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LACUNARY p-ABSOLUTELY SUMMABLE SEQUENCES OF FUZZY REAL NUMBERS

Abstract. In this paper we have introduced the sequence space $(\ell_p)_\theta^F$. We have established a relation between $(\ell_p)_\theta^F$ and $Ces(p)$. We have studied some algebraic properties like solidness, symmetricness, convergence free and provided some inclusion results.

Key words: Lacunary sequence, p-absolutely summable, solid, symmetric, convergence free, sequence algebra etc.

AMS Mathematics Subject Classification: 40A05, 40A25, 40A30, 40C05.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [7] in 1965. Since then many author have working on it and research have been taking place for its further development and application. In recent times the notion is applied in almost all the branches of science and technology. Subsequently fuzzy numbers are introduced and different properties and arithmetic operations with fuzzy numbers are established. It is found that in many respects fuzzy numbers depict the physical world more realistically than single-valued numbers. Many authors studied different mathematical notions in terms of fuzzy numbers and established some important results, which opens a new era in the field of mathematics. Subsequently fuzzy real-valued sequences are introduced and many results are established. We start here with some preliminaries definitions of fuzzy real number.

A fuzzy real number X is a fuzzy set on R i.e. a mapping $X : R \to I$ $(= [0,1])$, associating each real number t, with its grade of membership $X(t)$.

In general we consider those fuzzy numbers, which satisfy three conditions, viz. *normal*, *upper-semi-continuous* and *convex*.

A fuzzy real number X is said to be *upper-semi-continuous* if for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$, for all $a \in I$ is open in the usual topology of R.

If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

The class of all *upper-semi-continuous*, *normal*, *convex* fuzzy real number is denoted by $R(I)$.

The absolute value of $X \in R(I)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for} \quad t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The set R of all real numbers can be embedded in $R(I)$. For $r \in R$, $\overline{r} \in R(I)$ is defined by

$$\overline{r}(t) = \begin{cases} 1, & \text{for} \quad t = r, \\ 0, & \text{for} \quad t \neq r. \end{cases}$$

Accordingly the additive identity and multiplicative identity of $R(I)$ are denoted by $\overline{0}$ and $\overline{1}$ respectively.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$. Then we have

$$d(X, Y) = \max(|X^L - Y^L|, |X^R - Y^R|).$$

It is easy to prove that (D, d) is a complete metric space.

Also define $\overline{d} : R(I) \times R(I) \to R$ by

$$\overline{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha), \quad \text{for} \quad X, Y \in R(I).$$

It is well established that $(R(I), \overline{d})$ is a complete metric space.

2. Definitions and preliminaries

In this part we recall some of the preliminary notions, which are closely related to the article and reveal the main theme of the paper.

The set E^F of sequence of fuzzy real numbers is closed under addition and scalar multiplication defined as follows:

For (X_k) and $(Y_k) \in E^F, r \in R$,

$$(X_k) + (Y_k) = (X_k + Y_k) \in E^F$$

and

$$r(X_k) = (rX_k) \in E^F,$$

where

$$r(X_k)(t) = \begin{cases} X_k(r^{-1}t), & \text{if} \quad r \neq 0, \\ \overline{0}, & \text{if} \quad r = 0. \end{cases}$$

A fuzzy real-valued sequence is denoted by (X_k), where $X_k \in R(I)$, for all $k \in N$.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0, if for every $\epsilon > 0$, there exists $k_0 \in N$ such that $\overline{d}(X_k, X_0) < \epsilon$, for all $k \geq k_0$.

A fuzzy real valued sequence space E^F is said to be solid if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \leq |X_k|$, for all $k \in N$.

A fuzzy real valued sequence space E^F is said to be symmetric if $(X_{\pi(k)}) \in E^F$, whenever $(X_k) \in E^F$, where π is a permutation on N.

A fuzzy real valued sequence space E^F is said to be convergence free if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $X_k = \overline{0}$ implies $Y_k = \overline{0}$.

A sequence space E^F is said to be sequence algebra if $(X_k) \bigotimes (Y_k) \in E^F$ whenever $(X_k), (Y_k) \in E^F$.

By a lacunary sequence $\theta = (k_r)(r = 0, 1, 2, 3, ..)$, we mean an increasing sequence of non-negative integers with, $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The interval determined by θ is denoted be by $I_r = (k_{r-1}, k_r]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.

Different classes of lacunary sequences have been investigated by the workers on sequence spaces in the recent past. Among them Altin et. al [1], Altinok et. al [2], Jinlu Li [4] and Savas and Karakaya [5] are a few to be named.

Many authors have studied the sequence space $\ell(p)$ and established many important results. Tripathy and Dutta [6] introduced the sequence space ${}_2\ell_F^p$ and established some important results.

Karakaya [3] introduced the fuzzy real-valued sequence space $\ell(p, \theta)$ and studied some important properties of the space.

In this paper we have introduced the sequence space $(\ell_p)_\theta^F$ and proved some algebraic properties and some inclusion results between Cesaro sequence and lacunary p-absolutely summable sequence of fuzzy real numbers.

