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APPLICATIONS OF MINIMAL γ-SEMI-OPEN SETS

Abstract. In this paper, minimal γ-semi-open sets in topological spaces introduce and discuss. Also some properties of pre γ-semi-open sets using properties of minimal γ-semi-open sets obtain. As an application of a theory of minimal γ-semi-open sets, I obtain a sufficient condition for a γ-semi-locally finite space to be a pre γ-semi-T_2 space.

Key words: γ-semi-closed (open), γ-semi-closure, minimal γ-semi-open, pre γ-semi-open, finite γ-semi-open, γ-semi-locally finite, pre γ-semi-T_2 space.

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1. Introduction

N. Levine [11] introduced the notion of semi-open sets in the context of topological spaces, that is : $A \subseteq X$ is semi-open in a topological space X if $A \subseteq cl(int(A))$, where cl and int represents the closure and interior. S. Kasahara [12] introduced and discussed an operation γ of a topology τ into the power set P(X) of a space X . Several known characterizations of compact spaces, nearly compact spaces and H-closed spaces are unified by generalizing the notion of compactness with the help of a certain operation γ. H. Ogata [13], introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ by using operation γ. He introduced the notions of γ-T_i (i = 0, 1/2, 1, 2) spaces which generalize T_i - spaces (i = 0, 1/2, 1, 2) respectively. Moreover, he investigated general operator approaches of the closed graph mappings.

Carpintero, et al [7], introduced the notion of α-semi-open set as a generalization of semi-open set: Let $\alpha : P(X) \to P(X)$, $U \subseteq \alpha(U)$, $\forall\ U \in \tau$. $A \subseteq X$ is α-semi-open if there exists $U \in \tau$, such that $U \subseteq A \subseteq \alpha(U)$. It is clear that under this notation, if τ_α (τ_γ in the sense of H. Ogata [13]) is the set of α-open set (γ-open sets in the sense of H. Ogata [13]) and $\alpha - SO(X)$ the set α-semi-open then, $\tau_\alpha \subseteq \tau \subseteq \alpha - SO(X)$.

S. Hussain and B. Ahmad [1-6] and [8-10] continued studying the properties of γ-operations on topological spaces and investigated many interesting results. Recently B. Ahmad and S. Hussain [2] defined and discussed γ-semi-open sets in topological spaces. They explored many characterizations and properties of γ-semi-open(closed) sets, γ-semi-interior(exterior), γ-semi-closure, γ-semi-boundary and semi-regular operation. It is interesting to mention that γ-semi-open sets generalized γ-open sets introduced by H. Ogata [13].

In this paper, minimal γ-semi-open sets in topological spaces introduce and discuss. Also some properties of pre γ-semi-open sets using properties of minimal γ-semi-open sets obtain. As an application of a theory of minimal γ-semi-open sets, I obtain a sufficient condition for a γ-semi-locally finite space to be a pre γ-semi-T_2 space.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Definition 1 ([12]). *Let X be a space. An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ.*

Definition 2 ([12]). *Let $A \subseteq X$. A point $x \in A$ is said to be γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus*

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [13] if and only if $A = int_\gamma(A)$. A set A is called γ- closed [1] if and only if X-A is γ-open.

Definition 3 ([13]). *A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.*

Definition 4 ([12]). *An operation γ on τ is said e regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.*

Definition 5 ([13]). *An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \supseteq B$.*

Definition 6 ([2]). *A subset A of a space X is said to be a γ-semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed if and only if $X - A$ is γ-semi-open in X. Note that A is γ-semi-closed if and only if $int_\gamma cl_\gamma(A) \subseteq A$.*

Definition 7 ([2]). *Let A be a subset of a space X. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure of A and is denoted by $scl_\gamma(A)$. Note that A is γ-semi-closed if and only if $scl_\gamma(A) = A$. The set of all γ-semi-closed subsets of A is denoted by $SC_\gamma(A)$.*

Definition 8 ([2]). *Let A be a subset of a space X. The union of γ-semi-open subsets contained in A is called γ-semi-interior of A and is denoted by $sint_\gamma(A)$.*

Definition 9 ([2]). *An operation γ on τ is said be semi-regular, if for any semi-open sets U, V of $x \in X$, there exists a semi-open W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.*

Note that, if γ is semi-regular operation, then intersection of two γ-semi-open sets is γ-semi-open [2]. To explore the characterizations and properties of γ-semi-open set and semi-regular operation, I refer to [2].

