the range of every function of the class S contains the disc $\{w : |w| < \frac{1}{4}\}$, i.e. $\frac{1}{4}z \prec f$. The strengthend version of the Koebe-One-Quarter theorem says that the range of every convex function $f \in K$ contains the disc $|w| < \frac{1}{2}$, i.e. $\frac{1}{2}z \prec f$. Let $\mathrm{Re}(z)$ be the real part of z, and let $\mathrm{Im}(z)$ be the imaginary part of z. Again by P. L. Duren [4], let $f(\rho,\theta) = u(\rho,\theta) + iv(\rho,\theta)$. Assume that $u(\rho,\theta)$ is harmonic in $D = \{z : |z| < 1\}$. Assume further that $u(\rho,\theta)$ is continuous on $\bar{D} = \{z : |z| \leq 1\}$. The minimum principal for harmonic functions asserts that $u(\rho,\theta)$ attains its minimum on the boundary of D. By A. M. Mathai [6], the Chebychev's polynomials of the first kind $T_n(x)$, and of the second kind $U_n(x)$ are respectively defined by:

$$T_n(x) = \cos n\theta, \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta.$$

A continuous function $f(x)$ in $|x| \leq 1$ can have a Generalized Fourier Series expansion or a Fourier Chebychev Series expansion of the form

$$f(x) = \sum_{r=1}^{\infty} a_r T_r(x), \quad \text{where} \quad a_r = \frac{2}{\pi}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) T_r(x)dx.$$

The orthogonality conditions are given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_s(x)T_r(x)dx = \begin{cases} 0, & \text{if} \quad s \neq r \\ \pi, & \text{if} \quad s = r = 0 \\ \frac{\pi}{2}, & s = r = 1, 2, \cdots \end{cases}$$

Similarly,

$$g(x) = \sum_{r=1}^{\infty} b_r U_r(x), \quad \text{where,} \quad b_r = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\, g(x) U_r(x)dx$$

with the orthogonality conditions

$$\int_{-1}^{1} U_r(x)U_s(x)\sqrt{1-x^2}dx = \begin{cases} 0, & \text{if } r \neq s \\ \frac{\pi}{2}, & r = s \end{cases}$$

2. Means connected with power series

Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is regular for $|z| < 1$. Let

$S_n(z, f) = \sum_{k=0}^{n} a_k z^k$ be the sequence of partial sums of f,

$\sigma_n(z, f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z, f)$ be the Cesaro means or the $(C, 1)$ means of f,

$T_n(z, f) = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} S_k(z, f)$ be the Nörlund means of f, and let

$V_n(z, f) = \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} a_k z^k$ be the de la Vallee Poussin means of f.

3. Known results

In [7] G. Polya, and I.J. Schonberg proved the following theorem, and corollary:

Theorem 1. *For $f(z) \in K$, it is necessary and sufficient that $V_n(z, f) \in K$ for $n = 1, 2 \dots$.*

Corollary 1. *For $f(z) \in K$, $V_n(z, f) \prec f$ for $n = 1, 2, \dots$.*

In [3] T. Basgoze, J.L. Frank, and F.R. Keogh proved the following theorem:

Theorem 2. (i) *Suppose that the values taken by $f(z)$ for z in D lie in a convex domain D_w. Then the values taken by $\sigma_n(z, f)$ also lie in D_w for all n, and all z in D.*

(ii) *Conversely, suppose that the values taken by $\sigma_n(z, f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.*

In [2] Ziad S. Ali proved the following theorems:

Theorem 3. (i) *Let (N, q_n) be a regular Nörlund transformation such that $\{q_n\}_0^{\infty}$ is a non-decreasing sequence of positive numbers. Suppose that the values taken by $f(z)$, for z in D, lie in a convex domain D_w,then the values taken by $T_n(z, f)$, also lie in D_w for all n, and all z in D.*

(ii) *Conversely, suppose that the values taken by $T_n(z, f)$ lie in a convex domain D_w; then the values taken by $f(z)$ lie in D_w for all z in D.*

In [1] Ziad S. Ali proved the following theorem:

Theorem 4. *(i) Let* $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $(c_1 = 1)$ *be regular in the unit disc* $|z| < 1$.

