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R-WEAKLY COMMUTING MAPS IN G-METRIC SPACES

Abstract. In this paper, we prove a common fixed point theorem for R-weakly commuting maps in G- metric spaces.

Key words: G-metric space, weakly commuting maps, R-weakly commuting maps.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 1992, Dhage [1] introduced the concept of D-metric space. Recently, Mustafa and Sims [4] showed that most of the results concerning Dhage's D-metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure and called it as G-metric space. For more details on G-metric spaces, one can refer to the papers [4]-[7].

Now we give basic definitions and some basic results ([4]-[7]) which are helpful for proving our main result.

In 2006, Mustafa and Sims [5] introduced the concept of G-metric spaces as follows:

Definition 1 ([5]). *Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:*

($G1$) $G(x, y, z) = 0$ *if* $x = y = z$,

($G2$) $0 < G(x, x, y)$, *for all* $x, y \in X$ *with* $x \neq y$,

($G3$) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$,

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (*symmetry in all three variables*) *and*

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$, (*rectangle inequality*).

Then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2 ([5]). *Let (X, G) be a G-metric space then for $x_0 \in X$, $r > 0$, the G-ball with centre x_0 and radius r is*

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}$$

Proposition 1 ([5]). *Let (X, G) be a G-metric space then for any $x_0 \in X$, $r > 0$, we have,*

(a) if $G(x_0, x, y) < r$ then $x, y \in B_G(x_0, r)$,

(b) if $y \in B_G(x_0, r)$ then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

It follows from (2) of the above proposition that the family of all G-balls, $B = \{B_G(x, r) : x \in X, r > 0\}$ is the base of a topology $\tau(G)$ on X, the G-metric topology.

Proposition 2 ([5]). *Let (X, G) be a G-metric space then for all $x_0 \in X$ and $r > 0$, we have,*

$$B_G(x_0, \frac{1}{3}r) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r)$$

where $d_G(x, y) = G(x, y, y) + G(x, x, y)$, for all x, y in X. Consequently, the G-metric topology $\tau(G)$ coincides with the metric topology arising from d_G. Thus, while 'isometrically' distinct, every G-metric space is topologically equivalent to a metric space. This allows us to readily transport many results from metric spaces into G-metric spaces settings.

Definition 3 ([5]). *Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point 'x' in X is said to be the limit of the sequence $\{x_n\}$ if $G(x, x_n, x_m) \to 0$ as $n, m \to \infty$ and one says that sequence $\{x_n\}$ is G-convergent to x. Thus, that if $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ in a G-metric space (X, G) then for each $\epsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \epsilon$ for all $m, n \geq N$.*

Proposition 3 ([5]). *Let (X, G) be a G-metric space. Then the following are equivalent:*

(a) $\{x_n\}$ is G-convergent to x,

(b) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(c) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(d) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 4 ([5]). *Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if, for each $\epsilon > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$; i.e. $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.*

Proposition 4 ([5]). *If (X, G) is a G-metric space then the following are equivalent:*

(a) The sequence $\{x_n\}$ is G-Cauchy,

(b) for each $\epsilon > 0$, there exist a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq N$.

Proposition 5 ([5]). *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 5 ([5]). *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in X.*

Proposition 6 ([5]). *A G-metric space (X, G) is G-complete if and only if (X, d_G) is a complete metric space.*

Proposition 7 ([5]). *Let (X, G) be a G-metric space. Then, for any x, y, z, a in X it follows that:*
(i) *If* $G(x, y, z) = 0$, *then* $x = y = z$,
(ii) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \le 2G(y, x, x)$,
(iv) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \le \frac{2}{3}(G(x, y, a) + G(x, a, z) + G(a, y, z))$,
(vi) $G(x, y, z) \le (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

2. Main results

There has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the "fixed point arena" when the notion of commutativity was used by Jungck [2] to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors. In particular, now we look in the context of a common fixed point theorem in G-metric spaces. We start with the following contraction conditions:

Let T be a mapping from a complete G-metric space (X, G) into itself and consider the following conditions:

(1) $\quad G(Tx, Ty, Tz) \le \alpha G(x, y, z)$, for all $x, y, z \in X$, where $0 \le \alpha < 1$.

It is clear that every self mapping T of X satisfying condition (1) is continuous. Now we focus to generalize the condition(1) for a pair of self maps S and T of X in the following way:

(2) $G(Sx, Sy, Sz) \le \alpha G(Tx, Ty, Tz)$, for all $x, y, z \in X$, where $0 \le \alpha < 1$.

To prove the existence of common fixed points for (2), it is necessary to add additional assumptions of the following type: (i) construction of the sequence $\{x_n\}$ (ii) some mechanism to obtain common fixed point and this problem was overcome by imposing additional hypothesis on a pair of $\{S, T\}$.

Most of the theorems followed a similar pattern of maps: (i) contraction (ii) continuity of functions (either one or both) and (iii) some conditions on pair of mappings were given. In some cases, condition (ii) can be relaxed but condition (i) and (iii) are unavoidable.

In 1982, Sessa [9] introduced the concept of weakly commuting maps in metric spaces as follows:

The mappings f and g are said to be weakly commuting if

$$d(fgx, gfx) \leq d(fx, gx) \text{ for all } x \in X.$$

In 1994, Pant [8] introduced the notion of R-weakly commuting mappings in metric spaces as follows:

The mappings f and g are said to be R-weakly commuting if there exist some positive real number R such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all x in X.

Remark 1. R-weakly commuting maps are not necessarily continuous at the fixed point.