3. Minimal γ-semi-open sets

In view of the definition of minimal γ-open sets [8], we define minimal γ-semi-open sets as:

Definition 10. *Let X be a space and $A \subseteq X$ a γ-semi-open set. Then A is called a minimal γ-semi-open set, if ϕ and A are the only γ-semi-open subsets of A.*

The following is immediate:

Proposition 1. *Let X be a space. Then*

1) *Let A be a minimal γ-semi-open set and B a γ-semi-open set. Then $A \cap B = \phi$ or $A \subseteq B$, where γ is semi-regular.*

2) *Let B and C be minimal γ-semi-open sets. Then $B \cap C = \phi$ or $B = C$, where γ is semi-regular.*

Proposition 2. *Let X be a space and A a minimal γ-semi-open set. If $a \in A$, then for any γ-semi-open nbd B of a, $A \subseteq B$, where γ is semi-regular.*

Proof. Suppose on the contrary that B is a γ-semi-open nbd B of $a \in A$ such that $A \nsubseteq B$. Since γ is a semi-regular operation, therefore $A \cap B$ is a γ-semi-open set [2] with $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This is a contradiction to our supposition that A is a minimal γ-semi-open se. Hence the proof. ■

Example 1. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by $\gamma(A) = cl(A)$. Then calculations show that the operation γ is semi-regular. The semi-open sets are ϕ, X, $\{a\}$, $\{a, b\}$, $\{a, c\}$ and $SO_\gamma(X) = \{\phi, X\}$. Note that the above proposition is true in this case.

The following proposition easily follows from Proposition 2:

Proposition 3. *Let X be a space and A a minimal γ-semi-open set. Then for any $y \in A$, $A = \cap\{B : B$ is γ-semi-open nbd of $y\}$, where γ is semi-regular.*

Similarly we have:

Proposition 4. *Let A be a minimal γ-semi-open set in X and $x \in X$ such that $x \notin A$. Then for any γ-semi-open nbd C of x, $C \cap A = \phi$ or $A \subseteq C$, where γ is semi-regular.*

Corollary 1. *Let A be a minimal γ-semi-open set in X and $x \in X$ such that $x \notin A$. If $A_x = \{B : B$ is a γ-semi-open nbd of $x\}$. Then $A_x \cap A = \phi$ or $A \subseteq A_x$, where γ is semi-regular.*

Proposition 5. *Let X be a space and A be an open subset of X. If A is a nonempty minimal γ-semi-open set of X, then for a nonempty subset C of A, $A \subseteq cl_\gamma(C)$, where γ is semi-regular and monotone.*

Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any γ-semi-open nbd B of y. By Proposition 2, we have $A \subseteq B$. Also since γ is monotone, $C \subseteq A \cap C \subseteq A^\gamma \cap C \subseteq B^\gamma \cap C$. Thus we have $B^\gamma \cap C \neq \phi$ and hence $y \in cl_\gamma(C)$ [1]. This implies that $A \subseteq cl_\gamma(C)$. This completes the proof. ■

Proposition 6. *Let A be a nonempty γ-semi-open subset of a space X. If $A \subseteq cl_\gamma(C)$, then $cl_\gamma(A) = cl_\gamma(C)$, for any nonempty subset C of A, where γ is open.*

Proof. Since for any nonempty C such that $C \subseteq A$ implies $cl_\gamma(C) \subseteq cl_\gamma(A)$. On the other hand, by supposition, we have, $A \subseteq cl_\gamma(C)$. Since γ is open, $cl_\gamma(A) \subseteq cl_\gamma(cl_\gamma(C)) = cl_\gamma(C)$ [1] implies $cl_\gamma(A) \subseteq cl_\gamma(C)$. Hence the proof. ■

The following example shows that the condition that γ is open is necessary for the above proposition.

Example 2. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } \quad b \in A \\ int(cl(A)), & \text{if } \quad b \notin A. \end{cases}$$

Then the operation γ is not open. The γ-semi-open sets are ϕ, X, $\{b\}$, $\{a,c\}$. Let $A = \{b\}$ and $C = \{a,b\}$, then $cl_\gamma(A) = \{b\} \neq X = cl_\gamma(C)$.

