F A S C I C U L I   M A T H E M A T I C I

Nr 47 2011

Bessem Samet and Calogero Vetro *

COMMENTS ON THE PAPER "COINCIDENCE THEOREMS FOR SOME MULTIVALUED MAPPINGS" BY B. E. RHOADES, S. L. SINGH AND CHITRA KULSHRESTHA

Abstract. The aim of this note is to point out an error in the proof of Theorem 1 in the paper entitled "Coincidence theorems for some multivalued mappings" by B. E. Rhoades, S. L. Singh and Chitra Kulshrestha [Internat. J. Math.& Math. Sci., 7(1984), 429-434], and to indicate a way to repair it.

Key words: coincidence point, multivalued mapping, Hausdorff distance.

AMS Mathematics Subject Classification: 47H10; 54H25.

1. Preliminaries

The following definitions and the notations are the same as in [1]:

Let (X, d) be a metric space. We denote by $CL(X)$ the collection of non-empty closed subsets of X. Let $A \in CL(X)$, we define

$$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A,\ \varepsilon > 0\},$$

$$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \\ \qquad\qquad \text{and } B \subseteq N(\varepsilon, A)\} & \text{if the infimum exists,} \\ \infty & \text{otherwise,} \end{cases}$$

for each $A, B \in CL(X)$. H is called the generalized Hausdorff distance function for $CL(X)$ induced by d. $D(x, A)$ will denote the ordinary distance between $x \in X$ and A, a non-empty subset of X.

Let $f : X \to X$ and $T : X \to CL(X)$.

The notation $TX \subseteq fX$ means that $Tx \subseteq fX$ for each $x \in X$.

Definition 1. *A point $z \in X$ is said to be a coincidence point of f and T if $fz \in Tz$. We define $C(f, T) := \{z \in X : fz \in Tz\}$.*

* The second author is supported by University of Palermo, Local University Project R. S. ex 60 %.

Definition 2. *If for a point $x_0 \in X$ there exists a sequence $\{x_n\} \subset X$ such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \cdots$, then*

$$O_f(x_0) = \{fx_n : n = 1, 2, \cdots\}$$

is an orbit for (T, f) at x_0.

Definition 3. *If for $x_0 \in X$ there exists a sequence $\{x_n\} \subset X$ such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \cdots$ and every Cauchy sequence of the form $\{fx_{n_i}\}$ converges in X, then X is called (T, f)-orbitally complete with respect to x_0 or simply (T, f, x_0)-orbitally complete.*

Definition 4. *T is said to be asymptotically regular at $x_0 \in X$ if, for each sequence $\{x_n\} \subset X$, $x_n \in Tx_{n-1}$, $\lim_{n \to +\infty} d(x_n, x_{n+1}) = 0$.*

Let

$$\Psi = \{\phi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that } \phi \text{ is upper semicontinuous and nondecreasing}\}.$$

In this note, we focus on the following theorem of [1]:

Theorem 1. *Let T be a multi-valued mapping from a metric space (X, d) to $CL(X)$ and $\phi \in \Psi$. If there exists a mapping $f : X \to X$ such that $TX \subseteq fX$, and for each $x, y \in X$,*

(1) $H(Tx, Ty) \le \phi(\max\{D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx), d(fx, fy)\})$,

(2) $\phi(t) < qt$ *for each* $t > 0$, *for some fixed* $0 < q < 1$,

(3) *there exists an* $x_0 \in X$ *such that* T *is asymptotically regular at* x_0,

(4) X *is* (T, f, x_0)*-orbitally complete,*

then T and f have a coincidence point.

We point out an error in the proof of Theorem 1 and indicate a way to repair it.

2. Discussions and results

In the proof of Theorem 1, in order to prove the Cauchy character of the sequence $\{y_n\}$, the constructive schema of the sequence is abusively used. To be precise, it is not true that

$$d(y_{2n(k)+1}, y_{2m(k)}) \le q^{-1} H(Tx_{2n(k)}, Tx_{2m(k)-1}),$$

where $y_{2n(k)+1} = fx_{2n(k)+1} \in Tx_{2n(k)}$ and $y_{2m(k)} = fx_{2m(k)} \in Tx_{2m(k)-1}$. In fact, if $A, B \in CL(X)$ and $a \in A$, the inequality

$$d(a, b) \le q^{-1} H(A, B)$$

is incorrect for arbitrary $b \in B$. As a counterexample, let $X = \mathbb{R}$ endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Let $A = [0, 9]$, $B = [2, 10]$, $a = 1 \in A$ and $b = 9 \in B$. Then,

$$d(a, b) = 8 > q^{-1} \cdot 2 = q^{-1} H(A, B),$$

for any $1/4 < q < 1$.

