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ON NEW SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

Abstract. The purpose of this paper is to introduce the notions $\widetilde{g}$-R_0, $\widetilde{g}$-R_1, $\widetilde{g}$-T_0, $\widetilde{g}$-T_1 and $\widetilde{g}$-T_2 in bitopological spaces.

Key words: bitopological spaces, $\widetilde{g}$-closed set, $\widetilde{g}$-open set, $\widetilde{g}$-closure, $\widetilde{g}$-kernal.

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1. Introduction

The notion of R_0 topological spaces introduced by Shanin [14] in 1943. Later, A. S. Davis [3] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e.g. [4], [5], [10]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, A. S. Davis also introduced the notion of R_1 topological space which is independent of both T_0 and T_1 but strictly weaker than T_2. Some basic properties of the class of R_1 in topological spaces were discussed by Murdeshwar and Naimpally [9]. Bitopological forms of these concepts have appeared in the definitions of pairwise R_0 and pairwise R_1 spaces given by Mr$\widehat{s}$evi$\widehat{c}$ [8]. Recently, Jafari et al. [6] introduced the notion of $\widetilde{g}$-closed set and Sarasak and Rajesh [13] and Jafari and Rajesh [1] respectively introduced the notions of $\widetilde{g}$-R_i $(i = 1, 2)$ and $\widetilde{g}$-T_j $(j = 0, 1, 2)$ topological spaces as a generalization of the known notions of R_0, R_1, T_0, T_1 and T_2 topological spaces. In this paper, we offer the pairwise version of $\widetilde{g}$-R_0, $\widetilde{g}$-R_1, $\widetilde{g}$-T_0, $\widetilde{g}$-T_1, $\widetilde{g}$-T_2 in bitopological spaces and (X, τ_1, τ_2) and (Y, σ_1, σ_2) represent bitopological spaces on which no separation axioms are assumed unless otherwise explicitly mentioned.

2. Preliminaries

First we recall the following definitions and results, which are entering to our work.

For a subset A of a topological space (X, τ), $cl(A)$ and $int(A)$ denote the closure of A and the interior of A, respectively.

Definition 1. *A subset A of a topological space (X, τ) is called:*

(i) *semi-open [7] if $A \subset cl(int(A))$. The complement of semi-open set is called semi-closed. The intersection of all semi-closed sets containing A is called the semi-closure [2] of A and is denoted by $scl(A)$.*

(ii) *$\widehat{g}$-closed [16] if $cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in (X, τ). The complement of $\widehat{g}$-closed set is called $\widehat{g}$-open.*

(iii) *${}^*g$-closed [15] if $cl(A) \subset U$ whenever $A \subset U$ and U is $\widehat{g}$-open in (X, τ). The complement of ${}^*g$-closed set is called ${}^*g$-open.*

(iv) *${}^\#g$-semi-closed (briefly ${}^\#gs$-closed) [17] if $scl(A) \subset U$ whenever $A \subset U$ and U is ${}^*g$-open in (X, τ). The complement of ${}^\#gs$-closed set is called ${}^\#gs$-open.*

(v) *$\widetilde{g}$-closed [6] if $cl(A) \subset U$ whenever $A \subset U$ and U is ${}^\#gs$-open in (X, τ). The complement of $\widetilde{g}$-closed set is called $\widetilde{g}$-open. The family of all $\widetilde{g}$-open subsets of (X, τ) is denoted by $\widetilde{G}O(X, \tau)$.*

Definition 2. *Let (X, τ) be a topological space. The intersection of $\widetilde{g}$-closed (resp. $\widetilde{g}$-open) sets, each contained in a set A in X is called the $\widetilde{g}$-closure [11] (resp. $\widetilde{g}$-kernal [1]) of A and is denoted by $\widetilde{g}$-$cl(A)$ (resp. $\widetilde{g} - ker(A)$).*

Definition 3 ([1])**.** *A subset B_x of a topological space (X, τ) is said to a be $\widetilde{g}$-neighbourhood of a point $x \in X$ [12] if there exists a $\widetilde{g}$-open set U such that $x \in U \subset B_x$.*

Theorem 1 ([1])**.** *Let (X, τ) be a topological space and $x \in X$. Then $y \in \widetilde{g}$-$ker(\{x\})$ if and only if $x \in \widetilde{g}$-$cl(\{y\})$.*

Lemma 1 ([1])**.** *Let (X, τ) be a topological space and A be a subset of X. Then $\widetilde{g}$-$ker(A) = \{x \in X | \widetilde{g}$-$cl(\{x\}) \cap A \neq \varnothing\}$.*

