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# ON THE REFINED MEASURES OF GROWTH OF GENERALIZED BIAXIALLY SYMMETRIC POTENTIALS HAVING INDEX - $q$

Abstract. To obtain more refined measure of growth the q-proximate type is constructed for a class of generalized biaxially symmetric potentials (GBASP's). Finally, we obtain lower $q-$proximate type for GBASP's. Our results generalize some results of Kumar [7].

Key words: generalized biaxially symmetric potentials, index - $q$, $\lambda-$proximate type, irregular growth.

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## 1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in finite disc $D_R \equiv \{z : |z| < R, 0 < R < \infty\}$. The concept of index - $q$, the $q$-order $\rho_R(q)$ and lower $q$-order $\lambda_R(q)$ are introduced by Kapoor and Gopal [5] in order to obtain a measure of growth of the maximum modulus $M(r) \equiv M(r,f) = \max_{|z|=r} |f(z)|$, $0 < r < R$, of $f(z)$ when it is rapidly increasing. Thus let $M(r) \to \infty$ as $r \to R$ and for $q = 2, 3, ...,$ set

$$\begin{matrix} \rho_R(q) \\ \lambda_R(q) \end{matrix} = \begin{matrix} \lim & \sup \\ r \to R & \inf \end{matrix} \frac{\log^{[q]} M(r)}{-\log(\frac{R-r}{R})}, \tag{1}$$

where $\log^{[0]} M(r) = M(r)$ and $\log^{[q-1]} M(r) = \log[\log^{[q-2]} M(r)]$. The function $f(z)$ is said to have the index - $q$ if $\rho_R(q) < \infty$ and $\rho_R(q-1) = \infty$. If $q$ is the index of $f(z)$ then $\rho_R(q)$ is called the $q$-order of $f(z)$.

The notions of the index and $q$-order play a significant role in classifying the rapidly increasing functions analytic in $D_R$. However, these concepts fail to compare the rates of growth of any two functions analytic in $D_R$ that have same $q$-order. To refine this scale Gopal and Kapoor [4] studied the distinct parameters such as $q$-type for the rates of growth of such functions as:

**Definition 1.** *A function analytic in $D_R$ and having q-order $\rho_R(q)$ is said to be of q-type $T_R(q)$ and lower q-type $t_R(q)$ if*

$$\lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[q-1]} M(r)}{((R-r)/R)^{-\rho_R(q)}} = \begin{matrix} T_R(q) \\ t_R(q)\end{matrix}. \tag{2}$$

*In studying a more refined measure of growth of functions analytic in $D_R$ having index - q and q-order $\rho_R(q)$, we consider a real-valued comparison function $\rho_R(q,r)$ $(0<r<R)$ having the following properties:*

($i$) *$\rho_R(q,r)$ is positive,continuous and piecewise differentiable in* $0\le r_0<r<R$.

($ii$) $\lim_{r\to R^-}\rho_R(q,r)\to\rho_R(q)$; $(0\le\rho_R(q)<\infty)$

($iii$) $\lim_{r\to R^-} -\rho'_R(q,r)(R-r)\log((R-r)/R)=0$, *where $\rho'_R(q,r)$ is either the right or left hand derivative at points where these are different, and*

($iv$) $\lim_{r\to R}\sup \frac{\log^{[q-1]} M(r)}{(R/(R-r))^{\rho_R(q,r)}}=1.$

A function $\rho_R(q,r)$ satisfying the conditions ($i$)-($iv$) is said to be a $q$-proximate order. It is evident that $\rho_R(q,r)$ has been linked with the $q$-order $\rho_R(q,r)$ and $M(r)$ to give information about the growth of $f(z)$. Since the $q$-proximate order $\rho_R(q,r)$ is not linked with the $q$-type $T_R(q)$, so it becomes a natural question to the existence of another constant which should take into account the $q$-type of the function and is closely related with its maximum modulus. In analogy with the $q$-proximate order we call this function $T_R(q,r)$ as a $q$-proximate type or simply proximate type of an analytic function $f(z)$ with index - $q$.

