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COMMON FIXED POINT THEOREMS FOR HYBRID PAIRS OF MAPPINGS WITH SOME WEAKER CONDITIONS IN 2-METRIC SPACES

Abstract. In this paper, we prove common fixed point theorems for hybrid pairs of mappings satisfying an implicit relation in 2-metric spaces by using a new commutativity condition i.e. weak commutativity of type (KB). We also prove common fixed point theorem, of Gregus type for hybrid pairs of maps by using weak commutativity of type (KB) in 2-metric spaces. We extend, improve and generalize many known results. We also give examples to validate our results.

Key words: common fixed point, coincidence point, weak commutativity of type (KB).

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The notion of 2-metric space was introduced by Gahler ([12], [13]). It is generalization of usual notion of metric space (X, d). It has been developed extensively by Gahler and many other mathmaticians ([22], [23]). The topology induced by 2-metric space is called 2-metric topology, which is generated by the set of all open sphers with two centres. Many authors used this topology in many applications for example EI Nachie [10] used this sort of topology in physical applications. Many authors studied fixed point theorems in 2-metric spaces (Hsiao [14], Iseki [15]).

In 1992, Dhage [4] introduced a new class of generalized metric spaces called D-metric spaces. Dhage attempted to develop topological structures in such spaces ([5], [6], [7]). But in 2003, Mustafa and Sims [22], proved that the most of the claims concerning the fundamental topological structures of D-metric spaces are incorrect.

Sessa [30], introduced the concept of weakly commuting maps. Jungck [16] defined the notion of compatible maps in order to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible but the converse is not true.

Jungck and Rhoades ([17], [18]) defined the concepts of δ-compatibility between a set valued mappings and a single valued mappings.

Monsef et. al. [1] generalized some concept in 2-metric spaces for set valued mappings. They also proved some common fixed point theorems in 2-metric spaces.

Fixed point theorems for set valued and single valued mappings provide technique for solving variety of applied problems in mathmatical science and engineering (e.g. Krzyska and Kubiaczyk [19], Sessa and khan [31]).

Pant ([25], [27]) initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings in [25]. He also showed that pointwise R-weak commutativity is a necessary, hence minimal, condition for the existence of a common fixed point of contractive type maps [26].

In 1998, Jungck and Rhoades [18] defined the concept of weak compatibility. Pathak, Cho and Kang [28] introduced the concept of R-weakly commuting maps of type (A) and showed that they are not compatible.

Recently, Kubiaczyk and Bhavana Deshpande [20] extended the concept of R-weakly commutativity of type (A) for a pair of single valued mappings to a single valued and a set valued mappings i.e. a hybrid pair of mappings and introduced weak commutativity fo type (KB) which is weaker condition than δ-compatibility.

It is well known that a δ-compatible pair of hybrid maps is weakly commuting of type (KB) but converse need not true for examples see Kubiazyk, Deshpande [20], Sharma, Deshpande [32], Sharma, Deshpande, Pathak, [33]

In this paper, we prove common fixed point theorem of two hybrid pairs of weakly commuting mappings of type (KB) satisfying an implicit relation in 2-metric spaces our result generalizes and extends results of Aliouche [2], Aliouche and Djoudi [3] and others.

Djoudi and Nisse [8], proved a common fixed point theorem of Gregus type for weakly compatible single valued maps in a Banach space. We extend, improve and generalize result of Djoudi and Nisse for hybrid pairs of maps in 2-metric spaces by using weak commutativity of type (KB).

2. Preliminaries

Definition 1 ([11]). *Let X denotes a nonempty set and R, the set of all nonnegative numbers. Then X together with a function $d : X \times X \times X \to R$, is called a 2-metric space if it satisfies the following properties :*

1. *For distinct point $x, y \in X$, there exists a point $c \in X$ such that $d(x, y, c) \neq 0$ and $d(x, y, c) = 0$ if at least two of x, y and c are equal.*

2. $d(x, y, c) = d(x, c, y) = d(y, x, c) = d(y, c, x) = d(c, x, y) = d(c, y, x)$ (*symmetry*).

3. $d(x,y,c) \leq d(x,y,z)+d(x,z,c)+d(z,y,c)$ *for* $x,y,z,c \in X$. (*Rectangle in equality*)

The function d *is called a 2-metric for the space* X *and the pair* (X,d) *denotes 2-metric space. It has been shown by Gahler in [11] that 2-metric* d *is non-negative and although* d *is a continuous function of any of its three arguments, it need not be continuous in two arguments. A 2-metric* d *which is continuous in all of its arguments is said to be continuous.*

Geometrically, the value of 2-metric $d(x,y,c)$ *represents the area of a triangle with vertices* x, y *and* c.

Throughout this paper, let (X,d) be a 2-metric space unless mentioned, otherwise and $B(X)$ is the class of all nonempty bounded subsets of X.

Definition 2 ([29])**.** *A sequence* $\{x_n\}$ *in* (X,d) *is said to be convergent to a point* x *in* X, *denoted by* $\lim_{n\to\infty} x_n = x$ *if* $\lim_{n\to\infty} d(x_n,x,c) = 0$ *for all* c *in* X. *The point* x *is called the limit of the sequence* $\{x_n\}$ *in* x.

Definition 3 ([29])**.** *A sequence* $\{x_n\}$ *in* (X,d) *is said to be a Cauchy sequence if* $\lim_{n\to\infty} d(x_m,x_n,c) = 0$ *for all* c *in* X.

Definition 4 ([29])**.** *The space* (X,d) *is said to be complete if every Cauchy sequence in* X *converges to a point of* X.

Remark 1. We note that, in a metric space a convergent sequence is a Cauchy sequence and in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X [24].

