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WEIGHTED SUBSTITUTION OPERATORS BETWEEN L^p-SPACES OF VECTOR-VALUED FUNCTIONS

Abstract. In this paper we characterize weighted substitution operators between L^p-spaces of vector-valued functions and also make an attempt to characterize isometry and partial isometry of these operators.

Key words: weighted substitution operator, isometry, partial isometry, adjoint of an operator.

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1. Introduction and preliminaries

Let (X, S, μ) be a σ-finite measure space. Then for $1 \leq p < \infty$, $L^p(X, \mathbb{C}^n)$ denotes the class of all S-measurable $\mathbb{C}^n$-valued functions whose pth power is integrable on X with respect to the measure μ i.e.

$$L^p(X, \mathbb{C}^n) = \left\{ f | f : X \to \mathbb{C}^n \text{ is a measurable and } \int_X ||f(\cdot)||^p d\mu < \infty \right\}.$$

Then $L^p(X, \mathbb{C}^n)$ is a Banach space under the norm,

$$||f|| = \left(\int_X ||f(\cdot)||^p d\mu \right)^{\frac{1}{p}}$$

and for $p = 2$, $L^2(X, \mathbb{C}^n)$ is a Hilbert space under the inner product,

$$\langle f, g \rangle = \int_X \langle f(\cdot), g(\cdot) \rangle d\mu.$$

Let $w : X \to \mathbb{C}^n$ be a vector-valued measurable function and let $T : X \to X$ be a non-singular measurable transformation. Then a bounded linear transformation $S_{w,T} : L^p(X, \mathbb{C}^n) \to L^p(X, \mathbb{C}^n)$ defined by

$$(S_{w,T}, f)(x) = w(x) f(T(x))$$

is called a weighted composition operator or a weighted substitution operator induced by the pair (w, T). If we take $w(x) = 1$, the constant one function on X, we write $S_{w,T}$ as C_T and call it a composition operator or substitution operator induced by T. In case $T(x) = x$, for every $x \in X$, we write $S_{w,T}$ as M_w and call it a multiplication operator induced by w.

An atom of a measure μ is an element $A \in S$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure with no atoms is called non atomic. We can easily check the following well known facts see [15].

(a) Every σ-finite measure space (X, S, μ) can be decomposed into disjoint sets B and Z, such that μ is non atomic over B and Z is atmost countable union of atoms A_n of finite measure. So we can write X as follows:

$$X = B \cup (\cup_{n \in N}\{A_n\}).$$

(b) For each $f \in L^s(X, S, \mu)$, there exists two functions $f_1 \in L^p(X, S, \mu)$ and $f_2 \in L^q(X, S, \mu)$ such that $f = f_1 \cdot f_2$ and $||f||_s^s = ||f_1||_p^p = ||f_2||_q^q$ where $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$.

(c) Suppose $1 \leq p < q < \infty$. If a S_0-measurable set K, is non-atomic and s.t. $\mu(K) > 0$, there exists a function $g_0 \in L^p(X, S_0, \mu)$ with $\int_K |g_0|^q \, d\mu = \infty$.

Let (X, S, μ) be a σ-finite measure space and $S_0 \subset S$ be a σ-finite subalgebra. Then the conditional expectation $E(\cdot|S_0)$ is defined as a linear transformation from certain S-measurable function spaces (i.e. L^1, L^2 etc) into their S_0-measurable counterparts. In particular the conditional expectation with respect to the σ-algebra $T^{-1}(S)$ is a bounded projection from $L^p(X, S, \mu)$ onto $L^p(X, T^{-1}(S), \mu)$. We denote this transformation by E. The transformation E has the following properties:

(i) $E(f \cdot g \circ T) = E(f) \cdot (g \circ T)$

(ii) if $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere

(iii) $E(1) = 1$

(iv) $E(f)$ has the form $E(f) = g \circ T$ for exactly one σ-measurable function g. In particular $g = E(f) \circ T^{-1}$ is a well defined measurable function.

(v) $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$

(vi) For $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.

(vii) If ϕ is a convex function, then $\phi(E(f)) \leq E(\phi(f))$ μ-almost everywhere. For deeper study of the properties of E see [12].

Campbell ([1], [2]) made use of the expectation operator to study some properties of weighted composition operators on $L^2(X, \mathbb{C})$. Also T is a mapping from X into itself is a non-singular measurable transformation such that $\mu \circ T^{-1}$ is absolutely continuous with respect to μ (i.e. $\mu \circ T^{-1} \ll \mu$). Hence by Radon-Nikodym derivative theorem there exists a positive measurable function f_0 such that $\mu(T^{-1}(E)) = \displaystyle\int_E f_0 d\mu$, for every $E \in S$. The function

f_0 is called the Radon-Nikodym derivative of the measure μT^{-1} with respect to the measure μ. It is denoted by $\frac{d\mu T^{-1}}{d\mu}$.