**Definition 2.** *A real valued function $T_R(q,r)$ is said to be a q-proximate type of a function analytic in $D_R$ having index - q, q-order $\rho_R(q)$ and q-type $T_R(q)$, if for given $\eta(0<\eta<\infty), T_R(q,r)$ satisfies the following properties:*

($a$) *$T_R(q,r)$ is continuous and piecewise differentiable in* $0\le r_0<r<R$.

($b$) $T_R(q,r)\to T_R(q)$ *as* $r\to R^-$.

($c$) $(R-r)\frac{T'_R(q,r)}{T_R(q,r)}\to 0$ *as* $r\to R$, *where $T'_R(q,r)$ can be interpreted as either $T'_R(q,r^-)$ or $T_R(q,r^+)$ when these are different, and*

($d$) $\limsup_{r\to R}\frac{\log^{[q-2]} M(r)}{\exp\{(R/R-r)^{\rho_R(q)}T_R(q,r)\}}=\eta.$

To establish the existence of $q$-proximate type of a function analytic in $D_R$. We note the following:

**Proposition 1.** $\exp\{(R/(R-r))^{\rho_R(q)}T_R(q,r)\}$ *is monotonically increasing function for $r>r_0$.*

**Proof.** This proposition can be obtain following the lines of a result of Kumar[7, Thm. 2]. ■

## 2. Definitions and some basic results about generalized biaxially symmetric potentials (GBASP's)

Let $F^{(\alpha,\beta)}$ be a real-valued regular solution to the generalized biaxially symmetric potential equation

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{(2\alpha+1)}{x}\frac{\partial}{\partial x}+\frac{(2\beta+1)}{y}\frac{\partial}{\partial y}\right]F^{(\alpha,\beta)}=0,\quad \alpha>\beta>-\frac{1}{2},$$

subject to the Cauchy data $F_x^{(\alpha,\beta)}(0,y)=F_y^{(\alpha,\beta)}(x,0)=0$ which is satisfied along the singular lines in the open hypersphere $\sum_R^{(\alpha,\beta)}: x^2+y^2<R^2$ of finite radius $R$ about the origin. Such functions with even harmonic extensions are referred to as generalized biaxisymmetric potentials (GBSAP's) having local expansions of the form

$$F^{(\alpha,\beta)}(x,y)=\sum_{n=0}^{\infty}a_nR_n^{(\alpha,\beta)}(x,y)$$

in terms of the complete set

$$R_n^{(\alpha,\beta)}(x,y)=(x^2+y^2)^nP_n^{(\alpha,\beta)}\left[\left[(x^2-y^2)/(x^2+y^2)\right]/P_n^{(\alpha,\beta)}(1),\right.$$

$n=0,1,2,3,\cdots$, of biaxisymmetric harmonic potentials, where $P_n^{(\alpha,\beta)}$ are Jacobi Polynomials ([1], [10]). Let the operator $K_{\alpha,\beta}$ uniquely associated even analytic function

$$f(z)=\sum_{n=0}^{\infty}a_nz^{2n},\quad z=x+iy\in C,\quad \text{onto GBASP}$$

$$F^{(\alpha,\beta)}(x,y)=\sum_{n=0}^{\infty}a_nR_n^{(\alpha,\beta)}(x,y).$$

Following McCoy [9], from Karoonwinder's integral for Jacobi polynomials,

$$F^{(\alpha,\beta)}(x,y)=K_{\alpha,\beta}(f)=\int_0^1\int_0^{\pi}f(\zeta)\mu_{\alpha,\beta}(t,s)dsdt$$

where

$$\mu_{\alpha,\beta}(t,s)=\gamma^1_{\alpha,\beta}(1-t^2)^{\alpha-\beta-1}t^{2\beta+1}(\sin s)^{2\alpha}$$

$$\zeta^2=x^2-y^2t^2-i2xyt+\cos s$$

$$\gamma^1_{\alpha,\beta}=2\lceil(\alpha+1)\lceil\left(\frac{1}{2}\right)\lceil(\alpha,\beta)\lceil\left(\beta+\frac{1}{2}\right).$$