For all $A,B,C \in B(X)$, let $\delta(A,B,C)$ and $D(A,B,C)$ be the functions defined by

$$\delta(A,B,C) = \sup\{d(a,b,c) : a \in A, b \in B, c \in C\}$$
$$D(A,B,C) = \inf\{d(a,b,c) : a \in A, b \in B, c \in C\}$$

If A consists of a single point a we write $\delta(A,B,C) = \delta(a,B,C)$. If B and C also consists of single points b and c, respectively, we write $\delta(A,B,C) = D(A,B,C) = d(a,b,c)$. It follows immediately from the definition that:

$$\begin{aligned} \delta(A,B,C) &= \delta(A,C,B) = \delta(C,B,A) = \delta(C,A,B) = \delta(B,C,A) \\ &= \delta(B,A,C) \geq 0, \\ \delta(A,B,C) &\leq \delta(A,B,E) + \delta(A,E,C) + \delta(E,B,C) \\ &\qquad \text{for all } A,B,C,E \in B(X). \end{aligned}$$

$\delta(A,B,C) = 0$ if atleast two of A, B and C are equal singleton sets.

Definition 5 ([1])**.** *A sequence* $\{A_n\}$ *of subsets of a 2-metric space* (X,d) *is said to be convergent to a subset* A *of* X *if -*

(i) given $a \in A$, there is sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, 3, ...$ and $\lim_{n\to\infty} d(a_n, a, c) = 0$.

(ii) given $\varepsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε.

Definition 6 ([1]). *The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be weakly commuting on X if $fFx \in B(x)$ and $\delta(Ffx, fFx, C) \leq \max\{\delta(fx, Fx, c), \delta(fFx, fFx, C)\}$.*

Note that if F is a single valued mapping, then the set fFx consists of a single piont. Therefore $\delta(fFx, fFx, C) = d(fFx, fFx, C) = 0$ and the above inequality reduces to the condition given by Khan [21], that is $d(Ffx, fFx, C) \leq d(fx, Fx, C)$.

Definition 7 ([1]). *The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be δ-compatible if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n, C) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx \in B(X), Fx_n \to \{t\}$ and $fx_n \to t$ for some t in X.*

Definition 8 ([9]). *The maps $f : X \to X$ and $F : X \to B(X)$ are said to be D-maps iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some t in X.*

The authors of [2] and [3] proved common fixed point theorems for weakly compatible single-valued maps in metric spaces. Our purpose here is to extend their results to the setting of single and set-valued maps.

Definition 9. *The mappings $F : X \to B(X)$ and $f : X \to X$ are said to be weakly commuting of type (KB) at x if there exists some positive real number R such that $\delta(ffx, Ffx, C) \leq R\delta(fx, Fx, C)$. Here F and f are weakly commuting of type (KB) on X if the above inequality holds for all $x \in X$.*

Example 1. Let $X = [0, 10]$ Define $d(x, y, z) = \min\{|x-y|, |y-z|, |z-x|\}$ for all x, y, z in X. Define $F : X \to B(X)$ and $f : X \to X$ by $Fx = [1, x]$ and $fx = x$, for all $x \in X$, consider the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ in X, then $\lim_{n\to\infty} fx_n = 1 \in \{1\} = \lim_{n\to\infty} Fx_n$. Therefore f and F are D-maps. also $\lim_{n\to\infty} \delta(Ffx_n, fFx_n, C) = 0$.

Thus the pair $\{f, F\}$ is $\delta-$compatible. We can see that there exists positive real number R such that $\delta(ffx, Ffx, C) \leq R\delta(fx, Fx, C)$, for all $x \in X$. Thus the pair $\{f, F\}$ is weakly commuting of type (KB).

Example 2. Let $X = [0, 10]$ Define $d(x, y, z) = min\{|x-y|, |y-z|, |z-x|\}$ for all x, y, z in X. Define $F : X \to B(X)$ and $f : X \to X$ by $Fx = [1, 2x]$ and $fx = 2x$, for all $x \in X$. Consider the sequence $\{x_n\} = \{1 - \frac{1}{n}\}$ in X, then $\lim_{n\to\infty} fx_n = 2 \in [1, 2] = \lim_{n\to\infty} Fx_n$. Therefore f and F are D-maps. also $\lim_{n\to\infty} \delta(Ffx_n, fFx_n, C) \neq 0$.

Thus the pair $\{f, F\}$ is not $\delta-$compatible. We can observe that the pair $\{f, F\}$ is weakly commuting of type (KB) at $x = 0$ for all real $R \geq 0$.

Lemma 1 ([1]). *If $\{A_n\}$ and $\{B_n\}$ are sequence in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n, C)\}$ converges to $\delta(A, B, C)$.*

3. Implicit relation

Let Φ be the family of all continuous maps $\phi : R_+^6 \to R$ such that
(ϕ_1): for all $u, v \geq 0$ with
$(\phi_a) : \phi(u, v, v, u, u + v, 0) \leq 0$
or
$(\phi_b) : \phi(u, v, u, v, 0, u + v) \leq 0$
we have $u \leq v$
$(\phi_2) : \phi(u, u, 0, 0, u, u) > 0, \forall u > 0.$

Example 3. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2, \beta(t_5^2 + t_6^2)\}$ where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ then
(ϕ_1) : if $u > 0$ and $v \geq 0$ we have

$$\begin{aligned}(\phi_a) : \phi(u, v, v, u, u + v, 0) &= u^2 - \alpha \max\left\{v^2, v^2, u^2, \beta\left((u + v)^2 + 0\right)\right\}\\ &= u^2 - \alpha \max\{v^2, v^2, u^2, \beta(u + v)^2\} \leq 0\end{aligned}$$

and

$$\begin{aligned}(\phi_b) : \phi(u, v, u, v, 0, u + v) &= u^2 - \alpha \max\left\{v^2, u^2, v^2, \beta\left(0 + (u + v)^2\right)\right\}\\ &= u^2 - \alpha \max\{v^2, u^2, v^2, \beta(u + v)^2\} \leq 0.\end{aligned}$$