Theorem 2 ([13])**.** *A space (X, τ) is $\widetilde{g}$-T_1 if and only if each singleton is $\widetilde{g}$-closed.*

3. Pairwise $\widetilde{g}$-R_0 space

Definition 4. *A bitopological space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0 if for each τ_i-$\widetilde{g}$-open set G, $x \in G$ implies τ_j-$\widetilde{g}$-$cl(\{x\}) \subset G$, where $i, j = 1, 2$ and $i \neq j$.*

Example 1. (a) Let $X = \{a, b, c\}$, $\tau_1 = \{\varnothing, X\}$ and $\tau_2 = \{\varnothing, \{a\}, X\}$. Clearly, the space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0.

(b) Let X=$\{a, b, c\}$, τ_1=$\{\varnothing$, $\{a\}$, X$\}$ and τ_2=$\{\varnothing$, $\{b\}$, X$\}$. Then the space (X, τ_1, τ_2) is not a pairwise $\widetilde{g}$-R_0.

Theorem 3. *In a bitopological space (X, τ_1, τ_2) the following statements are equivalent:*

(i) (X, τ_1, τ_2) *is pairwise* $\widetilde{g} - R_0$.

(ii) For any τ_i-$\widetilde{g}$-*closed set* F *and a point* $x \notin F$, *there exists a* $U \in \widetilde{G}O(X, \tau_j)$ *such that* $x \notin U$ *and* $F \subset U$ *for* $i, j = 1, 2$ *and* $i \neq j$.

(iii) For any τ_i-$\widetilde{g}$-*closed set* F *and* $x \notin F$, τ_j-$\widetilde{g}$-$cl(\{x\}) \cap F = \varnothing$, *for* $i, j = 1, 2$ *and* $i \neq j$.

Proof. $(i) \Rightarrow (ii)$: Let F be a τ_i-$\widetilde{g}$-closed set and $x \notin F$. Then by (i) τ_j-$\widetilde{g}$-$cl(\{x\}) \subset X - F$, where $i, j = 1, 2$ and $i \neq j$. Let $U = X - \tau_j$-$\widetilde{g}$-$cl(\{x\})$, then $U \in \widetilde{G}O(X, \tau_j)$ and also $F \subset U$ and $x \notin U$.

$(ii) \Rightarrow (iii)$: Let F be a τ_i-$\widetilde{g}$-closed set and a point $x \notin F$. Suppose the given conditions hold. Since $U \in \widetilde{G}O(X, \tau_j)$, $U \cap \tau_j$-$\widetilde{g}$-$cl(\{x\}) = \varnothing$. Then $F \cap \tau_j$-$\widetilde{g}$-$cl(\{x\}) = \varnothing$, where $i, j = 1, 2$ and $i \neq j$.

$(iii) \Rightarrow (i)$: Let $G \in \widetilde{G}O(X, \tau_i)$ and $x \in G$. Now $X - G$ is τ_j-$\widetilde{g}$-closed and $x \notin X - G$. By (iii), τ_j-$\widetilde{g}$-$cl(\{x\}) \cap (X - G) = \varnothing$ and hence τ_j-$\widetilde{g}$-$cl(\{x\}) \subset G$ for $i, j = 1, 2$ and $i \neq j$. Therefore, the space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0. ■

Theorem 4. *A bitopological space* (X, τ_1, τ_2) *is pairwise* $\widetilde{g}$-R_0 *if and only if for each pair x, y of distinct points in X,* τ_1-$\widetilde{g}$-$cl(\{x\}) \cap \tau_2$-$\widetilde{g}$-$cl(\{y\}) = \varnothing$ *or* $\{x, y\} \subset \tau_1$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_2$-$\widetilde{g}$-$cl(\{y\})$.