The inverse operator $K^{-1}_{\alpha,\beta}$ applies orthogonality of Jacobi polynomials ([1],p.8) and Poisson Kernel ([1], p. 11) to uniquely define the transform

$$f(z) = K^{-1}_{\alpha,\beta}(F^{\alpha,\beta}) = \int_{-1}^{+1} F^{(\alpha,\beta)}(r\xi, r(1-\xi^2)^{1/2})\nu_{\alpha,\beta}\left((z/r)^2, \xi\right) d\xi$$

where $\nu_{\alpha,\beta}(\Im, \xi) = S_{\alpha,\beta}(\Im, \xi)(1-\xi)^{\alpha}(1+\xi)^{\beta}$

$$S_{\alpha,\beta}(\Im, \xi) = \eta_{\alpha,\beta}\frac{1-\Im}{(1+\Im)^{\alpha+\beta+2}}F\left(\frac{\alpha+\beta+2}{2}; \frac{\alpha+\beta+3}{2}; \beta+1; \frac{2\Im(1+\xi)}{(1+\Im)^2}\right)$$

$$\eta_{\alpha,\beta} = \lceil(\alpha+\beta+2)/2^{\alpha+\beta+1}\lceil(2\alpha+1)\lceil(\beta+1).$$

Here the normalization $K_{\alpha,\beta}(1) = K^{-1}_{\alpha,\beta}(1) = 1$ is taken place. The Kernel $S_{\alpha,\beta}(\Im, \xi)$ is analytic on $||\Im|| < 1$ for $-1 \leq \xi \leq 1$.

Let a real valued generalized biaxially symmetric potentials GBASP's $F^{(\alpha,\beta)}$ regular in $\Sigma_R^{(\alpha,\beta)}$ having $q$-order $\rho_R(q)(0 < \rho_R(q) < \infty)$, $q$-type $T_R(q)(0 \leq T_R(q) \leq \infty)$ and satisfying in addition $(a)$-$(c)$. Then, for a given $\eta(0 < \eta < \infty), T_R(q, r)$ satisfies also:

$$\limsup_{r \to R} \frac{\log^{[q-2]} M(r, F^{(\alpha,\beta)})}{\exp\{(R/(R-r))^{\rho_R(q)}T_R(q, r)\}} = \eta,$$

$$M(r, F^{(\alpha,\beta)}) = \max_{x^2+y^2=r^2} |F^{(\alpha,\beta)}(x, y)|.$$

The $q$-proximate type of a real valued GBASP is not uniquely determined. For example, if we add $\log^{[q-2]}\gamma/(R/(R-r))^{\rho_R(q)}$, $0 < \log^{[q-2]}\gamma < \infty$ in the $q$-proximate type $T_R(q)$, we again obtain a new $q$-proximate type for the same GBASP and the corresponding value of $\eta$ is divided by $\log^{[q-3]}\gamma$.

Since $R_n^{(\alpha,\beta)}(x, y)$ form complete sets for even harmonic, respectively analytic functions, regular at the origin. The GBASP's, then, are the natural extensions of harmonic or analytic functions.Hence we anticipate properties similar to those of the harmonic functions found from associated analytic $f$, by taking $Ref$, the real part of $f$. The Envelope Method ([2], [3]) easily establishes that the GBASP $F^{(\alpha,\beta)}$ is regular in the hypersphere $\sum_R^{(\alpha,\beta)} : x^2 + y^2 < R^2$ of finite radius $R$ about the origin if, and only if its associate $f$ is analytic in the disk $D_R : x^2 + y^2 < R^2$. On the singular axis $y = 0$, the identity $f(x + i0) = F^{(\alpha,\beta)}(x, 0)$, $|x| < R$ can be analytically continued as $f(z) = F^{(\alpha,\beta)}(z, 0)$, $|z| < R$. By the Hadamard three circle theorem,we know, if $f(z)$ is analytic in finite disc, $\log M(r, f)$ is an increasing convex function of $\log r$ in $0 < r < R$. Using this theorem for $F^{(\alpha,\beta)}(x, y)$ we have if $F^{(\alpha,\beta)}(x,y)$ is regular in open hypersphere $\Sigma_R^{(\alpha,\beta)}, \log M(r,F^{(\alpha,\beta)})$ is an increasing convex function of $\log r$ in $0<r<R$. It has the representation