Suppose that $u > v$, then $u^2 \leq \alpha u^2 < u^2$, which is a contradiction. Therefore $u \leq v$. If $u = 0$ then $u \leq v$
$(\phi_2) : \phi(u, u, 0, 0, u, u) = u^2(1 - \alpha) > 0\ \forall u > 0.$

Example 4. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2 t_3, t_3 t_4, t_5 t_6\}$ where $0 < \alpha < 1$ then
(ϕ_1) : if $u > 0$ and $v \geq 0$
we have

$$\begin{aligned}(\phi_a) : \phi(u, v, v, u, u + v, 0) &= u - \alpha \max\{v^2, uv, 0\}\\ &= u - \alpha \max\{v^2, uv\} \leq 0\end{aligned}$$

and

$$\begin{aligned}(\phi_b) : \phi(u, v, u, v, 0, u + v) &= u - \alpha \max\{uv, uv, 0\}\\ &= u - \alpha \max\{uv, uv\} \leq 0.\end{aligned}$$

Suppose that $u > v$, then $u \leq 0$ and $(1 - \alpha u) \leq 0$, which implies that $\alpha u \geq 1$, which is a contradiction. Therefore $u \leq v$. If $u = 0$ then $u \leq v$

$$\begin{aligned}(\phi_2) : \phi(u, u, 0, 0, u, u) &= u - \alpha \max\{u^2\}\\ &= u - \alpha u^2 = u(1 - \alpha u) > 0\ \ \forall u > 0.\end{aligned}$$

Example 5. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4, t_5 t_6\}$ where $0 < \alpha < 1$

(ϕ_1) : if $u > 0$ and $v \geq 0$
we have

(ϕ_a) : $\phi(u, v, v, u, u+v, 0) = u-\alpha \max\{v, v, u, 0\} = u-\alpha \max\{v, v, u\} \leq 0$
and

(ϕ_b) : $\phi(u, v, u, v, 0, u+v) = u-\alpha \max\{v, u, v, 0\} = u-\alpha \max\{v, u, v\} \leq 0.$

Suppose that $u > v$, then $u \leq \alpha u$, which is a contradiction. Therefore $u \leq v$. If $u = 0$ then $u \leq v$

$$(\phi_2) : \phi(u, u, 0, 0, u, u) = u - \alpha \max\{u, 0, 0, u^2\} = u(1 - \alpha u) > 0 \forall u > 0.$$

Example 6. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2\} - c_2 \max\left\{t_3^2, t_4^2, t_5^2\right\} - c_3 \max\{t_3 t_5, t_4 t_6\}$ where $c_1, c_2 > 0$, $c_3 \geq 0$ & $c_1 + c_2 + c_3 < 1$ then

(ϕ_1) : if $u > 0$ and $v \geq 0$ we have

$$(\phi_a) : \phi(u, v, v, u, u+v, 0) = u^2 - c_1 \max\{v^2, v^2\} - c_2 \max\{v^2, v^2, (u+v)^2\} - c_3 \max\{v(u+v), 0\} \leq 0$$
and

$$(\phi_b) : \phi(u, v, u, v, 0, u+v) = u^2 - c_1 \max\{v^2, u^2\} - c_2 \max\{u^2, v^2, 0\} - c_3 \max\{0, v(u+v)\} \leq 0.$$

Suppose that $u > v$, then $u^2(1 - c_1 - c_2 - c_3) \leq 0$, which implies that $c_1 + c_2 + c_3 \geq 1$, which is a contradiction. Thus $u < v$ and $u \leq (c_1 + c_2 + c_3)^{\frac{1}{2}} v = hv$ where $h = (c_1 + c_2 + c_3)^{\frac{1}{2}} < 1$

$$(\phi_2) : \phi(u, u, 0, 0, u, u) = u^2 - c_1 \max\{u^2, 0\} - c_2 \max\{0, 0, u^2\} - c_3 \max\{0, 0\} = u^2(1 - c_1 - c_2) > 0 \text{ for all } u > 0.$$

Example 7. Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - at_2 - b\frac{t_3 t_5 + t_4 t_6}{t_5 + t_6} - ct_4$ where $a + b + c = 1$, $0 < a < 1$, $0 < b < 1$, $c < 1$ then

(ϕ_1) : if $u \geq 0$ and $v \geq 0$ we have

$$(\phi_a) : \phi(u, v, v, u, u + v, 0) = u - av - bv - cu = u(1 - c) - (a + b)v \leq 0,$$

which implies that

$$u \leq \frac{a + b}{1 - c} v.$$

Therefore $u \leq v$

$$(\phi_b) : \phi(u, v, u, v, 0, u + v) = u - av - bv - cv = u - (a + b + c)v \leq 0,$$

which implies that

$$u \leq (a + b + c)v.$$

Therefore $u \leq v$

$$(\phi_2) : \phi(u, u, 0, 0, u, u) = u - au = u(1 - a) > 0, \quad \forall u > 0.$$

4. Main results

Theorem 1. *Let f and g be self maps of a 2-metric space (X, d) and let $F, G : X \to B(X)$ be set valued maps satisfying the following conditions.*

$$FX \subseteq gX \quad and \quad GX \subseteq fX \tag{1}$$

$$\phi(\delta(Fx, Gy, C), d(fx, gy, C), \delta(fx, Fx, C), \delta(gy, Gy, C), \delta(fx, Gy, C), \delta(gy, Fx, C)) \leq 0 \tag{2}$$

for all x, y in $X, \phi \in \Phi$ and $C \in B(X)$. If either

$$\text{If } f \text{ and } F \text{ are } D-\text{maps and } FX \text{ is closed.} \tag{3}$$

or

$$\text{If } g \text{ and } G \text{ are } D-\text{maps and } GX \text{ is closed.} \tag{4}$$

Then

(i) g and G have a coincidence point and f and F have a coincidence point.