Proof. Suppose that τ_i-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{y\}) \neq \varnothing$ and $\{x, y\} \not\subseteq \tau_i$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{y\})$. Let $z \in \tau_i$- $\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{y\})$ and $x \notin \tau_i$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{y\})$. Then $x \notin \tau_j$-$\widetilde{g}$-$cl(\{y\})$ which implies that $x \in X - \tau_j$-$\widetilde{g}$-$cl(\{y\}) \in \widetilde{G}O(X, \tau_j)$. But τ_i-$\widetilde{g}$-$cl(\{x\}) \not\subseteq X - (\tau_j$-$\widetilde{g}$-$cl(\{y\}))$, because z $\in \tau_j$-$\widetilde{g}$-$cl(\{y\})$, so the bitopological space (X, τ_1, τ_2) is not pairwise $\widetilde{g}$-R_0. Conversely, let U be a τ_i-$\widetilde{g}$-open set and $x \in U$. Suppose τ_j-$\widetilde{g}$-$cl(\{x\}) \not\subseteq U$. So there is a point $y \in \tau_j$-$\widetilde{g}$-$cl(\{x\})$ such that $y \notin U$ and τ_i-$\widetilde{g}$-$cl(\{y\}) \cap U = \varnothing$. Since $X - U$ is τ_i-$\widetilde{g}$-closed and $y \in X - U$. Hence, $\{x, y\} \not\subseteq \tau_i$-$\widetilde{g}$-$cl(\{y\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{x\})$ and so τ_i-$\widetilde{g}$-$cl(\{y\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{x\}) \neq \varnothing$. ■

Theorem 5. *In a bitopological space* (X, τ_1, τ_2) *the following statements are equivalent:*

(i) (X, τ_1, τ_2) *is pairwise* $\widetilde{g}$-R_0.

(ii) For any $x \in X$, τ_i-$\widetilde{g}$-$cl(\{x\}) \subset \tau_j$-$\widetilde{g}$-$ker(\{x\})$, *for* $i, j = 1, 2$ *and* $i \neq j$.

(iii) For any $x, y \in X$ *and* $y \in \tau_i$-$\widetilde{g}$-$ker(\{x\})$ *if and only if* $x \in \tau_j$-$\widetilde{g}$-$ker(\{y\})$, *for* $i, j = 1, 2$ *and* $i \neq j$.

(iv) For any $x, y \in X$ *and* $y \in \tau_i$-$\widetilde{g}$-$cl(\{x\})$ *if and only if* $x \in \tau_j$-$\widetilde{g}$-$cl(\{y\})$, *for* $i, j = 1, 2$ *and* $i \neq j$.

(v) For any τ_i-$\widetilde{g}$-*closed set* F, *and a point* $x \notin F$, *there exists a* τ_j-$\widetilde{g}$-*open set* G *and* $F \subset G$, *for* $i, j = 1, 2$ *and* $i \neq j$.

(vi) Each τ_i-$\widetilde{g}$-*closed set* F *can be expressed as* $F = \cap\{G | G$ *is a* τ_j-$\widetilde{g}$-*open set and* $F \subset G\}$, *for* $i, j = 1, 2$ *and* $i \neq j$.

(vii) Each τ_i-$\widetilde{g}$*-open set* G, $G = \cup\{F|F$ *is a* τ_j-$\widetilde{g}$*-closed set and* $F \subset G\}$ *for* $i, j = 1, 2$ *and* $i \neq j$.

(viii) For each τ_i-$\widetilde{g}$*-closed set* F, $x \notin F$ *implies* τ_j-$\widetilde{g}$-$cl(\{x\}) \cap F = \emptyset$, *for* $i, j = 1, 2$ *and* $i \neq j$.

Proof. $(i) \Rightarrow (ii)$: By Theorem 1, for any $x \in X$ we have τ_j-$\widetilde{g}$-$ker(\{x\}) = \cap\{G|G$ is τ_j-$\widetilde{g}$-open and $x \in G\}$ and by Definition 4, each τ_j-$\widetilde{g}$-open set G containing x contains τ_i-$\widetilde{g}$-$cl(\{x\})$. Hence τ_i-$\widetilde{g}$-$cl(\{x\}) \subset \tau_j$-$\widetilde{g}$-$ker(\{x\})$ for $i, j = 1, 2$ and $i \neq j$.

$(ii) \Rightarrow (iii)$: For any $x, y \in X$, if $y \in \tau_i$-$\widetilde{g}$-$ker(\{x\})$ then $x \in \tau_i$-$\widetilde{g}$-$cl(\{y\})$ and hence by (ii), $y \in \tau_j$-$\widetilde{g}$-$ker(\{y\})$.

$(iii) \Rightarrow (iv)$: For $x, y \in X$, if $y \in \tau_i$-$\widetilde{g}$-$cl(\{x\})$, then by (iii), $y \in \tau_j$-$\widetilde{g}$-$ker(\{x\})$ and hence, by Theorem 1, $x \in \tau_j$-$\widetilde{g}$-$cl(\{y\})$ for $i = 1, 2$ and $i \neq j$.