$$(3) \qquad \log M(r,F^{(\alpha,\beta)}) = \log M(r_0,F^{(\alpha,\beta)}) + \int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x} dx, \quad 0<r_0<r<R,$$

where $w(x,F^{(\alpha,\beta)})$ is a positive, continuous and piecewise differentiable function of $x$. In view of the above properties of GBASP's the existence of (3) established by using [11].

McCoy [8] considered the approximation of Pseudoanalytic functions on the disc. Pseudoanalytic functions are constructed as complex combinations of real-valued analytic solutions to the Stokes-Beltrami system. These solutions include the GBASP's. McCoy obtained some coefficient and Bernstein type growth theorems on the disc. Also, Kapoor and Nautiyal [6] characterized the order and type of GBASP's (not necessarily entire) in terms of rates of decay of approximation errors on both sup norm and $L^p$-norm, $1\le p<\infty$, for $q=2$.To obtain more refined measure of growth of GBASP's ,in this paper, we define $q$-proximate type of GBASP's having index - $q$ and then prove its existence. The idea is further extended by defining the $\lambda_q$-proximate type and establish its existence also. It is significant to mention here that Kumar[7] obtained some results for $q=2$. Our results and methods in the present paper are different from all those of authors mentioned above.

Now we prove

**Lemma 1.** *For a real valued GBASP $F^{(\alpha,\beta)}$, regular in hyper sphere $\sum_R^{(\alpha,\beta)}$ and having q-order $\rho_R(q)$ and lower q-order $\lambda_R(q)$, such that*

$$(4) \qquad \begin{matrix}\lim & \sup \\ r\to R & \inf\end{matrix} \; \frac{(R-r)w(r,F^{(\alpha,\beta)})M(r,F^{(\alpha,\beta)})}{\Delta_{[q-1]}M(r,F^{(\alpha,\beta)})} = \begin{matrix}\gamma_R(q) \\ \delta_R(q)\end{matrix},$$

*then*

$$(5) \qquad \delta_R(q) \le \lambda_R(q) \le\le \rho_R(q) \le \gamma_R(q),$$

*where*

$$\Delta_{[q-1]}M(r,F^{(\alpha,\beta)}) = \Pi_{i=0}^{q-1} \log^{[i]} M(r,F^{(\alpha,\beta)}).$$

**Proof.** For given $\varepsilon>0$ and $0<r_0<r<R$, we have from (4)

$$(6) \qquad \delta_R(q)-\varepsilon < \frac{(R-r)w(r,F^{(\alpha,\beta)})M(r,F^{(\alpha,\beta)})}{r\Delta_{[q-1]}M(r,F^{(\alpha,\beta)})} < \gamma_R(q)+\varepsilon.$$

On differentiating (3), we get

$$\frac{M'(r, F^{(\alpha,\beta)})}{M(r, F^{(\alpha,\beta)})} = \frac{w(r, F^{(\alpha,\beta)})}{r} \tag{7}$$

On using (7) in (6), we obtain

$$\frac{\delta_R(q) - \varepsilon}{(R-r)} < \frac{M'(r, F^{(\alpha,\beta)})}{\Delta_{[q-1]}M(r, F^{(\alpha,\beta)}} < \frac{\gamma_R(q) + \varepsilon}{(R-r)}. \tag{8}$$

For $q = 2$ the result has been obtained by Kumar [7]. Now consider the case for $q > 2$, integrating (8), we get

$$(\delta_R(q) - \varepsilon)\log(1/(R-r)) < \log^{[q]} M(r, F^{(\alpha,\beta)}) < (\gamma_R(q) + \varepsilon)\log(1/(R-r)).$$