Further, if

(5) *The hybrid pairs $\{f, F\}$ and $\{g, G\}$are weakly commuting of type (KB) at coincidence points.*

Then

(ii) f, g, F and G have a unique common fixed point in X.

Proof. Suppose that f and F are D-maps. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$. Since FX is closed and $FX \subseteq gX$, there is a point $u \in X$ such that $gu = t$, using inequality (2) we get

$$\phi(\delta(Fx_n, Gu, C), d(fx_n, gu, C), \delta(fx_n, Fx_n, C), \delta(gu, Gu, C), \delta(fx_n, Gu, C), \delta(gu, Fx_n, C)) \leq 0$$

Since ϕ is continuous, we get on letting $n \to \infty$.

$$\begin{aligned}\phi(\delta(t, Gu, C), d(t, t, C), \delta(t, t, C), \delta(t, Gu, C), \delta(t, Gu, C), \delta(t, t, C))\\ = \phi(\delta(t, Gu, C), 0, 0, \delta(t, Gu, C), \delta(t, Gu, C), 0))\\ = \phi(\delta(gu, Gu, C), 0, 0, \delta(gu, Gu, C), \delta(gu, Gu, C), 0)) \leq 0.\end{aligned}$$

Using (ϕ_a) we have $Gu = \{gu\} = \{t\}$ and since the hybrid pair $\{g, G\}$ is weakly commuting of type (KB) at coincidence points, we have

$$\delta(ggu, Ggu, C) \leq R\delta(gu, Gu, C)$$

which gives $Ggu = \{ggu\}$ or $Gt = \{gt\}$.

If $g^2u \neq gu$, then using (2), we have

$$\begin{aligned}&\phi(\delta(Fx_n, Ggu, C), d(fx_n, g^2u, C), \delta(fx_n, Fx_n, C),\\ &\qquad \delta(g^2u, Ggu, C), \delta(fx_n, Ggu, C), \delta(g^2u, Fx_n, C)) \leq 0.\end{aligned}$$

Letting $n \to \infty$ and using the continuity of ϕ, we obtain

$$\begin{aligned}&\phi(\delta(t, Ggu, C), d(t, g^2u, C), \delta(t, t, C), \delta(t, t, C), \delta(t, Ggu, C), \delta(g^2u, t, C))\\ &= \phi(d(gu, g^2u, C), d(gu, g^2u, C), 0, 0, d(gu, g^2u, C), d(g^2u, gu, C)) \leq 0.\end{aligned}$$

which contradicts (ϕ_2). Then $g^2u = gu$ and so

$$Ggu = \{ggu\} = \{gu\} = \{t\}.$$

Since $GX \subseteq fX$, there exists an element $v \in X$ such that $\{fv\} = Gu$. We claim that $Fv = \{fv\}$. If not, then the use of condition (2) gives

$$\begin{aligned}&\phi(\delta(Fv, Gu, C), d(fv, gu, C), \delta(fv, Fv, C), \delta(gu, Gu, C), \delta(fv, Gu, C),\\ &\delta(gu, Fv, C)) = \phi(\delta(Fv, fv, C), 0, \delta(fv, Fv, C), 0, 0, \delta(fv, Fv, C)) \leq 0.\end{aligned}$$

By (ϕ_b), we get $fv = \{fv\}$ since the hybrid pair $\{f, F\}$ is weakly commuting of type (KB) at coincidence points, we have

$$\delta(ffv, Ffv, C) \leq R\delta(fv, Fv, C),$$

which gives

$$Ffv = \{ft\} \quad \text{or} \quad Ft = \{ft\}.$$

Suppose that $f^2v = fv$. Then using (2), we have

$$\begin{aligned}&\phi(\delta(Ffv, Gu, C), d(f^2v, gu, C), \delta(f^2v, Ffv, C), \delta(gu, Gu, C),\\ &\qquad \delta(f^2v, Gu, C), \delta(gu, Ffv, C))\\ &\quad = \phi(d(f^2v, fv, C), d(f^2v, fv, C), d(f^2v, f^2v, C), \delta(gu, Gu, C),\\ &\qquad d(f^2v, fv, C), d(fv, f^2v, C))\\ &\quad = \phi(d(f^2v, fv, C), d(f^2v, fv, C), 0, 0, d(f^2v, fv, C), d(fv, f^2v, C))\\ &\quad \leq 0,\end{aligned}$$

which is a contradiction of (ϕ_2). Hence $f^2v = fv$ and so

$$Ffv = \{ffv\} = \{fv\} = \{gu\} = \{t\}.$$

Therefore t is a common fixed point of maps f, g, F and G such that $t^{'} \neq t$, the use of inequality (2) gives.

$$\begin{aligned}
&\phi(\delta(Ft,Gt^{'},C),d(ft,gt^{'},C),\delta(ft,Ft,C),\delta(gt^{'},Gt^{'},C),\\
&\qquad\delta(ft,Gt^{'},C),\delta(gt^{'},Ft,C))\\
&\quad=\phi(d(t,t^{'},C),d(t,t^{'},C),d(t,t,C),d(t^{'},t^{'},C),d(t,t^{'},C),d(t^{'},t,C))\\
&\quad=\phi(d(t,t^{'},C),d(t,t^{'},C),0,0,d(t,t^{'},C),d(t^{'},t,C))\leq 0,
\end{aligned}$$

which is a contradiction of (ϕ_2). Hence $t^{'} = t$. ■

The proof is similar if we use (4) in lieu of (3).