This is to note that $(\ell_p)_\theta^F = Ces(p)$, for $\theta = (2^r)$.

We define the sequence space $(\ell_p)_\theta^F$, lacunary p-absolutely summable sequence of fuzzy real numbers as follows:

$$(\ell_p)_\theta^F = \left\{ X = (X_n) : \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p < \infty \right\},$$

where $1 \leq p < \infty$.

3. Main results

Theorem 1. *The sequence space $(\ell_p)_\theta^F$ is closed under addition and scalar multiplication.*

Proof. (i) Let $\theta = (k_r)$ be a lacunary sequence and $(X_n) \in (\ell_p)_\theta^F$, $\lambda \in R$. Then

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p < \infty$$

Then we write

$$\begin{aligned} \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(\lambda X_n, \overline{0}) \right)^p &= \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \{ |\lambda| \overline{d}(X_n, \overline{0}) \} \right)^p \\ &= |\lambda|^p \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p < \infty. \end{aligned}$$

Thus it implies that $(\lambda X_n) \in (\ell_p)_\theta^F$

(ii) Let $\theta = (k_r)$ be a lacunary sequence and $(X_n), (Y_n) \in (\ell_p)_\theta^F$. Then

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p < \infty \quad \text{and} \quad \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(Y_n, \overline{0}) \right)^p < \infty$$

We can write

$$\begin{aligned} \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}((X_n + Y_n), \overline{0}) \right)^p &\leq \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p \\ &+ \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(Y_n, \overline{0}) \right)^p < \infty. \end{aligned}$$

Thus $(X_n + Y_n) \in (\ell_p)_\theta^F$. ■

Theorem 2. *Let* $0 < p < 1$, *then* $\liminf q_r > 1$, *implies* $Ces(p) \subset (\ell_p)_\theta^F$.

Proof. If $\liminf q_r > 1$, there exist $\delta > 0$ such that $q_r = 1 + \delta$, for all $r > 1$.

Let $(X_n) \in Ces(p)$, then we can write

$$\sum_{r=1}^{\infty} \left(\frac{1}{k_r} \sum_{n=1}^{k_r} \overline{d}(X_n, \overline{0}) \right)^p < \infty.$$

Now we have

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n \in I_r} \overline{d}(X_n, \overline{0}) \right)^p = \sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{n=1}^{k_r} \overline{d}(X_n, \overline{0}) - \frac{1}{h_r} \sum_{n=1}^{k_{r-1}} \overline{d}(X_n, \overline{0}) \right)^p$$

$$\leq \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n=1}^{k_r}\overline{d}(X_n,\overline{0})\right)^p + \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n=1}^{k_{r-1}}\overline{d}(X_n,\overline{0})\right)^p$$

$$= \sum_{r=1}^{\infty}\frac{1}{k_r}\left(\frac{k_r}{h_r}\sum_{n=1}^{k_r}\overline{d}(X_n,\overline{0})\right)^p + \sum_{r=1}^{\infty}\frac{1}{k_{r-1}}\left(\frac{k_{r-1}}{h_r}\sum_{n=1}^{k_{r-1}}\overline{d}(X_n,\overline{0})\right)^p.$$

Since $h_r = k_r - k_{r-1}$ we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_{r-1}} \leq \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} \leq \frac{1}{\delta}$$

Hence

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p \leq \sum_{r=1}^{\infty}\left(\frac{1}{k_r}\sum_{n=1}^{k_r}\overline{d}(X_n,\overline{0})\right)^p + \sum_{r=1}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{n=1}^{k_{r-1}}\overline{d}(X_n,\overline{0})\right)^p \tag{1}$$

Since $(X_n) \in Ces(p)$, we get

$$\sum_{r=1}^{\infty}\left(\frac{1}{k_r}\sum_{n=1}^{k_r}\overline{d}(X_n,\overline{0})\right)^p < \infty \quad \text{and} \quad \sum_{r=1}^{\infty}\left(\frac{1}{k_{r-1}}\sum_{n=1}^{k_{r-1}}\overline{d}(X_n,\overline{0})\right)^p < \infty.$$

Thus (1) implies

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p < \infty.$$

■

Hence we get $(X_n) \in (\ell_p)_\theta^F$.

Theorem 3. *If* $1 < \limsup q_r < \infty$*, then* $(\ell_p)_\theta^F \subset Ces(p)$.

The proof of this theorem is similar to Theorem 2, so omitted.