(ii) Let T_n *be a transformation of the Nörlund type. Let*

$$\begin{aligned} Q_k^n &= \sum_{r=0}^{k} q_r^n = \sum_{r=0}^{k} \tfrac{(2n-2r+1)}{(2n-r+1)} \tbinom{2n}{r} q_0, \text{ and} \\ \omega(n) &= \tfrac{-2}{Q_n^n} \sum_{k=1}^{n} (-1)^k Q_{n-k}^n, \text{ then} \end{aligned}$$

$$\tfrac{1}{\omega(n)} T_n(z,f) \in K \quad \text{if and only if } f \in K.$$

4. The main theorems

In this section we prove the following theorems:

Theorem 5. *(i) Let* $f(z) = \sum_{r=1}^{\infty} a_r z^r$, $(a_1 = 1)$ *be regular in the unit disc* $|z| < 1$.

(ii) Let T_n *be a transformation of the Nörlund type. Let*

$$\begin{aligned} Q_k^n &= \sum_{r=0}^{k} q_r^n = \sum_{r=0}^{k} \tfrac{(2n-2r+1)}{(2n-r+1)} \tbinom{2n}{r} q_0, \text{ and} \\ \omega(n) &= \tfrac{-2}{Q_n^n} \min_{|z|\le 1} \ Re \sum_{r=1}^{n} Q_{n-r}^n z^r, \text{ then} \end{aligned}$$

$$\tfrac{1}{\omega(n)} T_n(z,f) \in K, \quad \text{if and only if } \quad f \in K.$$

Proof. First we note that

$$Q_j^n = \sum_{r=0}^{j} q_r^n = \sum_{r=0}^{j} \frac{(2n-2r+1}{(2n-r+1)} \binom{2n}{r} q_0 = \binom{2n}{j}, \quad \text{hence}$$

$$Q_n^n = \binom{2n}{n} \quad \text{and} \quad Q_{n-r}^n = \binom{2n}{n-r}.$$

Accordingly,

$$\frac{1}{\omega(n)} T_n(z,f) = \frac{1}{\frac{-2}{Q_n^n} \min\limits_{|z|\le 1} \ \mathrm{Re} \sum\limits_{r=1}^{n} Q_{n-r}^n z^r} \frac{1}{Q_n^n} \sum_{r=1}^{n} Q_{n-r}^n a_r z^r.$$

Second, by the minimum principal of harmonic functions

$$u(\rho,\theta)=\min_{|z|\le 1}\ \mathrm{Re}\sum_{r=1}^{n} Q^n_{n-r}\rho^r e^{ir\theta},$$

is harmonic since

$$\bigtriangledown^2 u=\frac{\partial^2 u}{\partial\rho^2}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{\rho}\frac{\partial u}{\partial\rho}=0,$$

it follows that $u(\rho,\theta)$ attains its minimum on the boundary of D.

Third, it can be shown that

$$\mathrm{Re}\left(\sum_{r=1}^{n}\binom{2n}{n-r}(e^{ir\theta}-e^{ir\pi})\right)=2^{2n-1}\left(\cos(\frac{\theta}{2})\right)^{2n},$$

it follows that $u(\rho,\theta)$ has its minimum at $\theta=\pi$.

