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COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS SATISFYING A-CONTRACTIVE CONDITIONS OF INTEGRAL TYPE

Abstract. In this paper, we establish the existence of coincidence and unique common fixed points of two pairs of weakly compatible maps satisfying A-contractive condition of integral type.

Key words: metric space, contractions, common fixed point, weakly compatible maps.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The well celebrated Banach's contraction mapping principle is of fundamental importance in the Metric fixed point theory in that it guarantees the existence of a unique fixed point, as well as the convergent approximation method, for a selfmap T of a metric space X satisfying $d(Tx,Ty) \leq ad(x,y)$, for any $x,y \in X$ and some $a \in [0,1)$. Any map T satisfying this condition is said to be a strict contraction. It is well known that strict contractions are necessarily continuous functions. Several authors have extended and generalized the Banach's contraction principle, especially with the view of introducing weaker contractive conditions which do not imply continuity of T. Some of these authors include Bianchini [5], Chatterjea [7], Ciric [9], Kannan [14], Reich [17], Zamfirescu [24]. For a detailed study of various contractive definitions, see Rhoades [19].

Few decades ago, Jungck [11] used the concept of commuting mappings as a tool to generalize the Banach fixed point theorem. This concept was improved upon by Sessa [21] by introducing the weak commutativity. The concept of compatibility was then introduced by Jungck [12] as a generalization of weakly commuting maps. Ultimately, Jungck and Rhoades [13] defined the notion of weak compatibility in 1998.

Definition 1. *Two selfmaps T and S of a metric space X are said to be weakly compatible if $TSu = STu$ whenever $Su = Tu$ for some $u \in X$.*

In 2002, Branciari [6] obtained an integral-type analogue of the Banach's contraction principle. Thereafter, Altun et al [4], Aliouche [3], Djoudi and Aliouche [10], Vijayaraju et al [23] and others obtained more general results for mappings satisfying weaker contractive conditions of integral type.

Recently, Akram et al [2] introduced a new class of contraction mappings referred to as the A-contractions.

Definition 2. *Let R_+ denote the set of all nonnegative real numbers and A the set of all functions $\alpha : R_+^3 \longrightarrow R_+$ satisfying the following conditions.*

(i) α is continuous on the set R_+^3

(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in R_+$.

A selfmap T on a metric space X is said to be an A-contraction if it satisfies

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)) \tag{1}$$

for all $x, y \in X$ and some $\alpha \in A$.

Theorem 1 ([2]). *Let T be an A-contraction on a complete metric space X. Then T has a unique fixed point in X such that the sequence $\{T^n x_0\}$ converges to the fixed point, for any $x_0 \in X$.*

It has been shown that the class of A-contractions contains the contractions studied by Bianchini [5], Khan [15], Reich and Kannan [18]. The purpose of this paper is to establish existence of unique common fixed point for two pairs of weakly compatible selfmaps of a metric space X, satisfying a generalized A-contractive condition of integral type, such that X needs not to be complete. Our result is a considerable improvement on the result of Akram et al [2].

2. Preliminaries

Let F, G, S and T be selfmaps of a metric space X satisfying

$$SX \subseteq FX \ ; \quad TX \subseteq GX. \tag{2}$$

Then for any point $x_0 \in X$, we can find points x_1, x_2, x_3 ..., all in X, such that

$$Sx_0 = Fx_1, \quad Tx_1 = Gx_2, \quad Sx_2 = Fx_3 \quad \ldots$$

Therefore, by induction, we can define a sequence $\{y_n\}$ in X as

$$y_n = \begin{cases} Sx_n = Fx_{n+1}, & \text{when } n \text{ is even} \\ Tx_n = Gx_{n+1}, & \text{when } n \text{ is odd,} \end{cases} \qquad n = 0, 1, 2, \ldots \ .$$

In this paper, we shall use the following contractive condition. For all $x, y \in X$,

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \alpha \left(\int_0^{d(Gx,Fy)} \varphi(t)dt, \int_0^{d(Gx,Sx)} \varphi(t)dt, \int_0^{d(Fy,Ty)} \varphi(t)dt \right),$$

where $\alpha \in A$ and $\varphi : R^+ \longrightarrow R^+$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \text{for each } \epsilon > 0.$$

Remark 1. If $\varphi(t) = 1$, $Gx = Fx = x$ and $Tx = Sx$ for all $x \in X$, our contractive condition reduces to that of Akram et al.

For the rest of this article, the set of coincidence points of T and F shall be denoted by $C(T, F)$; and the set of natural numbers by N.