Corollary 1. *Let (X,d) be a 2-metric space and $f : X \to X$ and $F : X \to B(X)$ be single and set valued maps, respectively suppose that*

(6) $$FX \subseteq fX$$

(7) $$\begin{aligned}&\phi(\delta(Fx,Fy,C),d(fx,fy,C),\delta(fx,Fx,C),\\&\qquad\delta(fy,Fy,C),\delta(fx,Fy,C),\delta(fy,Fx,C))\leq 0\end{aligned}$$

for all x, y in X, where $\phi \in \Phi$.

(8) *If f and F are $D-$ maps and FX is closed.*

Then

(i) f and F have a coincidence point.

Further if

(9) *The hybrid pairs $\{f,F\}$ are weakly commuting of type (KB) at coincidence points.*

Then

(ii) f and F have a unique common fixed point in X.

Corollary 2. *Let f be a map from a 2-metric space (X,d) into itself and let $F,G \to B(X)$ be two set-valued maps such that*

(10) $$FX \subseteq fX \quad \text{and} \quad GX \subseteq fX.$$

(11) $$\begin{aligned}&\phi(\delta(Fx,Gy,C),d(fx,fy,C),\delta(fx,Fx,C),\\&\qquad\delta(fy,Gy,C),\delta(fx,Gy,C),\delta(fy,Fx,C))\leq 0\end{aligned}$$

for all x,y in X,where $\phi \in \Phi$ if either

(12) *If f and F are $D-$ maps and $FX$is closed.*

or

(13) *If f and G are $D-$ maps and GX is closed.*

Then

(i) *f and G have a coincidence point and f and F have a coincidence point.*

Further if

(14) *The hybrid pairs $\{f, F\}$ and $\{f, G\}$are weakly commuting of type (KB) at coincidence points.*

Then

(ii) *f, F and G have a unique common fixed point in X.*

Corollary 3. *If in the hypothesis of Theorem1 we have instead of (2) the next inequality.*

$$\delta^2(Fx, Gy, C) \leq \alpha \max\{d^2(fx, gy, C), \delta^2(fx, Fx, C), \delta^2(gy, Gy, C), \\ \beta(\delta^2(fx, Gy, C) + \delta^2(gy, Fx, C))\}$$

for all x, y in X, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, Then f, g, F and G have a unique common fixed point in X.

Proof. Use Theorem 1 and Example 1. ■

Corollary 4. *If in Theorem1 we have in lieu of inequality (2) the next inequality*

$$\delta(Fx, Gy, C) \leq \alpha \max\{d(fx, gy, C) \cdot \delta(fx, Fx, C), \delta(fx, Fx, C) \cdot \delta(gy, \\ Gy, C), \delta(fx, Gy, C) \cdot \delta(gy, Fx, C)\}$$

for all x, y in X, where $0 < \alpha < 1$, Then f, g, F and G have a unique common fixed point.

Proof. Use Theorem 1 and Example 2. ■

Corollary 5. *If in Theorem1 we have in lieu of inequality (2) the next inequality*

$$\delta(Fx, Gy, C) \leq \alpha \max\{d(fx, gy, C), \delta(fx, Fx, C), \delta(fx, Fx, C), \\ \delta(gy, Gy, C), \delta(fx, Gy, C) \cdot \delta(gy, Fx, C)\}$$

for all x, y in X, where $0 < \alpha < 1$, Then f, g, F and G have a unique common fixed point.

Proof. Use Theorem 1 and Example 3. ∎

Corollary 6. *If in Theorem 1 we have in lieu of inequality (2) the next inequality*

$$\begin{aligned}\delta^2(Fx,Gy,C) \leq c_1 \max\{d^2(fx,gy,C),\delta^2(fx,Fx,C)\}\\ - c_2 \max\{\delta^2(fx,Fx,C),\delta^2(gy,Gy,C),\delta^2(fx,Gy,C)\}\\ - c_3 \max\{\delta(fx,Fx,C)\delta(fx,Gy,C),\delta(gy,Gy,C)\delta(gy,Fx,C)\}\end{aligned}$$

for all x, y in X, where $c_1, c_2 > 0$, $c_3 \geq 0$ and $c_1 + c_2 + c_3 < 1$. Then f, g, F and G have a unique common fixed point.

Proof. Use Theorem 1 and Example 4. ∎

Corollary 7. *If in the hypothesis of Theorem 1 we have instead of (2) the next inequality.*

$$\begin{aligned}\delta^2(Fx,Gy,C) \leq ad^2(fx,gy,C) + b\delta^2(fx,Fx,C)\\ + c\,\frac{\delta^2(fx,Fx,C)+\delta^2(gy,Gy,C)}{\delta(fx,Gy,C)+\delta(gy,Fx,C)}\end{aligned}$$

for all x, y in X, where $0 < a < 1$ and $0 < b < 1$, $a+b+c = 1$, $c < 1$ Then f, g, F and G have a unique common fixed point.