Boundedness of the composition operators in $L^p(X, S, \mu), (1 \leq p < \infty)$ spaces, where the measure spaces are σ- finite, appeared already in [13] and for two different L^p-spaces in [14]. Also boundedness of weighted operators on $C(X, E)$ has already been studied in [9]. More detailed classes of weighted composition operators on some function spaces are considered in ([3], [4], [5], [6], [9], [10], [11]). In this paper we plan to study weighted composition operators on vector valued L^p-spaces.

2. Weighted substitution operators

Theorem 1. *Suppose* $1 \leq p,\ q < \infty$. *Every weighted substitution transformation* $S_{w,T} : L^p(X, \mathbb{C}^n) \to L^q(X, \mathbb{C}^n)$ *is always bounded.*

Proof. It is easy to prove by using closed graph theorem and so we omit it. ■

Theorem 2. *Let* $S_{w,T} : L^2(X, \mathbb{C}^n) \to L^2(X, \mathbb{C}^n)$ *be a linear transformation. Then* $S_{w,T}$ *is bounded if and only if* $J \in L^\infty(X, \mathbb{C}^n)$, *where* $J = f_0 E_{n\times n}(w^* w) \circ T^{-1}$.

Proof. The proof is given by Hornor and Jamison [[6], p-3124]. ■

In the next theorem, we characterize the boundedness of weighted substitution operator for atomic measure spaces.

Theorem 3. $S_{w,T} \in \mathbb{C}(L^2(N, \mathbb{C}^n))$ *if and only if* $J : N \to \mathbb{C}^{n\times n}$ *is a bounded function, where*

$$J(n) = \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m) w^*(m) w(m)}{\mu(n)}.$$

Proof. For any $f \in L^2(N, \mathbb{C}^n)$, consider

$$\begin{aligned} ||S_{w,T} f||^2 &= \sum_{n=1}^{\infty} \langle (w \cdot f \circ T)(n), (w \cdot f \circ T)(n) \rangle \mu(n) \\ &= \sum_{n=1}^{\infty} \langle w(n) f(T(n)), w(n) f(T(n)) \rangle \mu(n) \\ &= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle w(m) f(n), w(m) f(n) \rangle \mu(m) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle w^*(m)w(m)f(n), f(n)\rangle \mu(m)$$

$$= \sum_{n=1}^{\infty} \langle \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m)w^*(m)w(m)}{\mu(n)} f(n), f(n)\rangle \mu(n)$$

$$= \sum_{n=1}^{\infty} \langle J(n)f(n), f(n)\rangle \mu(n)$$

$$= ||M_{J^{\frac{1}{2}}} f||^2.$$

Hence $S_{w,T}$ is a bounded operator if and only if $J : N \to \mathbb{C}^{n \times n}$ is a bounded function. ■

Theorem 4. *Let $S_{w,T} \in \mathbb{C}(L^2(N, \mathbb{C}^n))$. Define*

$$(Ag)(n) = \frac{1}{\mu(n)} \sum_{m \in T^{-1}(\{n\})} \mu(m)w^*(m)g(m) \quad \textit{for every } g \in L^2(N, \mathbb{C}^n).$$

Then $S^*_{w,T} = A$.

Proof. For any $f, g \in L^2(N, \mathbb{C}^n)$, consider

$$\begin{aligned} \langle S_{w,T} f, g\rangle &= \sum_{n=1}^{\infty} \langle (w \cdot f \circ T)(n), g(n)\rangle \mu(n) \\ &= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle w(m)f(T(m)), g(m))\rangle \mu(m) \\ &= \sum_{n=1}^{\infty} \sum_{m \in T^{-1}(\{n\})} \langle f(n), w^*(m)g(m))\rangle \mu(m) \\ &= \sum_{n=1}^{\infty} \langle f(n), \sum_{m \in T^{-1}(\{n\})} \frac{\mu(m)w^*(m)g(m)}{\mu(n)}\rangle \mu(n) \\ &= \sum_{n=1}^{\infty} \langle f(n), (S^*_{w,T} g)(n)\rangle \mu(n) \\ &= \langle f, Ag\rangle. \end{aligned}$$

Hence $S^*_{w,T} = A$. ■

Theorem 5. *Let $S_{w,T} \in \mathbb{C}(L^2(X, \mathbb{C}^n))$. Then $S_{w,T}$ is a partial isometry if and only if J is an idempotent.*

Proof. Suppose $S_{w,T}$ is a partial isometry. Then

$$S_{w,T} = S_{w,T}S^*_{w,T}S_{w,T}$$

and therefore

$$S^*_{w,T}S_{w,T} = S^*_{w,T}S_{w,T}S^*_{w,T}S_{w,T}$$

or

$$M_J = M_{J^2}.$$

Hence we can conclude that J is an idempotent.