Fourth, it follows by Ziad S. Ali in [1] that

$$\frac{1}{w(n)}T_n(z,f)=\frac{1}{\binom{2n}{n}}\sum_{n=r}^{n}\binom{2n}{n+r}a_r z^r=V_n(z,f),$$

which are the De La Vallee Poussin means of f, and the theorem follows by Pólya and Schoenberg [7]. ∎

Theorem 6. *(i)] Suppose that $f(z)=\sum_{k=0}^{\infty}a_k z^k$ is regular for $|z|<1$, and suppose that T_n are the Nörlund means.*

(ii) Let $Q^n_n=n+1$, and let

$$\omega(n)=\begin{cases}\frac{-2}{Q^n_n}\min\limits_{|z|\le 1} Re\sum\limits_{k=1}^{n}Q^n_{n-k}z^k, & n \textit{ is odd },\\ \\ \frac{-2}{Q^n_n}\min\limits_{|z|\le 1} Re\sum\limits_{k=1}^{n}Q^n_{n-k}z^k+1, & n \textit{ is even},\end{cases}$$

then

$$\frac{1}{\omega(n)}T_n(z,f)\in K \quad \textit{if and only if} \quad f\in K.$$

Proof. Clearly $Q^n_{n-k}=n-k+1$. Considering two separate cases for n even, and n odd an using the minimum principal for harmonic functions we can easily see that

$$n+1=\begin{cases}-2\min\limits_{|z|\le 1} Re\sum\limits_{k=1}^{n}(n-k+1)z^k, & n \text{ is odd}\\ \\ -2\min\limits_{|z|\le 1} Re\sum\limits_{k=1}^{n}(n-k+1)z^k+1, & n \text{ is even.}\end{cases}$$

Accordingly for any n we have:

$$\frac{1}{\omega(n)} T_n(z,f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(z,f) = \sigma_n(z,f),$$

which are the Cesaro means of f, and the result follows by T. Basgoze, J.L. Frank, and F.R. Keogh [3]. ■

Theorem 7. *(i) Let $f(z) = \sum\limits_{r=0}^{\infty} a_r z^r$ be regular in the unit disc $D = \{z : |z| < 1\}$.*

(ii) Let $T_n(z,f)$ be a regular Nörlund transformation defined by a non-decreasing sequence $\{q_r^n\}_{r=1}$ of positive real numbers such that $\sum\limits_{i \in odd} q_i^n = \sum\limits_{i \in even} q_i^n$, where i is a non-negative integer and let

$$\omega(n) = \frac{-2}{Q_n^n} \min_{|z| \le 1} Re \sum_{r=1}^{n} Q_{n-r}^n z^r, \text{ then:}$$

$$\frac{1}{\omega(n)} T_n(z,f) \in K \quad \text{if and only if } f \in K.$$

Proof. It can be shown in general that

$$\omega(n) = \frac{-2}{Q_n^n} \min_{|z| \le 1} \mathrm{Re} \sum_{r=1}^{n} Q_{n-r}^n z^r = 1, \text{ if and only if, } \sum_{i \in \text{odd}} q_i^n = \sum_{i \in \text{even}} q_i^n.$$

Now the proof of the theorem follows by the minimum principal of harmonic functions and by Ziad S. Ali [1] ■

5. An application to function expansion by Chebychev polynomials

Theorem 8. *(i) Let T_n be the Chebychev polynomials of the first kind. Let n be fixed and let $|x| \le 1$. The Fourier Chebychev Series expansion of $f(n,x) = \left(2^{n-1}(x+1)^n - \frac{1}{2}\binom{2n}{n}\right)$ is given by*

$$\sum_{r=1}^{\infty} \binom{2n}{n-r} T_r(x) = \left(2^{n-1}(x+1)^n - \frac{1}{2}\binom{2n}{n}\right), \quad x = \cos\theta.$$

(ii) Let U_n be the Chebychev polynomials of the second kind with n fixed and $|x| \le 1$. The Fourier Chebychev Series expansion of $g(n,x) = n2^{n-1}(x+1)^{n-1}$ is given by

$$\sum_{r=1}^{\infty} \binom{2n}{n-r} r U_{r-1}(x) = n2^{n-1}(x+1)^{n-1}, \quad x = \cos\theta$$

Proof. By the third step in the proof of Theorem 5, the theorem follows. We note further that by the same step of Theorem 5, we can generate all Chebychev polynomials of the first kind and of second kind of degrees $1, 2, \cdots$, giving us a different way of obtaining them. We note that the Fourier Chebychev Series given above in (i) and (ii) are truncated at $r = n+1$. ■