$(iv) \Rightarrow (v)$: Let F be a τ_i-$\widetilde{g}$-closed set and a point $x \notin F$. Then for any $y \in F$, τ_i-$\widetilde{g}$-$cl(\{y\}) \subset F$ and so $x \notin \tau_i$-$\widetilde{g}$-$cl(\{y\})$. Now, by (iv) $x \notin \tau_i$-$\widetilde{g}$-$cl(\{y\})$ implies $y \notin \tau_j$-$\widetilde{g}$-$cl(\{x\})$, that is there exists a τ_j-$\widetilde{g}$-open set G_y such that $y \in G_y$ and $x \notin G_y$. Let $G = \cup_{y \in F}\{G_y|G_y$ is τ_j-$\widetilde{g}$-open, $y \in G_y$ and $x \notin G_y\}$. Then G is τ_j-$\widetilde{g}$-open set such that $x \notin G$ and $F \subset G$.

$(v) \Rightarrow (vi)$: Let F be a τ_i-$\widetilde{g}$-closed set and $H = \cap\{G|G$ is a τ_i-$\widetilde{g}$-open set and $F \subset G\}$. Clearly, $F \subset H$ and it remains to show that $H \subset F$. Let $x \notin F$. Then by (v), there exists a τ_j-$\widetilde{g}$-open set G such that $x \notin G$ and $F \subset G$ and hence $x \notin H$. Therefore, each τ_i-$\widetilde{g}$-closed set F can be expressed as $F = \cap\{G|G$ is a τ_j-$\widetilde{g}$-open set and $F \subset G\}$, for $i, j = 1, 2$ and $i \neq j$.

$(vi) \Rightarrow (vii)$: Obvious.

$(vii) \Rightarrow (viii)$: Let F be a τ_i-$\widetilde{g}$-closed set and $x \notin F$. Then $X - F = G$ (say) is a τ_i-$\widetilde{g}$-open set containing x. Then by (vii), G can be written as the union of τ_j-$\widetilde{g}$-closed sets, and so there is a τ_j-$\widetilde{g}$-closed set H such that $x \in H \subset G$; and hence τ_j-$\widetilde{g}$-$cl(\{x\}) \subset G$. Thus, τ_j-$\widetilde{g}$-$cl(\{x\}) \cap F = \emptyset$.

$(viii) \Rightarrow (i)$: Let G be a τ_i-$\widetilde{g}$-open set and $x \in G$. Then by $(viii)$, there exists a τ_j-$\widetilde{g}$-closed set F such that $x \in F \subset G$ and τ_j-$\widetilde{g}$-$l(\{x\}) \cap F \neq \varnothing$, which implies that τ_j-$\widetilde{g}$-$cl(\{x\}) \subset G$, where $i, j = 1, 2$ and $i \neq j$. Therefore, (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0. ■

Remark 1. For each $x \in X$, we define (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) = \tau_1$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_2$-$\widetilde{g}$-$cl(\{x\})$ and (τ_1, τ_2)-$\widetilde{g}$-$ker(\{x\}) = \tau_1$-$\widetilde{g}$-$ker(\{x\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{x\})$.

Theorem 6. *For any* $x, y \in X$ *in a pairwise* $\widetilde{g}$-R_0 *space* (X, τ_1, τ_2) *we have either* (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) = (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\})$ *or* (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\}) = \varnothing$.

Proof. Let (X, τ_1, τ_2) be a pairwise $\widetilde{g}$-R_0 space. Suppose that (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) \neq (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\})$ and (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\}) \neq \varnothing$. Let $s \in (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\})$ and $x \notin (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\}) =$

τ_1-$\widetilde{g}$-$cl(\{y\}) \cap \tau_2$-$\widetilde{g}$-$cl(\{y\})$. Then $x \notin \tau_i$-$\widetilde{g}$-$cl(\{y\})$ and $x \in X - \tau_i$-$\widetilde{g}$-$cl(\{y\}) \in \widetilde{G}O(X, \tau_i)$. But τ_j-$\widetilde{g}$-$cl(\{x\}) \nsubseteq X - \tau_i$-$\widetilde{g}$-$cl(\{y\})$, because $s \in (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\})$. Which in its turn, contradicts the hypothesis of pairwise $\widetilde{g}$-R_0-ness of X. Hence we have either (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) = (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\})$ or (τ_1, τ_2)-$\widetilde{g}$-$cl(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{y\}) = \varnothing$. ■

Remark 2. The converse of Theorem 6 need not be true, in general. Let X, τ_1 and τ_2 be as in Example 1 (b). Let $b, c \in X$. Then (τ_1, τ_2)-$\widetilde{g}$-$cl(\{b\}) = (\tau_1, \tau_2)$-$\widetilde{g}$-$cl(\{c\}) = \{c\}$. However, the bitopological space (X, τ_1, τ_2) is not pairwise $\widetilde{g}$-R_0.