Proposition 7. *Let A be a nonempty γ-semi-open subset of a space X. If $cl_\gamma(A) = cl_\gamma(C)$, for any nonempty subset C of A, then A is a minimal γ-semi-open set.*

Proof. We suppose on the contrary that A is not a minimal γ-semi-open set. Then there exists a nonempty γ-semi-open set D such that $D \subsetneq A$ and hence there exists an element $x \in A$ such that $x \notin D$. Then we have $cl_\gamma(\{x\}) \subseteq D^\gamma$ implies that $cl_\gamma(\{x\}) \neq cl_\gamma(A)$. This contradiction proves the proposition. ■

Combining Propositions 5, 6 and 7, we have:

Theorem 1. *Let X be a space and A be an open subset of X. If A is a nonempty γ-semi-open subset of space X. Then the following are equivalent:*

(a) *A is minimal γ-semi-open set.*

(b) *For any nonempty subset C of A, $A \subseteq cl_\gamma(A)$, where γ is semi-regular and monotone.*

(c) *For any nonempty subset C of A, $cl_\gamma(A) = cl_\gamma(C)$, where γ is open.*

Definition 11. *Let X be a space and $A \subseteq X$. Then A is called a pre-γ-semi-open set, if $A \subseteq sint_\gamma(cl_\gamma(A))$. The family of all pre-$\gamma$-semi-open sets of X will be denoted by $PSO_\gamma(X)$.*

In view of the definition of a pre γ-T_2 space [8], we define a pre γ-semi-T_2 space as:

Definition 12. *A space X is called a pre γ-semi-T_2 space, if for any $x, y \in X$, $x \neq y$, there exist subsets U and V of $PSO_\gamma(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.*

Proposition 8. *Let X be a space and A be an open subset of X. If A is a minimal γ-semi-open set, then $\phi \neq C \subseteq A$ is a pre-γ-semi-open set, where γ is semi-regular and monotone.*

Proof. Let A be a minimal γ-semi-open set and $\phi \neq C \subseteq A$. By Proposition 5, we have $A \subseteq cl_\gamma(C)$ implies $sint_\gamma(A) \subseteq sint_\gamma(cl_\gamma(C))$. Since A is a γ-semi-open set, therefore $C \subseteq A = sint_\gamma(A) \subseteq sint_\gamma(cl_\gamma(C))$ or $C \subseteq sint_\gamma(cl_\gamma(C))$, that is, C is pre-γ-semi-open. Hence the proof. ■

We use Theorem 1(a) and prove the following:

Theorem 2. *Let X be a space, $\phi \neq B \subseteq X$, A a minimal γ-semi-open set and an open subset of X. If there exists a γ-semi-open set C containing B such that $C \subseteq cl_\gamma(B \cup A)$, then for any nonempty subset D of A, $B \cup D$ is a pre-γ-semi-open set, where γ is regular.*

Proof. Suppose A is a minimal γ-semi-open set in X. Since γ is regular, therefore for any nonempty subset D of A, we have

$$cl_\gamma(B \cup D) = cl_\gamma(B) \cup cl_\gamma(D) = cl_\gamma(B) \cup cl_\gamma(A) = cl_\gamma(B \cup A) \quad \text{with [1]}.$$

By supposition, we have $C \subseteq cl_\gamma(B \cup A) = cl_\gamma(B \cup D)$ implies $sint_\gamma(C) \subseteq sint_\gamma(cl_\gamma(B \cup D))$, C being γ-semi-open set such that $B \subseteq C$. It follows that

$$(1) \quad B \subseteq C = sint_\gamma(C) \subseteq sint_\gamma(cl_\gamma(B \cup D)) \quad \text{or} \quad B \subseteq sint_\gamma(cl_\gamma(B \cup D))$$

and

$$(2) \quad \begin{aligned} sint_\gamma(A) &= A \subseteq cl_\gamma(A) \subseteq cl_\gamma(B) \cup cl_\gamma(A) \\ &= cl_\gamma(B \cup A) \quad \text{implies} \quad sint_\gamma(A) \subseteq sint_\gamma(cl_\gamma(B \cup A)). \end{aligned}$$

Since A is a γ-semi-open set, therefore

$$(3) \qquad D \subseteq A = sint_\gamma(A) \subseteq sint_\gamma(cl_\gamma(B \cup A)) \subseteq sint_\gamma(cl_\gamma(B \cup D)).$$

From (1) and (3),

$$B \cup D \subseteq sint_\gamma(cl_\gamma(B \cup D))$$

implies $B \cup D$ is a pre-γ-semi-open set. This completes the proof. ■

Corollary 2. *Let X be a space, $\phi \neq B \subseteq X$, A a minimal γ-semi-open set and an open subset of X. If there exists a γ-semi-open set C containing B such that $C \subseteq cl_\gamma(A)$, then for any nonempty subset D of A, $B \cup D$ is a pre γ-semi-open set, where γ is regular.*