Proof. Use Theorem 1 and Example 5. ∎

Example 8. *Let $X = [0, 10)$ Define $d(x, y, z) = \min\{|x-y|, |y-z|, |z-x|\}$ for all x, y, z in X. Then (X, d) is 2-metric space. Let $F : G \to B(X)$ and $f, g : X \to X$ be defined by*

$$F(x) = \begin{cases} [0,x], & 0 \leq x \leq 5, \\ [1, \frac{3x+5}{10}], & 5 < x < 10, \end{cases} \qquad g(x) = \begin{cases} 2x, & 0 \leq x \leq 5, \\ \frac{x-1}{2}, & 5 < x < 10, \end{cases}$$

$$G(x) = \begin{cases} [0,\frac{x}{2}], & 0 \leq x \leq 5, \\ [1, \frac{x+5}{5}], & 5 < x < 10, \end{cases} \qquad f(x) = \begin{cases} x, & 0 \leq x \leq 5, \\ \frac{2x+4}{7}, & 5 < x < 10, \end{cases}$$

Then $F(x) \subseteq g(x)$ and $G(X) \subseteq f(x)$. Consider the sequence $\{x_n\}$ define by $x_n = \{5+\frac{1}{n}\}$ in X, then $\lim_{n\to\infty} fx_n = 2 \in [1, 2] = \lim_{n\to\infty} Fx_n$.Therefore f and F are D-maps and $\lim_{n\to\infty} gx_n = 2 \in [1, 2] = \lim_{n\to\infty} Gx_n$ Therefore g and G are D-maps but $\lim_{n\to\infty} \delta(Ffx_n, fFx_n, C) \neq 0$ and $\lim_{n\to\infty} \delta(Ggx_n, gGx_n, C) \neq 0$. Therefore the pair $\{f, F\}$ and $\{g, G\}$ are not δ-compatible.

We can observe that the hybrid pairs $\{f, F\}$ and $\{g, G\}$ are weakly commuting of type (KB) at coincidence point $0 \in X$ for all positive real number R. Condition (2) of Theorem1 is satisfied if we take $\alpha = 0.9$, $C = [1, 8]$(the bounded subset of $X = [0, 10)$). Let $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \alpha \max\{t_2, t_3, t_4, t_5 t_6\}$ where $\phi \in \Phi$ is that of Example 8.

Thus all the condition of Theorem 1 and Corollary 5 are satisfied and $0 \in X$ is unique common fixed point of f, g, F and G.

Theorem 2. *Let (X, d) be a 2-metric space let $f, g : X \to X$ be two single-valued maps and let $F_n : X \to B(X)$, where $n = 1, 2, \ldots$ are set-valued maps satisfying the conditions.*

$$F_n X \subseteq gX \quad \textit{and} \quad F_{n+1} X \subseteq fX. \tag{15}$$

$$\begin{aligned}\phi(\delta(F_n x, F_{n+1} y, C), d(fx, gy, C), \delta(fx, F_n x, C), \delta(gy, F_{n+1} y, C),\\ \delta(fx, F_{n+1} y, C), \delta(gy, F_n x, C)) \leq 0\end{aligned} \tag{16}$$

for all x, y in $X, \phi \in \Phi$ if either

$$\textit{If } f \textit{ and } F_n \textit{ are } D - \textit{maps and } F_n X \textit{ is closed.} \tag{17}$$

or

$$\textit{If } g \textit{ and } F_{n+1} \textit{ are } D - \textit{maps and } F_{n+1} X \textit{ is closed.} \tag{18}$$

Then

(i) g and F_{n+1} have a coincidence point and f and F_n have a coincidence point.

Further if

$$\begin{aligned}&\textit{The hybrid pairs } \{f, F_n\} \textit{ and } \{g, F_{n+1}\} \textit{ are weakly commuting}\\ &\textit{of type (KB).}\end{aligned} \tag{19}$$

Then

(ii) f, g and F_n have a unique common fixed point in X.

Theorem 3. *Let f, g be single-valued maps of a 2-metric space (X, d) into itself and let $F, G : X \to B(X)$ be set-valued maps such that*

$$FX \subseteq gX \quad \textit{and} \quad GX \subseteq fX. \tag{20}$$

$$\begin{aligned}\delta^p(Fx, Gy, C) \leq \phi[a d^p(fx, gy, C) + (1 - a) \max\{\alpha \delta^p(fx, Fx, C),\\ \beta \delta^p(gy, Gy, C), \delta^{\frac{p}{2}}(fx, Fx, C) \delta^{\frac{p}{2}}(gy, Fx, C), \delta^{\frac{p}{2}}(gy, Fx, C)\\ \delta^{\frac{p}{2}}(fx, Gy, C), \frac{1}{2}(\delta^p(fx, Fx, C) + \delta^p(gy, Gy, C))\}]\end{aligned} \tag{21}$$

for all x, y in X, where $0 < a < 1$, $0 < \alpha$, $\beta \leq 1$, $p \in N^* = \{1, 2, ...\}$ and $\phi \in \Phi$ if either

(22) *If f and F are $D-$ maps and FX is closed.*

or

(23) *If g and G are $D-$ maps and GX is closed.*

Then

(*i*) *g and G have a coincidence point and f and F have a coincidence point.*

Further if

(24) *The hybrid pairs $\{f, F\}$ and $\{g, G\}$ are weakly commuting of type (KB) at coincidence points,*

Then

(*ii*) *f, g, F and G have a unique common fixed point in X.*

Proof. Suppose that f and F are D-maps. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$. Since FX is closed and $FX \subseteq gX$, there exists an element $u \in X$ such that $gu = t$

Suppose that $Gu \neq \{t\}$. Then use of (21) gives.