3. Main results

Theorem 2. *Let F, G, S and T be selfmaps of a metric space X satisfying (2) and, for all $x, y \in X$,*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq \alpha \left(\int_0^{d(Gx,Fy)} \varphi(t)dt, \int_0^{d(Gx,Sx)} \varphi(t)dt, \int_0^{d(Fy,Ty)} \varphi(t)dt \right), \tag{3}$$

where $\alpha \in A$ and $\varphi : R^+ \longrightarrow R^+$ is a Lebesgue integrable mapping which is summable, nonnegative and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \textit{for each } \epsilon > 0. \tag{4}$$

If $FX \cup GX$ is a complete subspace of X, and the pairs (T, F) and (S, G) are weakly compatible, then F, G, S and T have a unique common fixed point for any $x_0 \in X$.

Proof. We obtain the following from (3), for $n = 2, 4, 6, \ldots$.

$$\int_0^{d(y_n,y_{n+1})} \varphi(t)dt = \int_0^{d(Sx_n,Tx_{n+1})} \varphi(t)dt$$
$$\leq \alpha \left(\int_0^{d(Gx_n,Fx_{n+1})} \varphi(t)dt, \int_0^{d(Gx_n,Sx_n)} \varphi(t)dt, \int_0^{d(Fx_{n+1},Tx_{n+1})} \varphi(t)dt \right)$$
$$= \alpha \left(\int_0^{d(y_{n-1},y_n)} \varphi(t)dt, \int_0^{d(y_{n-1},y_n)} \varphi(t)dt, \int_0^{d(y_n,y_{n+1})} \varphi(t)dt \right),$$

which implies

$$\int_0^{d(y_n,y_{n+1})}\varphi(t)dt \leq k\int_0^{d(y_{n-1},y_n)}\varphi(t)dt.$$

On the other hand, for $n = 1, 3, 5, \ldots$,

$$\begin{aligned}
&\int_0^{d(y_n,y_{n+1})}\varphi(t)dt = \int_0^{d(Tx_n,Sx_{n+1})}\varphi(t)dt \\
&\leq \alpha\left(\int_0^{d(Gx_{n+1},Fx_n)}\varphi(t)dt, \int_0^{d(Gx_{n+1},Sx_{n+1})}\varphi(t)dt, \int_0^{d(Fx_n,Tx_n)}\varphi(t)dt\right) \\
&= \alpha\left(\int_0^{d(y_n,y_{n-1})}\varphi(t)dt, \int_0^{d(y_n,y_{n+1})}\varphi(t)dt, \int_0^{d(y_n,y_{n-1})}\varphi(t)dt\right)
\end{aligned}$$

which also implies

$$\int_0^{d(y_n,y_{n+1})}\varphi(t)dt \leq k\int_0^{d(y_{n-1},y_n)}\varphi(t)dt.$$

Thus, for all $n \in \mathbf{N}$, we have

$$\int_0^{d(y_n,y_{n+1})}\varphi(t)dt \leq k\int_0^{d(y_{n-1},y_n)}\varphi(t)dt.$$

for some $k \in [0, 1)$. Hence, by induction, we obtain

$$\int_0^{d(y_n,y_{n+1})}\varphi(t)dt \leq k^n\int_0^{d(y_0,y_1)}\varphi(t)dt,$$

for some $k \in [0, 1)$.

Now, letting $n \to \infty$,

$$\lim_{n\to\infty}\int_0^{d(y_n,y_{n+1})}\varphi(t)dt \leq 0.$$

And by virtue of (4), we conclude that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. In order to show that the sequence $\{y_n\}$ is Cauchy in X, it suffices to show that the subsequence $\{y_{2n}\}$ is Cauchy. Suppose $\{y_{2n}\}$ is not Cauchy, then we can find $\epsilon > 0$ such that for each positive integer $2i$ there exist positive integers $2i < 2n_i < 2m_i$ such that

$$d(y_{2n_i}, y_{2m_i}) \geq \epsilon.$$

Assuming $2m_i$ is the least positive integer satisfying the last inequality, then $d(y_{2n_i}, y_{2m_i-2}) < \epsilon$ so that

$$\begin{aligned}\epsilon &\le d(y_{2n_i}, y_{2m_i}) \\ &\le d(y_{2n_i}, y_{2m_i-2}) + d(y_{2m_i-2}, y_{2m_i-1}) + d(y_{2m_i-1}, y_{2m_i}) \\ &< \epsilon + d(y_{2m_i-2}, y_{2m_i-1}) + d(y_{2m_i-1}, y_{2m_i}).\end{aligned}$$

Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, we have $\epsilon \le \lim_{i\to\infty} d(y_{2n_i}, y_{2m_i}) \le \epsilon$, where $2i < 2n_i < 2m_i$ and $d(y_{2n_i}, y_{2m_i}) \ge \epsilon$.