Proof. Let A be a minimal γ-semi-open set and $B \subseteq X$. Suppose there exists a γ-semi-open set C containing B such that $C \subseteq cl_\gamma(A)$. Then we have $C \subseteq cl_\gamma(B) \cup cl_\gamma(A) = cl_\gamma(A \cup B)$[1], since γ is regular. By Theorem 3.16, it follows that for any nonempty subset D of A, $B \cup D$ is a pre γ-semi-open set. This completes the proof. ■

4. Finite γ-semi-open sets

Proposition 9. *Let X be a space and $\phi \neq B$ a finite γ-semi-open set in X. Then there exists at least one (finite) minimal γ-semi-open set A such that $A \subseteq B$.*

Proof. Suppose that B is a finite γ-semi-open set in X. Then we have the following two possibilities:

(a) B is a minimal γ-semi-open set.

(b) B is not a minimal γ-semi-open set.

In case (a), if we choose $B = A$, then the theorem is proved. If the case (b) is true, then there exists a nonempty (finite) γ-semi-open set B_1 which is properly contained in B. If B_1 is minimal γ-semi-open, we take $A = B_1$. In case (b), if B_1 is not a minimal γ-semi-open set, then there exists a nonempty (finite) γ-semi-open set B_2 such that $B_2 \subset B_1 \subset B$. We continue this process and have a sequence of γ-semi-open sets $... \subset B_m \subset ... \subset B_2 \subset B_1 \subset B$. Since B is a finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal γ-semi-open set B_k such that $B_k = A$. This completes the proof. ■

In view of the Definition of γ-locally finite space [8], we define γ-semi-locally finite space as:

Definition 13. *A space X is said to be a γ-semi-locally finite space, if for each $x \in X$ there exists a finite γ-semi-open set A in X such that $x \in A$.*

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} A, & \text{if} \quad b \in A \\ cl(A), & \text{if} \quad b \notin A. \end{cases}$$

Then calculations show that the γ-semi-open sets are ϕ, X, $\{b\}$, $\{a, b\}$, $\{a, c\}$ [2]. Clearly X is γ-semi-locally finite space.

Proposition 10. *Let $\phi \neq B$ be a γ-semi-open set in a γ-semi-locally finite space X. Then there exists at least one (finite) minimal γ-semi-open set A which is contained in B, where γ is semi-regular.*

Proof. Let $y \in B$. Since X is a γ-semi-locally finite space, then there exists a finite γ-semi-open set B_y such that $y \in B_y$. Semi-regularity of γ implies that $B \cap B_y$ is a finite γ-semi-open set [2], therefore by Proposition 9 there exists a minimal γ-semi-open set A such that $A \subseteq B \cap B_y \subseteq B$. This completes the proof. ■

Proposition 11. *Let X be a γ-semi-locally finite space and for any $\alpha \in I$, B_α a γ-semi-open set and $\phi \neq A$ a finite γ-semi-open set. Then $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a finite γ-semi-open set, where γ is semi-regular.*

Proof. Since X is a γ-semi-locally finite space, then there exists an integer k such that $A \cap (\bigcap_{\alpha \in I} B_\alpha) = A \cap (\bigcap_{i=1}^{k} B_i)$. Since γ is semi-regular [2], $A \cap (\bigcap_{\alpha \in I} B_\alpha)$ is a finite γ-semi-open set. This completes the proof. ■

Using Proposition 11, we can prove the following:

Theorem 3. *Let X be a space and for any $\alpha \in I$, B_α a γ-semi-open set and for any $\beta \in J$, A_β a nonempty finite γ-semi-open set. Then $(\bigcup_{\beta\in J} A_\beta) \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a γ-semi-open set, where γ is semi-regular.*

5. Applications

Let A be a nonempty finite γ-semi-open set. It is clear, by Proposition 1 and Proposition 10, that if γ is semi-regular, then there exists a natural number m such that $\{A_1, A_2, ..., A_m\}$ is the class of all minimal γ-semi-open sets in A satisfying the following two conditions:

(a) For any l, n with $1 \leq l$, $n \leq m$ and $l \neq n$, $A_l \cap A_n = \phi$.

(b) If C is a minimal γ-semi-open set in A, then there exists l with $1 \leq l \leq m$ such that $C = A_l$.