We now have the following rather interesting theorem:

Theorem 9. *Let n be a positive fixed integer, let $r \leq n$, and let $|x| \leq 1$ then we have:*

$$(i) \qquad \binom{2n}{n-r} = \frac{2}{\pi}\int_{-1}^{1} \frac{\left(2^{n-1}(x+1)^n - \frac{1}{2}\binom{2n}{n}\right)}{\sqrt{1-x^2}} T_r(x)dx.$$

$$(ii) \qquad r\binom{2n}{n-r} = \frac{2}{\pi}\int_{-1}^{1} \sqrt{1-x^2}\left(n2^{n-1}(x+1)^{n-1}\right) U_{r-1}(x)dx$$

Proof. Follows by the definition of the coefficients in each expansion. ■

6. An application to bionomial trigonometric formulas

Theorem 10. *Let n be fixed, $r \leq n$, and let α be real. We have:*

$$(i) \quad (\cos^n \alpha)^2 + (\sin^n \alpha)^2 = \frac{1}{2^{2n-1}}\left(2\sum_{r \in even}^{n}\binom{2n}{n-r}\cos(2\,r\alpha) + \binom{2n}{n}\right), \quad r \geq 2$$

$$(ii) \quad (\cos^n \alpha)^2 - (\sin^n \alpha)^2 = \frac{1}{2^{2n-1}}\left(2\sum_{r \in odd}^{n}\binom{2n}{n-r}\cos(2\,r\alpha)\right), \quad r \geq 1$$

Proof. Follows by the third step in the proof of Theorem 5, and noting the following:

$$\mathrm{Re}\left(\sum_{r=1}^{n}(-1)^r\binom{2n}{n-r}e^{ir\theta}\right) = 2^{2n-1}\left(\sin(\frac{\theta}{2})\right)^{2n} - \frac{1}{2}\binom{2n}{n}.$$

■

7. An application to the subordination principle

Theorem 11. *(i) Let K denote the class of "Schlicht" power series wich map $|z| < 1$ onto some convex domain, and let $f \in K$.*

(ii) Let T_n be a transformation of the Nörlund type. Let

$$Q_k^n = \sum_{r=0}^{k} q_r^n = \sum_{r=0}^{k} \frac{(2n-2r+1)}{(2n-r+1)} \binom{2n}{r} q_0, \ \textit{and}$$

$$\omega(n) = \frac{-2}{Q_n^n} \min_{|z|\leq 1} \ Re \sum_{r=1}^{n} Q_{n-r}^n z^r, \ \textit{then}$$

$$\frac{1}{\omega(n)} T_n(z, f) \prec f.$$

Proof. Follows by Theorem 5 given above on page 20, and by Corollary 1 which is given above on page 19. ■

Acknowledgement. I would like to thank the referee for his valuable comments.

References

[1] ALI Z.S., A note on a multiplier of the Nörlund means and convex maps of the unit disc, *International Journal of Mathematics and Mathematical Sciences (IJMMS),* December 20(11)(2005), 3351-3357.

[2] ALI Z.S., Convexity of Nörlund and progressive means, *Indian Journal of Mathematics,* May 15(2)(1973).

[3] BASGOZE T., FRANK J.L., KEOGH F.R., On convex univalent functions, *Canadian Journal of Mathematics,* 22(1970), 123-127.

[4] DUREN P.L., *Univalent Functions,* Springer-Verlag, Berlin and NewYork, (1983).

[5] HARDY G.H., *Divergent Series,* Oxford Uni. Press, 1949.

[6] MATHAI A.M., *A Handbook of Generalized Special Functions for statistical and physical sciences,* Oxford Science Publications, 1993.

[7] PÓLYA G., SCHONBERG I.J., Remarks on the de la Vallee Poussin means and convex conformal maps of the circle, *Pacific J. Math.,* 8(1958), 295-334.

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Received on 16.05.2010 and, in revised form, on 25.11.2010.