$$\begin{aligned}&\delta^p(Fx_n, Gu, C) \leq \phi[ad^p(fx_n, gu, C) + (1-a)\max\{\alpha\delta^p(fx_n, Fx_n, C),\\ &\quad \beta\delta^p(gu, Gu, C), \delta^{\frac{p}{2}}(fx_n, Fx_n, C)\delta^{\frac{p}{2}}(gu, Fx_n, C),\\ &\quad \delta^{\frac{p}{2}}(gu, Fx_n, C)\delta^{\frac{p}{2}}(fx_n, Gu, C), \frac{1}{2}(\delta^p(fx_n, Fx_n, C) + \delta^p(gu, Gu, C))\}].\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\delta^p(t, Gu, C) &\leq \phi[(1-a)\max\{0, \beta\delta^p(t, Gu, C), \frac{1}{2}\delta^p(t, Gu, C)\}]\\ &= \phi[(1-a)\max\{(\beta, \frac{1}{2})\delta^p(t, Gu, C)\}]\\ &< (1-a)\max\{(\beta, \frac{1}{2})\delta^p(t, Gu, C)\}\\ &< \delta^p(t, Gu, C)\},\end{aligned}$$

which is a contradiction. Therefore

$$Gu = \{t\} = \{gu\}.$$

Since the pair $\{g, G\}$ is weakly commuting of type (KB) at coincidence point in X, we have

$$\delta(ggu, Ggu, C) \leq R\delta(gu, Gu, C),$$

which gives

$$Ggu = \{ggu\} \quad or \quad Gt = \{gt\}.$$

If $Gt \neq \{t\}$ then by (21) we get

$$\delta^p(Fx_n, Gt, C) \leq \phi[ad^p(fx_n, gt, C) + (1-a)\max\{\alpha\delta^p(fx_n, Fx_n, C), \\ \beta\delta^p(gt, Gt, C), \delta^{\frac{p}{2}}(fx_n, Fx_n, C)\delta^{\frac{p}{2}}(gt, Fx_n, C), \\ \delta^{\frac{p}{2}}(gt, Fx_n, C)\delta^{\frac{p}{2}}(fx_n, Gt, C), \frac{1}{2}(\delta^p(fx_n, Fx_n, C) + \delta^p(gt, Gt, C))\}].$$

Letting $n \to \infty$, it follows that

$$\begin{aligned} d^p(t, gt, C) &= \delta^p(t, Gt, C) \\ &\leq \phi[ad^p(t, gt, C) + (1-a)d^p(t, gt, C)] \\ &= \phi[d^p(t, gt, C)] \\ &< d^p(t, gt, C), \end{aligned}$$

which is a contradiction and so $Gt = \{gt\} = \{t\}$. Since $GX \subseteq fx$,there is a point $v \in X$ such that

$$\{fv\} = gt = \{t\}.$$

We claim that $Fv = \{fv\}$. If not, then using (21) we obtain

$$\delta^p(Fv, Gt, C) \leq \phi[ad^p(fv, gt, C) + (1-a)\max\{\alpha\delta^p(fv, Fv, C), \\ \beta\delta^p(gt, Gt, C), \delta^{\frac{p}{2}}(fv, Fv, C)\delta^{\frac{p}{2}}(gt, Fv, C), \\ \delta^{\frac{p}{2}}(gt, Fv, C)\delta^{\frac{p}{2}}(fv, Gt, C)\frac{1}{2}(\delta^p(fv, Fv, C) + \delta^p(gt, Gt, C))\}].$$

That is

$$\begin{aligned} \delta^p(Fv, t, C) \leq \phi[(1-a)\max\{\alpha\delta^p(t, Fv, C), 0, \delta^p(t, Fv, C), \\ \frac{1}{2}(\delta^p(t, Fv, C))\}] = \phi[(1-a)\delta^p(t, Fv, C)] \\ < (1-a)\delta^p(t, Fv, C) \\ < \delta^p(t, Fv, C), \end{aligned}$$

a contradiction. This implies that $Fv = \{t\} = \{fv\}$ and since the pair $\{f, F\}$ is weakly commuting we have $fFV = Ffv$ i.e. $Ft = \{ft\}$.

Suppose that $Ft \neq \{t\}$ then by (21) we have

$$\begin{aligned}\delta^p(Ft,Gt,C) \leq \phi[ad^p(ft,gt,C) + (1-a)\max\{\alpha\delta^p(ft,Ft,C),\\ \beta\delta^p(gt,Gt,C), \delta^{\frac{p}{2}}(ft,Ft,C)\delta^{\frac{p}{2}}(gt,Ft,C),\\ \delta^{\frac{p}{2}}(gt,Ft,C)\delta^{\frac{p}{2}}(ft,Gt,C), \frac{1}{2}(\delta^p(ft,Ft,C) + \delta^p(gt,Gt,C))\}]\end{aligned}$$

that is

$$\begin{aligned}d^p(ft,t,C) &= \delta^p(Ft,t,C)\\ &\leq \phi[ad^p(ft,t,C) + (1-a)\max\{0, d^p(ft,t,C)\}]\\ &= \phi[d^p(ft,t,C)]\\ &< d^p(ft,t,C),\end{aligned}$$

which is a contradiction. Hence $Ft = \{t\} = \{ft\}$. Therefore t is a common fixed point of both f, g, F and G.

Finally, we prove that t is unique.

Suppose that $t' \neq t$, is another common fixed point of both f, g, F and G by using inequality (21) we obtain

$$\begin{aligned}d^p(t,t',C) &= \delta^p(Ft,Gt',C)\\ &\leq \phi[ad^p(ft,gt',C) + (1-a)\max\{\alpha\delta^p(ft,Ft,C), \beta\delta^p(gt',Gt',C),\\ &\quad \delta^{\frac{p}{2}}(ft,Ft,C)\delta^{\frac{p}{2}}(gt',Ft,C), \delta^{\frac{p}{2}}(gt',Ft,C)\delta^{\frac{p}{2}}(ft,Gt',C),\\ &\quad \frac{1}{2}(\delta^p(ft,Ft,C) + \delta^p(gt',Gt',C))\}]\\ &= \phi(d^p(t,t',C)) < d^p(t,t',C),\end{aligned}$$

a contradiction. This implies that $t' = t$. ■

Note that if we let $F = G$ and $f = g$ in Theorem 2 we get the following corollary.