Theorem 4. *The sequence space* $(\ell_p)_\theta^F$ *is solid.*

Proof. Let $\theta = (k_r)$ be a lacunary sequence and (X_n) and (Y_n) be two sequences of fuzzy real numbers such that $\overline{d}(Y_n,\overline{0}) \leq \overline{d}(X_n,\overline{0})$ for all $n \in N$. Let $(X_n) \in (\ell_p)_\theta^F$, then

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p < \infty.$$

Now we have

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(Y_n,\overline{0})\right)^p \le \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p < \infty.$$

Thus $(Y_n) \in (\ell_p)_\theta^F$. Hence the sequence space $(\ell_p)_\theta^F$ is solid. ■

Theorem 5. *The sequence space $(\ell_p)_\theta^F$ is sequence algebra.*

Proof. Let $\theta = (k_r)$ be a lacunary sequence and (X_n) and (Y_n) be two sequences of fuzzy real numbers in $(\ell_p)_\theta^F$. Then

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p < \infty \quad \text{and} \quad \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(Y_n,\overline{0})\right)^p < \infty,$$

for all $n \in N$.

Thus we can write

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n\otimes Y_n,\overline{0})\right)^p \le \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\overline{d}(Y_n,\overline{0})\right)^p$$
$$\le \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(X_n,\overline{0})\right)^p \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{n\in I_r}\overline{d}(Y_n,\overline{0})\right)^p < \infty$$

Thus $(X_n \bigotimes Y_n) \in (\ell_p)_\theta^F$.

Hence the sequence space $(\ell_p)_\theta^F$ is sequence algebra. ■

Theorem 6. *The sequence space $(\ell_p)_\theta^F$ is not symmetric.*

Proof. We shall prove it by the following example.

Example 1. Let $p = 1$ and $\theta = (2^r)$ be a lacunary sequence. Consider the sequence (X_k) taken from $(\ell_p)_\theta^F$, defined as follows:

For $k = 2^r$, $r \in N$.

$$(X_k)(t) = \begin{cases} (k^2t+1), & \text{for} \quad \frac{-1}{k^2} \le t \le 0, \\ (1-k^2t), & \text{for} \quad 0 \le t \le \frac{1}{k^2}, \\ 0, & \text{otherwise.} \end{cases}$$

$X_k = \overline{0}$, otherwise.

Then

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\overline{d}(X_k,\overline{0})\right)^p = \sum_{r=1}^{\infty}\left(\frac{1}{2^r}\sum_{k\in I_r}k^{-2}\right) < \infty.$$

Now consider the rearrangement of (X_k) defined as $(Y_k) = (X_1, \overline{0}, X_2, \overline{0}, X_3, \overline{0}, \ldots)$.

Then we have

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} \overline{d}(Y_k, \overline{0}) \right)^p \to \infty.$$

Thus $(Y_k) \notin (\ell_p)_\theta^F$.

■

Hence $(\ell_p)_\theta^F$ is not symmetric.

Theorem 7. *The sequence space $(\ell_p)_\theta^F$ is not convergence free.*

Proof. We shall prove it by the following example.

Example 2. Let $\theta = (3^r)$ be a lacunary sequence and $p = 1$. Consider the sequence (X_k) defined as follows:

$$X_k(t) = \begin{cases} (k^2 t + 1), & \text{for } \frac{-1}{k^2} \le t \le 0, \\ (1 - k^2 t), & \text{for } 0 \le t \le \frac{1}{k^2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we get

$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} \overline{d}(X_k, \overline{0}) \right)^p = \sum_{r=1}^{\infty} \left(\frac{1}{2.3^{r-1}} \sum_{k \in I_r} k^{-2} \right) < \infty.$$

Thus $(X_k) \in (\ell_p)_\theta^F$.

Now consider the sequence (Y_k) defined by

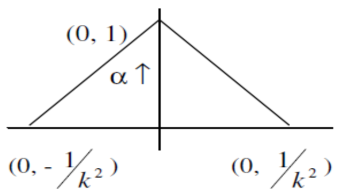
$$Y_k(t) = \begin{cases} (t\sqrt{k} + 1), & \text{for } \frac{-1}{\sqrt{k}} \le t \le 0, \\ (1 - t\sqrt{k}), & \text{for } 0 \le t \le \frac{1}{\sqrt{k}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

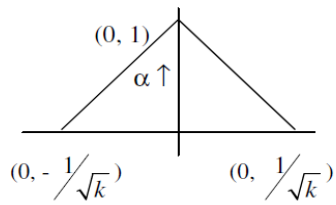
$$\sum_{r=1}^{\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} \overline{d}(Y_k, \overline{0}) \right)^p = \sum_{r=1}^{\infty} \left(\frac{1}{2.3^{r-1}} \sum_{k \in I_r} k^{-1/2} \right) = \infty.$$

Thus $(Y_k) \notin (\ell_p)_\theta^F$.

Hence the sequence space $(\ell_p)_\theta^F$ is not convergence free.



Graph of $X_k(t)$



Graph of $Y_k(t)$

■

Theorem 8. *Let* $0 < p < q$. *Then* $(\ell_p)_\theta^F \subset (\ell_q)_\theta^F$.

Proof. The proof is clear from the following inclusion relation:
For $(X_k) \in (\ell_p)_\theta^F$,

$$\sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\overline{d}(X_k,\overline{0})\right)^p \subset \sum_{r=1}^{\infty}\left(\frac{1}{h_r}\sum_{k\in I_r}\overline{d}(X_k,\overline{0})\right)^q .$$

■

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