Conversely, if J is an idempotent mapping, then,since $kerS_{w,T} = kerM_J$, so for any $f \in (kerS_{w,T})^\perp = \overline{ranM_J}$, we have

$$\begin{aligned}\langle S^*_{w,T}S_{w,T}f, g\rangle &= \int_X \langle w \cdot f \circ T, w \cdot g \circ T\rangle d\mu \\ &= \int_X \langle w^* w \cdot f \circ T, g \circ T\rangle d\mu \\ &= \int_X \langle hE_n(w^* w) \circ T^{-1} f, g\rangle d\mu \\ &= \int_X \langle Jf, g\rangle d\mu \\ &= \langle f, g\rangle.\end{aligned}$$

Hence $S_{w,T}$ is a partial isometry. ■

Theorem 6. *Let $S_{w,T} : L^2(X, \mathbb{C}^n) \to L^2(X, \mathbb{C}^n)$ be a bounded operator. Then $S_{w,T}$ is an isometry if and only if $J^{\frac{1}{2}}(x)$ is an isometry for μ-almost all $x \in X$.*

Proof. The proof follows from the equality,

$$||S_{w,T}f|| = ||M_{J^{\frac{1}{2}}}f|| \text{ for every } f \in^2 (X, \mathbb{C}^n).$$

■

Theorem 7. *Let $S_{w,T} \in \mathbb{C}(L^p(X, \mathbb{C}^n))$. Then $S_{w,T}$ is an idempotent operator if and only if $w \cdot w \circ T = w$ and $T^2 = T$ on* supp $w \cap$ supp $(w \circ T)$.

Proof. Suppose $S_{w,T}$ is an idempotent operator. Then for $e_k \in \mathbb{C}^n$, we have for any $E \in S$, with $\mu(E) < \infty$,

$$S_{w,T}S_{w,T}(\chi_E e_k) = S_{w,T}(\chi_E e_k),$$

which implies that

$$w \cdot w \circ T(\chi_{(T^2)^{-1}(E)} e_k) = w(\chi_{T^{-1}(E)} e_k).$$

Hence $T^2 = T$ and $w \cdot w \circ T = w$ on supp $w \cap$ supp $w \circ T$. The converse is easy to prove. ■

Example 1. Let $X = \mathbb{R}$, $T(x) = x + 1$ and

$$w(x) = \begin{cases} e^{\frac{1}{1-(x-1)^2}}, & \text{for} \quad |x| < 1 \\ 0, & \text{for} \quad |x| \geq 1. \end{cases}$$

Then $T^{-1}(S) = S$, $f_0 = 1$, $E[f] = f$ for every $f \in L^p(X, \mathbb{C}^n)$. Now

$$||f_0^{\frac{1}{q}} |w| \circ T^{-1}||_r^r = \int_X |w(x-1)|^r d\mu = 1.$$

Hence $S_{w,T}$ is a weighted substitution operator from $L^p(X, \mathbf{C}^n)$ into $L^p(X, \mathbb{C}^n)$ in view of Theorem 2.

Example 2. Let $X = [0, 1], \ldots$

$$T(x) = \begin{cases} 2x, & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ -2x, & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases}$$

And $w(x) = 2x$ for every $x \in X$. Then $f_0 = 1$ almost everywhere, so

$$(EF)(x) = \frac{1}{2}[f(x) + f(1-x)]$$

$$E(f) \circ T^{-1}(x) = \frac{1}{2}\left[f(x) + f\left(\frac{1-x}{2}\right)\right].$$

Now

$$\begin{aligned} ||f_0^{\frac{1}{q}} E(|w| \circ T^{-1})||_r^r &= \int_0^1 |E(|w| \circ T^{-1}(x)|^r d\mu \\ &= \int_0^1 \frac{|w(\frac{x}{2}) + w(\frac{1-x}{2})|^r}{2} d\mu = 1. \end{aligned}$$

Hence $S_{w,T}$ is a bounded operator.

If $w(x)$ is an isometry for almost all x, then we present a characterization for boundedness of weighted substitution operators by using the Radyon - Nikodym derivative.

Theorem 8. *Let $w : X \to \mathbb{C}^{n \times n}$ be a measurable function and let $T : X \to X$ be a non-singular transformation. Then for $1 \leq p$, $q < \infty$, $S_{w,T} : L^p(X, \mathbb{C}^n) \to L^q(X, \mathbb{C}^n)$ is continuous if and only if $v \in L^\infty(X, \mathbb{C}^n)$.*

Proof. For each $E \in S$, set $\lambda(E) = \int_{T^{-1}(E)} ||w(x)||^q d\mu(x)$. Then

$$\begin{aligned} ||S_{w,T}f||^q &= \int_X ||w(x)(f \circ T)(x)||^q d\mu(x) \\ &= \int_X ||w(x)||^q ||f \circ T(x)||^q d\mu(x) \\ &= \int_X ||f(T(x))||^q d\lambda(x) \\ &= \int_X ||f(x)||^q d\lambda T^{-1} \\ &= \int_X ||f(x)||^q v(x) d\mu(x), \quad \text{where } v = \frac{d\lambda T^{-1}}{d\mu}. \end{aligned}$$

Hence, we can conclude that $S_{w,T}$ is a bounded linear transformation if and only if v is essentially bounded. ■

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