Hence,

$$\lim_{i\to\infty} \int_0^{d(y_{2n_i}, y_{2m_i})} \varphi(t)dt = \int_0^{\epsilon} \varphi(t)dt.$$

Therefore,

$$\begin{aligned}&\int_0^{d(y_{2n_i}, y_{2m_i+1})} \varphi(t)dt = \int_0^{d(Sx_{2n_i}, Tx_{2m_i+1})} \varphi(t)dt \\ &\le \alpha\left(\int_0^{d(Gx_{2n_i}, Fx_{2m_i+1})} \varphi(t)dt, \int_0^{d(Gx_{2n_i}, Sx_{2n_i})} \varphi(t)dt, \right. \\ &\qquad\qquad \left. \int_0^{d(Fx_{2m_i+1}, Tx_{2m_i+1})} \varphi(t)dt\right) \\ &= \alpha\left(\int_0^{d(y_{2n_i-1}, y_{2m_i})} \varphi(t)dt, \int_0^{d(y_{2n_i-1}, y_{2n_i})} \varphi(t)dt, \int_0^{d(y_{2m_i}, y_{2m_i+1})} \varphi(t)dt\right).\end{aligned}$$

As $i \to \infty$, we obtain

$$\int_0^{\epsilon} \varphi(t)dt \le \alpha\left(\int_0^{\epsilon} \varphi(t)dt, 0, 0\right).$$

That is, there exists $k \in [0, 1)$ such that

$$\int_0^{\epsilon} \varphi(t)dt \le k \cdot 0 = 0.$$

This is a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence. Consequently, $\{y_n\}$ is Cauchy.

Now since $\{y_n\}$ is Cauchy in $FX \cup GX$, and $FX \cup GX$ is complete, there exists a point $p \in FX \cup GX$ such that $\lim_{n\to\infty} y_n = p$.

Without loss of generality, let $p \in GX$. It means we can find a point $q \in X$ such that $p = Gq$. Let m be odd, putting $x = q$, $y = x_m$ into (3) yields

$$\int_0^{d(Sq, Ty)} \varphi(t)dt \le \alpha\left(\int_0^{d(Gq, Fx_m)} \varphi(t)dt, \int_0^{d(Gq, Sq)} \varphi(t)dt, \int_0^{d(Fx_m, Tx_m)} \varphi(t)dt\right),$$

This implies,

$$\int_0^{d(Sq,y_m)} \varphi(t)dt \leq \alpha \left(\int_0^{d(p,y_{m-1})} \varphi(t)dt, \int_0^{d(p,Sq)} \varphi(t)dt, \int_0^{d(y_{m-1},y_m)} \varphi(t)dt \right).$$

As $m \to \infty$, we obtain

$$\int_0^{d(Sq,p)} \varphi(t)dt \leq \alpha \left(\int_0^{d(p,p)} \varphi(t)dt, \int_0^{d(p,Sq)} \varphi(t)dt, \int_0^{d(p,p)} \varphi(t)dt \right).$$

That is,

$$\int_0^{d(Sq,p)} \varphi(t)dt \leq \alpha \left(0, \int_0^{d(p,Sq)} \varphi(t)dt, 0 \right),$$

which implies that

$$\int_0^{d(Sq,p)} \varphi(t)dt \leq k \cdot 0 = 0.$$

Therefore,

$$\int_0^{d(Sq,p)} \varphi(t)dt = 0,$$

leading to $Sq = p$. Since $SX \subseteq FX$, there exists $u \in X$ such that $Fu = Sq = p = Gq$. Substituting $x = q$, $y = u$, into (3) gives

$$\int_0^{d(p,Tu)} \varphi(t)dt) \leq \alpha\,(0, 0, d(p, Tu))$$

so that

$$\int_0^{d(p,Tu)} \varphi(t)dt \leq k \cdot 0 = 0.$$

Hence, $u \in C(F, T)$, $q \in C(S, G)$ and $Fu = Tu = p = Sq = Gq$.

If (F, T) and (S, G) are weakly compatible pairs, then F and T commute at u. G and S also commute at q to give

$$\begin{aligned} &Fp = FFu = FTu = TFu = Tp \quad \text{and} \\ &Sp = SSq = SGq = GSq = Gp. \end{aligned} \tag{5}$$

Now with $x = p$, $y = u$, from (3) and (5) we easily obtain $p = Sp = Gp$. Similarly, if we let $x = y = p$, (3) and (5) ultimately yield $p = Tp = Fp$. Hence, $Sp = Gp = p = Tp = Fp$.

Finally, the uniqueness of p is an easy consequence of (3), for if p' is a common fixed point of S, G, T and F such that $p \neq p'$, then substituting $x = p$ and $y = p'$ into (3) eventually leads us to a contradiction, namely $d(p, p') \leq 0$. ∎

Corollary 1. *Let S and T be selfmaps of a complete metric space X such that for all $x, y \in X$, and some $\beta \in [0, \frac{1}{2})$,*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \ \leq \ \beta \max\left\{ \int_0^{d(x,Sx)+d(y,Ty)} \varphi(t)dt, \int_0^{d(y,Ty)+d(Sx,Ty)} \varphi(t)dt, \int_0^{d(x,Sx)+d(Sx,Ty)} \varphi(t)dt \right\},$$

where and φ is as in Theorem 2. If the pairs (T, F) and (S, G) are weakly compatible, then F, G, S and T have a unique common fixed point for any $x_0 \in X$.