Certainly, the reader can explore the literature to find many papers with the same mistake, but this enumeration is not the aim of the present note.

As we have not found yet any counterexample to Theorem 1 or a new proof, we do not know whether it (and the related results in the literature) are true or not. Until a better solution is found, we suggest a correction by replacing conditions (3) and (4) with TX is a closed subset of $CL(X)$" and condition (0) below:

(0) f and T satisfy the property (E.A).

Definition 5 ([2]). *Two mappings $f : X \to X$ and $T : X \to CL(X)$ are said to satisfy the property (E.A) if there exist a sequence $\{x_n\} \subset X$, some $t \in X$ and $A \in CL(X)$ such that*

$$\lim_{n \to +\infty} f x_n = t \in A = \lim_{n \to +\infty} T x_n.$$

Example 1. Let $X = [1, +\infty)$ endowed with the usual metric $d(x, y) = | x - y |$ for every $x, y \in X$. Clearly (X, d) is a complete metric space. Define $f : X \to X$ as $fx = x^2$ and $T : X \to CL(X)$ as $Tx = [1, 3 + x]$, for all $x \in X$. Now, for the sequence $\{x_n\} = \{1 + 1/n\}$, we have:

$$\lim_{n \to +\infty} f x_n = 1 \in [1, 4] = \lim_{n \to +\infty} T x_n.$$

Therefore f and T satisfy property (E.A).

Remark 1. By virtue of the property (E.A), the proof of Theorem 1 with condition (0) can be concluded following the lines in Theorem 3.4 of [2].

Then, we give the proof of the following theorem.

Theorem 2. *Let T be a multi-valued mapping from a metric space X to $CL(X)$. We assume that there exist $\phi \in \Psi$ and a mapping $f : X \to X$ such that $TX \subseteq fX$, and for each $x, y \in X$,*

(0) f and T satisfy the property (E.A),

(1) $H(Tx, Ty) \leq \phi(\max\{D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx), d(fx, fy)\})$,

(2) $\phi(t) < qt$ for each $t > 0$, for some fixed $0 < q < 1$.

If TX is a closed subset of $CL(X)$, then T and f have a coincidence point.

Proof. By virtue of the property (E.A), there exist a sequence $\{x_n\} \subset X$, some $t \in X$ and $A \in CL(X)$ such that $\lim_{n \to +\infty} fx_n = t \in A = \lim_{n \to +\infty} Tx_n$. Since TX is a closed subset of $CL(X)$ and $TX \subseteq fX$, we have $\lim_{n \to +\infty} fx_n = fz$ for some $z \in X$. Thus $t = fz \in A$. We claim that $fz \in Tz$. If not, then condition (1) implies

$$H(Tx_n, Tz) \leq \phi(\max\{D(fx_n, Tx_n), D(fz, Tz), D(fx_n, Tz), D(fz, Tx_n), d(fx_n, fz)\}).$$

Taking the limit as $n \to +\infty$, we obtain

$$H(A, Tz) \leq \phi(\max\{D(fz, A), D(fz, Tz), D(fz, Tz), D(fz, A), d(fz, fz)\}).$$

Since $\phi(t) < qt$ for each $t > 0$, we get

$$H(A, Tz) \leq \phi(D(fz, Tz)) < q\, D(fz, Tz).$$

Now, by the definition of Hausdorff distance function, since $fz \in A$, it follows that

$$D(fz, Tz) < q\, D(fz, Tz),$$

which is a contradiction as $0 < q < 1$. Therefore $fz \in Tz$. ■

Remark 2. The conclusion of Theorem 2 remains true if we assume that fX is closed instead of TX. In this case, we can remove that $TX \subseteq fX$.

Example 2. Consider $X = [1, \infty)$ equipped with the usual metric $d(x, y) =\mid x - y \mid$ for every $x, y \in X$. Clearly (X, d) is a complete metric space. Define $f : X \to X$ as $fx = x^2$, $T : X \to CL(X)$ as $Tx = [1, 1 + x/4]$, $q = 2/3$ and $\phi(t) = t/2$ for each $t > 0$.