Corollary 8. *Let (X,d) be a 2-metric space and let $f : X \to X$ a single-valued map and let $F : X \to B(X)$ be set-valued map satisfying the conditions.*

$$FX \subseteq fX \tag{25}$$

$$\begin{aligned}\delta^p(Fx,Fy,C) \leq \phi[ad^p(fx,fy,C) + (1-a)\max\{\alpha\delta^p(fx,Fx,C),\\ \beta\delta^p(fy,Fy,C), \delta^{\frac{p}{2}}(fx,Fx,C)\delta^{\frac{p}{2}}(fy,Fx,C),\\ \delta^{\frac{p}{2}}(fy,Fx,C)\delta^{\frac{p}{2}}(fx,Fy,C), \frac{1}{2}(\delta^p(fx,Fx,C) + \delta^p(fy,Fy,C))\}]\end{aligned} \tag{26}$$

for all x, y *in* X, *where* $0 < a < 1$, $0 < \alpha$, $\beta \leq 1$, $p \in N^*$ *and* $\phi \in \Phi$

$$\text{(27)} \qquad \textit{If } f \textit{ and } F \textit{ are } D-\textit{maps and } FX \textit{ is closed .}$$

Then
(i) f *and* F *have a coincidence point.*
Further if

(28) *The hybrid pairs* $\{f, F\}$ *are weakly commuting of type (KB) at coincidence points.*

Then
(ii) f *and* F *have a unique common fixed point in* X.

If we put $f = g$, in Theorem 2 then we obtain the next result.

Corollary 9. *Let* $f : X \to X$ *be a single-valued map and let* $F, G : X \to B(X)$ *be two set-valued maps such that*

$$\text{(29)} \qquad FX \subseteq fX \quad \textit{and} \quad GX \subseteq fX$$

$$\text{(30)} \quad \begin{aligned} \delta^p(Fx, Gy, C) \leq \phi[ad^p(fx, fy, C) + (1-a)\max\{\alpha\delta^p(fx, Fx, C),\\ \beta\delta^p(fy, Gy, C), \delta^{\frac{p}{2}}(fx, Fx, C)\delta^{\frac{p}{2}}(fy, Fx, C),\\ \delta^{\frac{p}{2}}(fy, Fx, C)\delta^{\frac{p}{2}}(fx, Gy, C), \frac{1}{2}(\delta^p(fx, Fx, C) + \delta^p(fy, Gy, C))\}] \end{aligned}$$

for all x, y *in* X, *where* $0 < a < 1$, $0 < \alpha$, $\beta \leq 1$, $p \in N^*$ *and* $\phi \in \Phi$ *if either*

$$\text{(31)} \qquad \textit{If } f \textit{ and } F \textit{ are } D-\textit{maps and } FX \textit{ is closed.}$$

or

$$\text{(32)} \qquad \textit{If } f \textit{ and } G \textit{ are } D-\textit{maps and } GX \textit{ is closed.}$$

Then
(i) f *and* G *have a coincidence point and* f *and* F *have a coincidence point.*
Further if

(33) *The hybrid pairs* $\{f, F\}$ *and* $\{f, G\}$*a re weakly commuting of type (KB) at coincidence points.*

Then
(ii) f, F and G have a unique common fixed point in X.

Corollary 10. *If in the hypothesis of Theorem 2 we have the following inequality instead of condition (21).*

$$\delta^p(Fx, Gy, C) \leq \phi[ad^p(fx, gy, C) + (1-a)\max\{\delta(fx, Fx, C),$$
$$\delta(gy, Gy, C), \delta^{\frac{1}{2}}(fx, Fx, C)\delta^{\frac{1}{2}}(gy, Fx, C),$$
$$\delta^{\frac{1}{2}}(gy, Fx, C)\delta^{\frac{1}{2}}(fx, Gy, C)\}^p]$$

for all x, y in X, where $0 < a < 1$, $p \in N^*$ and $\phi \in \Phi$

Then f, g, F and G have a unique common fixed point in X.

Proof. It is similar to the proof of Theorem 2. ■

Now, we generalize Theorem 2.

Theorem 4. *Let $f, g : X \to X$ be two single-valued maps and let $F_n : X \to B(X)$, where $n = 1, 2, \ldots$ be set-valued maps satisfying the following conditions.*

(34) $$F_n X \subseteq gX \quad \text{and} \quad F_{n+1}X \subseteq fX$$

(35) $$\delta^p(F_n x, F_{n+1}y, C) \leq \phi[ad^p(fx, gy, C) + (1-a)\max\{\alpha\delta^p(fx, F_n x, C),$$
$$\beta\delta^p(gy, F_{n+1}y, C), \delta^{\frac{p}{2}}(fx, F_n x, C)\delta^{\frac{p}{2}}(gy, F_n x, C),$$
$$\delta^{\frac{p}{2}}(gy, F_n x, C)\delta^{\frac{p}{2}}(fx, F_{n+1}y, C), \frac{1}{2}(\delta^p(fx, F_n x, C) + \delta^p(gy, F_{n+1}y, C))\}]$$

for all x, y in X, where $0 < a < 1$, $0 < \alpha$, $\beta \leq 1$, $p \in N^* = \{1, 2, \ldots\}$ and $\phi \in \Phi$

(36) *If f and F_n are $D-$ maps and FX is closed.*

or

(37) *If g and F_{n+1} are $D-$ maps and GX is closed.*

Then

(i) g and F_{n+1} have a coincidence point and f and F_n have a coincidence point.

Further if

(38) *The hybrid pairs $\{f, F_n\}$ and $\{g, F_{n+1}\}$ are weakly commuting of type (KB) at coincidence points,*

Then

(ii) f, g and F_n have a unique common fixed point in X.

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