Theorem 7. *Let (X, τ_1, τ_2) be pairwise $\widetilde{g}$-R_0 space. Then for any point $x, y \in X$, (τ_1, τ_2)-$\widetilde{g}$-$ker(\{x\}) \neq (\tau_1, \tau_2)$-$\widetilde{g}$-$ker(\{y\})$ implies (τ_1, τ_2)-$\widetilde{g}$-$ker(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$ker(\{y\}) = \varnothing$.*

Proof. Let (X, τ_1, τ_2) be a pairwise $\widetilde{g}$-R_0 space. Suppose that (τ_1, τ_2)-$\widetilde{g}$-$ker(\{x\}) \cap (\tau_1, \tau_2)$-$\widetilde{g}$-$ker(\{y\}) \neq \varnothing$ and $s \in \tau_1$-$\widetilde{g}$-$ker(\{x\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{x\}) \cap \tau_1$-$\widetilde{g}$-$ker(\{y\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{y\})$. Also by Theorem 1, $s \in \tau_1$-$\widetilde{g}$-$ker(\{x\})$ implies that $x \in \tau_1$-$\widetilde{g}$-$ker(\{s\})$ which in its turn by Theorem 5 (iv) implies that $x \in \tau_2$-$\widetilde{g}$-$ker(\{s\})$. Hence τ_2-$\widetilde{g}$-$ker(\{x\}) \subset \tau_2$-$\widetilde{g}$-$ker(\{s\}) \subset \tau_2$-$\widetilde{g}$-$ker(\{y\})$. Thus $s \in \tau_1$-$\widetilde{g}$-$ker(\{x\})$ implies that τ_2-$\widetilde{g}$-$ker(\{x\}) \subset \tau_2$-$\widetilde{g}$-$ker(\{y\})$. Similarly, $s \in \tau_2$-$\widetilde{g}$-$ker(\{x\})$ implies τ_2-$\widetilde{g}$-$ker(\{x\}) \subset \tau_2$-$\widetilde{g}$-$ker(\{y\})$ and $s \in \tau_1$-$\widetilde{g}$-$ker(\{y\})$ implies τ_1-$\widetilde{g}$-$ker(\{y\}) \subset \tau_1$-$\widetilde{g}$-$ker(\{x\})$ and $s \in \tau_2$-$\widetilde{g}$-$ker(\{y\})$ implies τ_2-$\widetilde{g}$-$ker(\{y\}) \subset \tau_2$-$\widetilde{g}$-$ker(\{x\})$. Therefore, τ_1-$\widetilde{g}$-$ker(\{x\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{x\}) \subset \tau_1$-$\widetilde{g}$-$ker(\{y\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{y\})$ and τ_1-$\widetilde{g}$-$ker(\{y\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{y\}) \subset \tau_1$-$\widetilde{g}$-$ker(\{x\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{x\})$. Hence, τ_1-$\widetilde{g}$-$ker(\{y\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{y\}) = \tau_1$-$\widetilde{g}$-$ker(\{x\}) \cap \tau_2$-$\widetilde{g}$-$ker(\{x\})$. Therefore, (τ_1, τ_2)-$\widetilde{g}$-$ker(\{x\}) = (\tau_1, \tau_2)$-$\widetilde{g}$-$ker(\{y\})$. ■

Corollary 1. *For any pair of points x and y in a pairwise $\widetilde{g}$-R_0 space (X, τ_1, τ_2), the following statements are equivalent:*

(i) (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0.

(ii) For any τ_i-$\widetilde{g}$-closed set $F \subset X$, $F = \tau_j$-$\widetilde{g}$-$ker(F)$, where $i, j = 1, 2$ and $i \neq j$.

(iii) For any τ_i-$\widetilde{g}$-closed set $F \subset X$ and $x \in F$, τ_j-$\widetilde{g}$-$ker(\{x\}) \subset F$, where $i, j = 1, 2$ and $i \neq j$.

(iv) For any $x \in X$, τ_j-$\widetilde{g}$-$ker(\{x\}) \subset \tau_i$-$\widetilde{g}$-$cl(\{x\})$, where $i, j = 1, 2$ and $i \neq j$.