Theorem 4. *Let X be a space and $\phi \neq A$ a finite γ-semi-open set such that A is not a minimal γ-semi-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal γ-semi-open sets in A and $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. If $A_y = \cap\{B : B$ is a γ-semi-open nbd of $y\}$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that A_k is contained in A_y, where γ is semi-regular.*

Proof. Suppose on the contrary that for any natural number $k \in \{1, 2, ..., m\}$, A_k is not contained in A_y. Therefore, for any minimal γ-semi-open set A_k in A, $A_k \cap A_y = \phi$. By Proposition 14, $\phi \neq A_y$ is a finite γ-semi-open set. Therefore, by Proposition 9, there exists a minimal γ-semi-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, then C is a minimal γ-semi-open set in A. By supposition, for any minimal γ-semi-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore, for any natural number $k \in \{1, 2, ..., m\}$, $C \neq A_k$. This is a contradiction to our supposition. Hence the proof. ∎

Proposition 12. *Let X be a space and $\phi \neq A$ be a finite γ-semi-open set which is not a minimal γ-semi-open set. Let $\{A_1, A_2, ..., A_m\}$ be a class of all minimal γ-semi-open sets in A and $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that for any γ-semi-open nbd B_y of y, A_k is contained in B_y, where γ is semi-regular.*

Proof. This follows from Theorem 4, as $\cap\{B : B$ is a γ-semi-open nbd of $y\} \subseteq B_y$. Hence the proof. ∎

Theorem 5. *Let X be a space and $\phi \neq A$ be a finite γ-semi-open set which is not a minimal γ-semi-open set. Let $\{A_1, A_2, ..., A_m\}$ be the class of all minimal γ-semi-open sets in A and $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. Then there exists a natural number $k \in \{1, 2, ..., m\}$ such that $y \in cl_\gamma(A_k)$, where γ is semi-regular.*

Proof. It follows from Proposition 11 that there exists a natural number $k \in \{1, 2, ..., m\}$ such that $A_k \subseteq B$ for γ-semi-open nbd B of x. Therefore $\phi \neq A_k \cap A_k \subseteq A_k \cap B \subseteq A_k \cap B^\gamma$ implies $y \in cl_\gamma(A_k)$. This completes the proof. ■

Proposition 13. *Let $\phi \neq A$ be a finite γ-semi-open set in a space X and for each $k \in \{1, 2, ..., m\}$, A_k a minimal γ-semi-open set in A. If the class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-semi-open sets in A, then for any $\phi \neq B_k \subseteq A_k$, $A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, where γ is regular, monotone and semi-regular.*

Proof. Let $\phi \neq A$ be a finite γ-semi-open set. We consider the following two cases:

Case 1. If A is a minimal γ-semi-open set, then this follows directly from Proposition 5.

Case 2. If A is not a minimal γ-semi-open set. $y \in A - (A_1 \cup A_2 \cup ... \cup A_m)$. Then by Theorem 5, it follows that $y \in cl_\gamma(A_1) \cup cl_\gamma(A_2) \cup ... \cup cl_\gamma(A_m)$. Therefore by Proposition 5, we have

$$\begin{aligned} A \subseteq cl_\gamma(A_1) \cup cl_\gamma(A_2) \cup ... \cup cl_\gamma(A_m) &= cl_\gamma(B_1) \cup cl_\gamma(B_2) \cup ... \cup cl_\gamma(B_m) \\ &= cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m), \end{aligned}$$

since γ is regular [1]. This completes the proof. ■

Proposition 14. *Let X be a space and $\phi \neq A$ a finite γ-semi-open set and A_k a minimal γ-semi-open set in A, for each $k \in \{1, 2, ..., m\}$. If for any $\phi \neq B_k \subseteq A_k$, $A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$ then $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, where γ is open.*

Proof. For any $\phi \neq B_k \subseteq A_k$, $k \in \{1, 2, ..., m\}$, we have $cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m) \subseteq cl_\gamma(A)$. Also, we have

$$cl_\gamma(A) \subseteq cl_\gamma(cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m),$$

since γ is open [1].