Now, we have:

$$H(Tx, Ty) = \frac{1}{4} \mid x - y \mid \leq \frac{1}{2} \mid x^2 - y^2 \mid = \phi(d(fx, fy)).$$

Then, all the conditions of Theorem 2 are easily verified. Therefore $f1 \in T1$, that is, f and T have a coincidence point.

By definition, an element $x \in X$ is said to be a common fixed point of f and T iff $x = fx \in Tx$. Moreover, as $1 = f1 \in T1$, then $x = 1$ is common fixed point of f and T.

We recall the following notion [2].

Two mappings $f : X \to X$ and $T : X \to CL(X)$ are noncompatible if $fTx \in CL(X)$ for all $x \in X$ and there exists at least one sequence $\{x_n\} \subset X$ such that $\lim_{n \to +\infty} Tx_n = A \in CL(X)$ and $\lim_{n \to +\infty} fx_n = t \in A$ but $\lim_{n \to +\infty} H(fTx_n, Tfx_n) \neq 0$ or the limit does not exist.

Remark 3. It is immediate to see that if f and T are noncompatible, then f and T satisfy the property (E.A).

We state the following corollary.

Corollary 1. *Let T be a multi-valued mapping from a metric space X to $CL(X)$. We assume that there exists a mapping $f : X \to X$ such that, for each $x, y \in X$,*

(0) *f and T are noncompatible,*

(1) $H(Tx, Ty) \leq \phi(\max\{D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx), d(fx, fy)\})$,

(2) *$\phi(t) < qt$ for each $t > 0$, for some fixed $0 < q < 1$, $\phi \in \Psi$.*

If fX is a closed subset of X, then T and f have a coincidence point.

Finally, motivated by [3], we state and prove the following theorem.

Theorem 3. *Let T be a multivalued mapping from a metric space X to $CL(X)$. We assume that there exists a mapping $f : X \to X$ such that, for each $x, y \in X$,*

(0) *f and T satisfy the property (E.A),*

(1) $H(Tx, Ty) \leq \phi(\max\{D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx), d(fx, fy)\})$,

(2) *$\phi(t) < qt$ for each $t > 0$, for some fixed $0 < q < 1$, $\phi \in \Psi$.*

If fX is a closed subset of X, then T and f have a common fixed point, provided that $ffz = fz$, for each $z \in C(f, T)$.

Proof. Essentially the same reasoning as in Theorem 2, keeping in mind Remark 2, establishes $t = fz \in Tz$. From this and $ffz = fz$ for $z \in C(f, T)$, we have $t = fz = ffz = ft \in Tz$. We claim that $z = t$. If not, then condition (1) implies

$$H(Tt, Tz) \leq \phi(\max\{D(ft, Tt), D(fz, Tz), D(ft, Tz), D(fz, Tt), d(ft, fz)\}).$$

Since $\phi(t) < qt$ for each $t > 0$, we get

$$H(Tt, Tz) \leq \phi(D(ft, Tt)) < q\, D(ft, Tt).$$

Now, by the definition of Hausdorff distance function, since $ft \in Tz$, it follows that

$$D(ft, Tt) < q\, D(ft, Tt),$$

which is a contradiction as $0 < q < 1$. Therefore $ft \in Tt$. Hence $z = t$ and so $t = ft \in Tt$. This makes end to the proof. ∎

References

[1] RHOADES B.E., SINGH S.L., KULSHRESTHA C., Coincidence theorems for some multivalued mappings, *Internat. J. Math. & Math. Sci.,* 7(1984) 429-434.
[2] KAMRAN T., Coincidence and fixed points for hybrid strict contractions, *J. Math. Anal. Appl.,* 299(2004) 235-241.
[3] SINTUNAVARAT W., KUMAM P., Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition, *Appl. Math. Lett.,* 22(2009) 1877-1881.

BESSEM SAMET
UNIVERSITÉ DE TUNIS, ECOLE SUPÉRIEURE DES SCIENCES
ET TECHNIQUES DE TUNIS
DÉPARTEMENT DE MATHÉMATIQUES
5, AVENUE TAHA HUSSEIN-TUNIS
B.P.:56, BAB MENARA-1008, TUNISIE
e-mail: bessem.samet@gmail.com

CALOGERO VETRO
DEPARTMENT OF MATHEMATICS AND INFORMATICS
UNIVERSITY OF PALERMO
VIA ARCHIRAFI 34, 90123 PALERMO, ITALY
e-mail: cvetro@math.unipa.it

Received on 14.11.2010 and, in revised form, on 04.01.2011.