Proof. $(i) \Rightarrow (ii)$: Let F be τ_i-$\widetilde{g}$-closed set and $x \notin F$. Then $X - F$ is τ_i-$\widetilde{g}$-open contianing x. Since (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0, τ_j-$\widetilde{g}$-$cl(\{x\}) \subset X - F$ where $i, j = 1, 2$ and $i \neq j$. Therefore, τ_j-$\widetilde{g}$-$cl(\{x\}) \cap F = \varnothing$ and by Lemma 1 $x \notin \tau_j$-$\widetilde{g}$-$ker(F)$. Hence τ_j-$\widetilde{g}$-$ker(F) \subset F$. Again by the definition of $\widetilde{g}$-kernel, $F \subset \tau_j$-$\widetilde{g}$-$ker(F)$, so $F = \tau_j$-$\widetilde{g}$-$ker(F)$, where $i, j = 1, 2$ and $i \neq j$.

$(ii) \Rightarrow (iii)$: Let F be a τ_i-$\widetilde{g}$-closed set containing x. Then $\{x\} \subset F$ and τ_j-$\widetilde{g}$-$ker(\{x\}) \subset \tau_j$-$\widetilde{g}$-$ker(F)$. From (ii), it follows that τ_j-$\widetilde{g}$-$ker(\{x\}) \subset F$, where $i, j = 1, 2$ and $i \neq j$.

$(iii) \Rightarrow (iv)$: Since $x \in \tau_i$-$\widetilde{g}$-$cl(\{x\})$ and τ_i-$\widetilde{g}$-$cl(\{x\})$ is $\widetilde{g}$-closed in X, which in turn ensures by (iii), that τ_j-$\widetilde{g}$-$ker(\{x\}) \subset \tau_i$-$\widetilde{g}$-$cl(\{x\})$, where $i, j = 1, 2$ and $i \neq j$.

$(iv) \Rightarrow (i)$: Let $x \in \tau_j$-$\widetilde{g}$-$cl(\{x\})$. Then by Theorem 1, $y \in \tau_j$-$\widetilde{g}$-$ker(\{x\})$. Hence by (iv) we have $y \in \tau_i$-$\widetilde{g}$-$cl(\{x\})$. Thus, $x \in \tau_j$-$\widetilde{g}$-$cl(\{x\}) \Rightarrow y \in \tau_i$-$\widetilde{g}$-$cl(\{x\})$. The reverse implication follows similarly. Hence by Theorem 5, (X, τ_1, τ_2) is a pairwise $\widetilde{g}$-R_0 space. ■

Definition 5. *A space (X, τ_1, τ_2) is said to be pairwise $\widetilde{g}$-R_1 if for each $x, y \in X$, τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$, there exist disjoint sets $U \in \widetilde{G}O(X, \tau_j)$ and $V \in \widetilde{G}O(X, \tau_i)$ such that τ_i-$\widetilde{g}$-$cl(\{x\}) \subset U$ and τ_j-$\widetilde{g}$-$cl(\{y\}) \subset V$ where $i, j = 1, 2$ and $i \neq j$.*

Theorem 8. *If (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1, then it is pairwise $\widetilde{g}$-R_0.*

Proof. Suppose that (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1. Let U be a τ_i-$\widetilde{g}$-open set and $x \in U$. If $y \notin U$, then $y \in X - U$ and $x \notin \tau_i$-$\widetilde{g}$-$cl(\{y\})$. Therefore, for each point $y \in X - U$, τ_j-$\widetilde{g}$-$cl(\{x\}) \neq \tau_i$-$\widetilde{g}$-$cl(\{y\})$. Since (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1, there exist a τ_i-$\widetilde{g}$-open set U_y and a τ_j-$\widetilde{g}$-open set V_y such that τ_j-$\widetilde{g}$-$cl(\{x\}) \subset U_y$, τ_i-$\widetilde{g}$-$cl(\{y\}) \subset V_y$ and $U_y \cap V_y = \varnothing$ where $i, j = 1, 2$ and $i \neq j$. Let $A = \bigcup\{V_y | y \in X - U\}$, then $X - U \subset A$, $x \notin A$ and A is τ_j-$\widetilde{g}$-open set. Therefore, τ_j-$\widetilde{g}$-$cl(\{x\}) \subset X - A \subset U$. Hence (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0. ■

Remark 3. The converse of Theorem 8 need not be true in general. The space (X, τ_1, τ_2) in Example 1 (a) is pairwise $\widetilde{g}$-R_0 but not pairwise $\widetilde{g}$-R_1.