This implies that for any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$. Hence the proof. ■

Proposition 15. *Let X be a space and $\phi \neq A$ be a finite γ-semi-open set and for each $k \in \{1, 2, ..., m\}$, A_k a minimal γ-semi-open set in A. If for any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$, then the class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-semi-open sets in A.*

Proof. Suppose on the contrary that C is a minimal γ-semi-open set in A and for $k \in \{1, 2, ..., m\}$, $C \neq A_i$. Therefore, for each $k \in \{1, 2, ..., m\}$, $C \cap cl_\gamma(A_k) = \phi$. This implies that any element of C is not contained in $cl_\gamma(A_1 \cup A_2 \cup ... \cup A_m)$. This is a contradiction to the fact that $C \subseteq A \subseteq cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$. This completes the proof. ∎

Combining Propositions 13, 14 and 15, we have the following theorem:

Theorem 6. *Let X be a space and $\phi \neq A$ be a finite γ-semi-open set and for each $k \in \{1, 2, ..., m\}$, A_k a minimal γ-semi-open set in A. Then the following three conditions are equivalent:*
(*a*) *The class $\{A_1, A_2, ..., A_m\}$ contains all minimal γ-semi-open sets in A,*
(*b*) *or any $\phi \neq B_k \subseteq A_k$, $A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$.*
(*c*) *For any $\phi \neq B_k \subseteq A_k$, $cl_\gamma(A) = cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m)$.*
Where γ is regular, monotone and semi-regular.

Remark 1. Suppose that $\phi \neq A$ is a finite γ-semi-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-semi-open sets in A such that for each $k \in \{1, 2, ..., m\}$, $y_k \in A_k$. Then by Theorem 5, it is clear that $\{y_1, y_2, ..., y_m\}$ is a pre-γ-semi-open set.

Theorem 7. *Let X be a space. Suppose that $\phi \neq A$ is a finite γ-semi-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-semi-open sets in A. If for any $B \subseteq A - \{A_1, A_2, ..., A_m\}$ and $\phi \neq B_k \subseteq A_k$, for each $k \in \{1, 2, ..., m\}$, then $B \cup B_1 \cup B_2 \cup ... \cup B_m$ is a pre-γ-semi-open set, where γ is regular, monotone and semi-regular.*

Proof. Suppose that $\phi \neq A$ is a finite γ-semi-open set and $\{A_1, A_2, ..., A_m\}$ is a class of all minimal γ-semi-open sets in A. Then by Proposition 13.

$$A \subseteq cl_\gamma(B_1 \cup B_2 \cup ... \cup B_m) \subseteq cl_\gamma(B \cup B_1 \cup B_2 \cup ... \cup B_m).$$

Also, A is γ-semi-open implies

$$B \cup B_1 \cup B_2 \cup ... \cup B_m \subseteq A = sint_\gamma(A) \subseteq sint_\gamma(cl_\gamma(B \cup B_1 \cup B_2 \cup ... \cup B_m)).$$

This follows that $B \cup B_1 \cup B_2 \cup ... \cup B_m$ is a pre-γ-semi-open set. This completes the proof. ∎

Theorem 8. *Let X be a γ-semi-locally finite space. If a minimal γ-semi-open set $A \subseteq X$ has more than one element, then X is a pre γ-semi-T_2 space, where γ is regular, monotone and semi-regular.*

Proof. Let $a, b \in X$ such that $a \neq b$. Since X is γ-semi-locally finite, therefore there exist finite γ-semi-open sets V and W containing a and b

respectively. Proposition 9 implies that there exist a class $\{V_1, V_2, ..., V_m\}$ of all minimal γ-semi-open sets in V and a class $\{W_1, W_2, ..., W_l\}$ of all minimal γ-semi-open sets in W. We consider three possibilities:

1. Suppose there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \in V_k$ and $b \in W_i$. Then Proposition 8 implies that $\{a\}$ and $\{b\}$ are pre-γ-semi-open sets such that $a \in \{a\}$, $b \in \{b\}$ and $\{a\} \cap \{b\} = \phi$.

2. Suppose there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \in V_k$ and $b \notin W_i$. Then by supposition, Proposition 8 and Theorem 7, we can find for each i, $b_i \in W_i$ such that $\{a\}$ and $\{b, b_1, b_2, ..., b_l\}$ are pre-γ-semi-open sets and $\{a\} \cap \{b, b_1, b_2, ..., b_l\} = \phi$.

3. Suppose that there exist $k \in \{1, 2, ..., m\}$ and $i \in \{1, 2, ..., l\}$ such that $a \notin V_k$ and $b \notin W_i$. Then by supposition and Theorem 7, for each k and i, we can find elements $a_k \in V_k$ and $b_i \in W_i$ such that $\{a, a_1, a_2, ..., a_m\}$ and $\{b, b_1, b_2, ..., b_l\}$ are pre-γ-semi-open sets and $\{a, a_1, a_2, ..., a_m\} \cap \{b, b_1, b_2, ..., b_l\} = \phi$. Hence X is a pre γ-semi-T_2 space. This completes the proof. ■

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