Proof. Define $\alpha : R_+^3 \longrightarrow R_+$ as $\alpha(a, b, c) = \beta \max\{a + b, b + c, a + c\}$, for all $a, b, c \in R_+$. It is easy to see that conditions (i) and (ii) of Definition 2 are satisfied (see Theorem 2 of [Akram et al]). Now with $Gx = Fx = x$ for all $x \in X$, the proof follows from Theorem 2. ■

Corollary 2. *Let S and T be selfmaps of a complete metric space X such that for all $x, y \in X$, and some nonnegative numbers p, q, r satisfying $p + q + r \leq 1$,*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \leq p \int_0^{d(x,Sx)} \varphi(t)dt + q \int_0^{d(y,Ty)} \varphi(t)dt, r \int_0^{d(x,y)} \varphi(t)dt.$$

If the pairs (T, F) and (S, G) are weakly compatible, then F, G, S and T have a unique common fixed point for any $x_0 \in X$.

Proof. Let $\alpha(a, b, c) = pa + qb + rc$ for all $a, b, c \in R_+$ and $Gx = Fx = x$ for all $x \in X$. ■

References

[1] Ahmad B., Rehman F.U., Some fixed point theorems in complete metric space, *Math. Japonica,* 36(2)(1991), 239-243.

[2] Akram M., Zafar A.A., Siddiqui A.A., A general class of contractions: A-contractions, *Novi Sad J. Math.,* 38(1)(2008), 25-33.

[3] Aliouche A., A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, *J. Math. Anal. Appl.,* 322(2)(2006), 796-802.

[4] Altun I., Turkoglu D., Rhoades B.E., Fixed points of weakly compatible maps satisfying a general contractive condition of integral type, *Fixed Point Theory Appl.,* (2007), 1-9.

[5] Bianchini R.M.T., Su un problema di S. Reich riguardante la teori dei punti fissi, *Boll. Un. Math. Ital.,* 5(1972), 103-108.

[6] Branciari A., A fixed point theorem for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* 29(2002), 531-536.
[7] Chatterjea S.K., Fixed point theorems, *C.R. Acad. Bulgare Sci.,* 25(1972), 727-730.
[8] Chuanyi Z., The generalized set-valued contraction and completeness of the metric space, *Math. Japonica,* 35(1)(1990), 111-118.
[9] Ciric Lj.B., A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.,* 45(1974), 267-273.
[10] Djoudi A, Aliouche A., Common fixed point theorems of Gregus-type for weakly compatible mappings satisfying contractive conditions of integral type, *J. Math. Anal. Appl.,* 329(2007), 31-45.
[11] Jungck G., Commuting mappings and fixed points, *Amer. Math. Monthly,* 83(1976), 261-263.
[12] Jungck G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.,* 9(1986), 771-779.
[13] Jungck G., Rhoades B.E., Fixed points for set-valued functions without continuity, *Indian J. Pure and Appl. Math.,* 29(3)(1998), 227-238.
[14] Kannan T., Some results on fixed points, *Bull. Calcutta Math. Soc.,* 60(1968), 71-76.
[15] Khan M.S., On fixed point theorems, *Math. Japonica,* 23(2)(1978/79), 201 -204.
[16] Pant R.P., Common fixed points of noncommuting mappings, *J. Math. Anal. Appl.,* 188(1994), 436-440.
[17] Reich S., Some remarks concerning contraction mappings, *Canad. Math. Bull.,* 14(1)(1971), 121-124.
[18] Reich S., Kannan T., Fixed point theorem, *Boll. Un. Math. Ital.,* 188(1994), 436-440.
[19] Rhoades B.E., A Comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.,* 226(1977), 257-290.
[20] Rhoades B.E., Sessa S., Common fixed point theorems for three mappings under a weak commutativity conditions, *Indian J. Pure Appl. Math.,* 329(1)(1986), 47-57.
[21] Sessa S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* (*N.S., Beograd*), 32(46)(1982), 149-153.
[22] Shioji N., Suzuki T., Takahashi W., Contractive mappings, Kannan mappings and metric completeness, *Proc. Amer. Math. Soc.,* 10(1998), 3117-3124.
[23] Vijarayaju P., Rhoades B.E., Mohanraj R., A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.,* (15)(2005), 2359-2364.
[24] Zamfirescu T., Fixed point theorems in metric spaces, *Arch. Math.*(*Basel*), 23(1972), 292-298.

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