Theorem 9. *A space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1 if and only if for every pair of points x and y of X such that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$, there exists a τ_i-$\widetilde{g}$-open set U and τ_j-$\widetilde{g}$-open set V such that $x \in V$, $y \in U$ and $U \cap V \neq \varnothing$, where $i, j = 1, 2$ and $i \neq j$.*

Proof. Suppose that (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1. Let x, y be points of X such that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$, where $i, j = 1, 2$ and $i \neq j$. Then there exist a τ_i-$\widetilde{g}$ open set U and τ_j-$\widetilde{g}$ open set V such that $x \in \tau_i$-$\widetilde{g}$-$cl(\{x\}) \subset V$ and $y \in \tau_j$-$\widetilde{g}$-$cl(\{y\}) \subset U$ and it follows that $U \cap V = \varnothing$, where $i, j = 1, 2$ and $i \neq j$. On the other hand, suppose there exist a τ_i-$\widetilde{g}$-open set U and a τ_j-$\widetilde{g}$-open set V such that $x \in V$, $y \in U$ and $U \cap V = \varnothing$, where $i, j = 1, 2$ and $i \neq j$. Since every pairwise $\widetilde{g}$-R_1 space is every pairwise $\widetilde{g}$-R_0, τ_j-$\widetilde{g}$-$cl(\{x\}) \subset$

V and τ_i-$\widetilde{g}$-$cl(\{y\}) \subset U$, from which we infer that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$, for $i = 1, 2$ and $i \neq j$. ■

Theorem 10. *A pairwise $\widetilde{g}$-R_0 space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1 if for each pair of points x and y of X with τ_i-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{y\}) = \varnothing$, there exist disjoint sets $U \in \widetilde{G}O(X, \tau_i)$ and $V \in \widetilde{G}O(X, \tau_j)$ such that $x \in U$ and $y \in V$ where $i, j = 1, 2$ and $i \neq j$.*

Proof. It follows directly from Theorems 6 and 9. ■

Theorem 11. *In a bitopological space (X, τ_1, τ_2) the following statements are equivalent:*

(i) *(X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1.*

(ii) *For any two distinct points $x, y \in X$, τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$ implies that there exist a τ_i-$\widetilde{g}$-closed set F_1 and a τ_j-$\widetilde{g}$-closed set F_2 such that $x \in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$ and $X = F_1 \cup F_2$, $i, j = 1, 2$ and $i \neq j$.*

Proof. $(i) \Rightarrow (ii)$: Suppose that (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1. Let $x, y \in X$ such that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$. By Theorem 9, then there exist disjoint sets $V \in \widetilde{G}O(X, \tau_i)$, $U \in \widetilde{G}O(X, \tau_j)$ such that $x \in U$ and $y \in V$ where $i, j = 1, 2$ and $i \neq j$. Then $F_1 = X - V$ is a τ_i-$\widetilde{g}$-closed set and $F_2 = X - U$ is a τ_j-$\widetilde{g}$-closed set such that $x \in F_1$, $x \notin F_2$, $y \notin F_1$, $y \in F_2$ and $X = F_1 \cup F_2$ where $i, j = 1, 2$ and $i \neq j$.

$(ii) \Rightarrow (i)$: Let $x, y \in X$ such that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$ where $i, j = 1, 2$ and $i \neq j$. By (ii), there exists a τ_i-$\widetilde{g}$-closed set F_1 and a τ_j-$\widetilde{g}$-closed set F_2 such that $X = F_1 \cup F_2$, $x \in F_1$, $y \in F_2$, $x \notin F_2$, $y \notin F_1$. Therefore, $x \in X - F_2 = U \in \widetilde{G}O(X, \tau_j)$ and $y \in X - F_1 = V \in \widetilde{G}O(X, \tau_j)$ which implies that τ_i-$\widetilde{g}$-$cl(\{x\}) \subset U$ and τ_j-$\widetilde{g}$-$cl(\{y\}) \subset V$ and $U \cap V = \varnothing$ where $i, j = 1, 2$ and $i \neq j$. ■

Definition 6. *A space (X, τ_1, τ_2) is said to be:*

(a) *a pairwise $\widetilde{g}$-T_0 (resp. pairwise $\widetilde{g}$-T_1) if for any pair of distinct points x and y in X, there exists a τ_i-$\widetilde{g}$-open set which contains one of them but not the other $i = 1$ or 2 (resp. there exist τ_i-$\widetilde{g}$-open set U and τ_j-$\widetilde{g}$-open set V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$, $i, j = 1, 2$, $i \neq j$).*