Now we introduce the concept of weakly commuting and R-weakly commuting maps in a G-metric space as follows:

Definition 6. *Let f and g be maps from a G-metric space (X, G) into itself. The mappings f and g are said to be weakly commuting if*

$$G(fgx, gfx, gfx) \leq G(fx, gx, gx) \quad \textit{for all } x \in X.$$

Definition 7. *Let f and g be maps from a G-metric space (X, G) into itself. The mappings f and g are said to be R-weakly commuting if there exist some positive real number R such that*

$$G(fgx, gfx, gfx) \leq RG(fx, gx, gx) \quad \textit{for all } x \in X.$$

Remark 2. If $R < 1$ then R-weakly commuting maps are weakly commuting.

Now we shall prove our main result:

Theorem 1. *Let (X, G) be a complete G-metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

$$f(X) \subseteq g(X); \tag{3}$$

$$f \textit{ or } g \textit{ is continuous}; \tag{4}$$

$$G(fx, fy, fz) \leq qG(gx, gy, gz), \textit{ for every } x, y, z \in X \textit{ and } 0 \leq q < 1. \tag{5}$$

Then f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (3), one can choose a point x_1 in X such that $fx_0 = gx_1$. In general choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$.

Now, we prove that $\{y_n\}$ is a G-Cauchy sequence in X. From (5), take $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$ we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq qG(gx_n, gx_{n+1}, gx_{n+1}) = qG(fx_{n-1}, fx_n, fx_n).$$

Continuing in the same way, we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) &\leq q^n\ G(fx_0, fx_1, fx_1) \\ \Rightarrow G(y_n, y_{n+1}, y_{n+1}) &\leq q^n\ G(y_0, y_1, y_1). \end{aligned}$$

Therefore, for all $n, m \geq N$ (set of natural numbers), $n < m$, we have by using $(G5)$

$$\begin{aligned} G(y_n, y_m, y_m) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) \\ &\quad + G(y_{n+2}, y_{n+3}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_m) \\ &\leq (q^n + q^{n+1} + q^{n+2} + \cdots + q^{m-1})G(y_0, y_1, y_1) \\ &\leq (q^n + q^{n+1} + q^{n+2} + \cdots)G(y_0, y_1, y_1) \\ &= \frac{q^n}{(1-q)} G(y_0, y_1, y_1) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $\{y_n\}$ is a G-Cauchy sequence in X. Since (X, G) is complete G-metric space, there exists a point z in X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = z$. Let us suppose that the mapping f is continuous. Therefore $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = fz$. Since f and g are R-weakly commuting,

$$G(fgx_n, gfx_n, gfx_n) \leq R\ G(fx_n, gx_n, gx_n).$$

By letting $n \to \infty$, we get $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = fz$.

We now prove that $z = fz$. Suppose $z \neq fz$, then $G(z, fz, fz) > 0$. From (5), on letting $x = x_n, y = fx_n, z = fx_n$, we have

$$G(fx_n, ffx_n, ffx_n) \leq q\ G(gx_n, gfx_n, gfx_n).$$

Proceeding limit as $n \to \infty$, we get

$$G(z, fz, fz) \leq qG(z, fz, fz) < G(z, fz, fz), \text{ a contradiction.}$$

Therefore, $z = fz$. Since $f(X) \subseteq g(X)$, we can find z_1 in X such that $z = fz = gz_1$. Now from (5), take $x = fx_n$, $y = z_1$, $z = z_1$, we have

$$G(ffx_n, fz_1, fz_1) \leq q\ G(gfx_n, gz_1, gz_1).$$

Taking limit as $n \to \infty$, we get

$$G(fz, fz_1, fz_1) \leq q\ G(fz, gz_1, gz_1) = q\ G(fz, fz, fz) = 0,$$

which implies that $fz = fz_1$, i.e. $z = fz = fz_1 = gz_1$. Also, by using definition of R-weakly commutativity,

$$G(fz, gz, gz) = G(fgz_1, gfz_1, gfz_1) \leq R\ G(fz_1, gz_1, gz_1) = 0,$$

which again implies that $fz = gz = z$. Thus z is a common fixed point of f and g.

Uniqueness. We assume that $z_1(\neq z)$ be another common fixed point of f and g.

Then $G(z, z_1, z_1) > 0$ and

$$\begin{aligned} G(z, z_1, z_1) &= G(fz, fz_1, fz_1)) \leq q\ G(gz, gz_1, gz_1) \\ &= q\ G(z, z_1, z_1) < G(z, z_1, z_1), \end{aligned}$$

a contradiction, therefore $z = z_1$. Hence uniqueness follows.

As an application of Theorem 1, we have the following result:

Corollary. *it Let (X, G) be a complete G- metric space and let f be a self-mappings of X. If there is $0 \leq q < 1$ such that*

$$G(fx, fy, fz) \leq q\ G(gx, gy, gz) \quad \textit{holds for all} \quad x, y, z \in X,$$

then f and g have a unique common fixed point in X.

Proof. Follow from Theorem 1 by taking g to be the identity map. ∎

The above Corollary is stated in [10] as Corollary 3.3 and proved by different way.

Example 1. Let $X = [-1, 1]$ and let $G : X \times X \times X \to R^+$ be the G-metric defined as follows:

$$G(x, y, z) = (\mid x - y \mid + \mid y - z \mid + \mid z - x \mid), \quad \text{for all } x, y, z \in X.$$

Then (X, G) is a G-metric space. Define $f(x) = x$ and $g(x) = 2x - 1$.

Here we note that,

(a) $f(X) \subseteq g(X)$,

(b) f is continuous on X,

(c) $G(fx, fy, fz) \leq qG(gx, gy, gz)$, holds for all $x, y, z \in X$, $\frac{1}{2} < q < 1$.

However, the maps f and g are R-weakly compatible and $x = 1$ is the unique common fixed point of f and g. Thus all the conditions of the Theorem 1 are satisfied.

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