(b) *a pairwise $\widetilde{g}$-T_2 if for any pair of distinct points x and y in X, there exist τ_i-$\widetilde{g}$-open set U and τ_j-$\widetilde{g}$-open set V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$, $i, j = 1, 2$, $i \neq j$.*

Theorem 12. *For a spcae (X, τ_1, τ_2), the following are equivalent:*

(i) *(X, τ_1, τ_2) is pairwise $\widetilde{g}$-T_0.*

(ii) *For every $x \in X$, $\{x\} = \tau_i$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{x\})$ $i, j = 1, 2$, $i \neq j$.*

(iii) *For each $x \in X$, the intersection of all τ_j-$\widetilde{g}$-neighbourhoods of x and all τ_j-$\widetilde{g}$-neighbourhoods of x is $\{x\}$ $i, j = 1, 2$, $i \neq j$.*

Proof. $(i) \Rightarrow (ii)$: Suppose $y \neq xinX$. There exists a τ_i-$\widetilde{g}$-open set V containing x but not y or τ_j-$\widetilde{g}$-open set U containing y but not x. In otherwords, either $x \notin \tau_i$-$\widetilde{g}$-$cl(\{y\})$ or $y \notin \tau_j$-$\widetilde{g}$-$cl(\{x\})$. Hence for a point $x, y \notin \tau_i$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{x\})$. Thus, $\{x\} = \tau_i$-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$cl(\{x\})$.

$(ii) \Rightarrow (iii)$: Straightforward.

$(iii) \Rightarrow (i)$: Let $x \neq y$ in X. By (iii), $\{x\}$ = the intersection of all τ_i-$\widetilde{g}$-neighbourhoods and τ_j-$\widetilde{g}$-neighbourhoods of x. Hence, there exists either one τ_i-neighbourhood of y but not containing x or a τ_j-neighbourhood of y but not containing x. Therefore, (X, τ_1, τ_2) is pairwise $\widetilde{g}$-T_0. ■

Theorem 13. *Let (X, τ_1, τ_2) be a pairwise $\widetilde{g}$-R_0 space. If for any $x \in X$, τ_i-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$ker(\{x\}) = \{x\}$, $i, j = 1, 2$ and $i \neq j$, then (X, τ_i) is $\widetilde{g}$-T_1 for $i = 1, 2$.*

Proof. Suppose that (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_0 and for any point $x \in X$, τ_i-$\widetilde{g}$-$cl(\{x\}) \cap \tau_j$-$\widetilde{g}$-$ker(\{x\}) = \{x\}$, where $i, j = 1, 2$ and $i \neq j$. By Theorem 5(ii), it follows that τ_i-$\widetilde{g}$-$cl(\{x\}) \cap \tau_i$-$\widetilde{g}$-$cl(\{x\}) = \{x\}$ where $i = 1, 2$. Therefore, τ_i-$\widetilde{g}$-$cl(\{x\}) = \{x\}$, where $i = 1, 2$. Hence each singletons is τ_i-$\widetilde{g}$-closed in (X, τ_i), where $i = 1, 2$. Hence by Theorem 2, (X, τ_i) is $\widetilde{g}$-T_1 for $i = 1, 2$. ■

Theorem 14. *If a space (X, τ_1, τ_2) is pairwise $\widetilde{g}$-T_2, then it is pairwise $\widetilde{g}$-R_1.*

Proof. Let (X, τ_1, τ_2) be pairwise $\widetilde{g}$-T_2. Then for any two distinct points x, y of X, their exist a τ_i-$\widetilde{g}$-open set U and a τ_j-$\widetilde{g}$-open set V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$ where $i, j = 1, 2$ and $i \neq j$. If (X, τ_1, τ_2) is pairwise $\widetilde{g}$-T_1, then $\{x\} = \tau_j$-$\widetilde{g} - cl(\{x\})$ and $\{y\} = \tau_i$-$\widetilde{g}$-$cl(\{y\})$ and thus τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$ $i, j = 1, 2$ and $i \neq j$. Thus for any distinct pair of points x, y of X such that τ_i-$\widetilde{g}$-$cl(\{x\}) \neq \tau_j$-$\widetilde{g}$-$cl(\{y\})$ where $i, j = 1, 2$ and $i \neq j$, there exist a τ_i-$\widetilde{g}$-open set U and τ_j-$\widetilde{g}$-open set V such that $x \in V$, $y \in U$ and $U \cap V = \varnothing$ where $i, j = 1, 2$ and $i \neq j$. Hence (X, τ_1, τ_2) is pairwise $\widetilde{g}$-R_1. ■

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