Dividing above equation by $\log(R/(R-r))$ we get

$$0(1) + (\delta_R(q) - \varepsilon) < \frac{\log^{[q]} M(r, F^{(\alpha,\beta)})}{\log(R/(R-r))} < (\gamma_R(q) + \varepsilon) + o(1)$$

as $r \to R$. Proceeding to limits, we get the requires results i.e., (5). ■

**Lemma 2.** *Let a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hyper sphere $\Sigma_R^{(\alpha,\beta)}$ having q-order $\rho_R(q)(0 < \rho_R(q) < \infty)$, q-type $T_R(q)$ and lower q-type $t_R(q)$ such that*

$$\lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{w(r, F^{(\alpha,\beta)})M(r, F^{(\alpha,\beta)})}{r\Delta_{[q-1]}M(r, F^{(\alpha,\beta)})(R/(R-r))^{\rho_R(q)+1}} = \begin{matrix} \gamma_R^*(q) \\ \delta_R^*(q) \end{matrix}, \tag{9}$$

*then*

$$\delta_R^*(q) \leq \rho_R(q)t_R(q) \leq \rho_R(q)T_R(q) \leq \gamma_R^*(q). \tag{10}$$

**Proof.** Using(9) for $\varepsilon > 0$ and $0 < r_0 < r < R$, we get

$$\delta_R^*(q) - \varepsilon < \frac{w(r, F^{(\alpha,\beta)})M(r, F^{(\alpha,\beta)})}{r\Delta_{[q-2]}M(r, F^{(\alpha,\beta)})(R/(R-r))^{\rho_R(q)+1}} < \gamma_R^*(q) + \varepsilon \tag{11}$$

In view of (7), (11) gives

$$\begin{aligned} (\delta_R^*(q) - \varepsilon)(R/(r-r))^{\rho_R(q)+1} &< \frac{M'(r, F^{(\alpha,\beta)})}{\Delta_{[q-2]}M(r, F^{(\alpha,\beta)})} \\ &< (\gamma_R^*(q) + \varepsilon)(R/(R-r))^{\rho_R(q)+1} \end{aligned} \tag{12}$$

For $q=2$, integrating (12) we get

$$(\delta_R^*(2)-\varepsilon)(R/(R-r))^{\rho(2)}/\rho_R(2) < \log M(r,F^{(\alpha,\beta)}) < (\gamma_R^*(2)+\varepsilon)(R/(R-r))^{\rho_R(2)}/\rho_R(2)$$

or

$$(\delta_R^*(2)-\varepsilon) < \rho_R(2)\frac{\log M(r,F^{(\alpha,\beta)})}{(R/R-r)^{\rho_R(2)}} < (\gamma_R^*(2)+\varepsilon).$$

Proceeding to limits, we get

$$\delta_R^*(2) \le \rho_R(2)t_R(2) \le \rho_R(2)T_R(2) \le \gamma_R^*(2). \tag{13}$$

Now for $q>2$, similarly on integrating (12) we get

$$(\delta_R^*(q)-\varepsilon)\frac{(R/(R-r))^{\rho_R(q)}}{\rho_R(q)} < \log^{[q-1]} M(r,F^{(\alpha,\beta)}) < (\gamma_R^*(q)+\varepsilon)\frac{(R/(R-r))^{\rho_R(q)}}{\rho_R(q)}$$

On applying the limits we get

$$\delta_R^*(q) \le \rho_R(q)t_R(q) \le \rho_R(q)T_R(q) \le \gamma_R^*(q) \tag{14}$$

combining (13) and (14), we get (10). Hence the proof is completed. ■

Now we prove

## 3. Main results

### 3.1. $q$ - Proximate type

**Theorem 1.** *Let a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hypersphere $\Sigma_R^{(\alpha,\beta)}$ having q-order $\rho_R(q)(0<\rho_R(q)<\infty)$ and q-type $T_R(q)(0\le T_R(q)\le\infty)$ such that limits in (4) and (9) exist. Then, for a positive real number $\eta, \log(\eta^{-1}\log^{[q-2]}$ $M(r,F^{(\alpha,\beta)})/(R/(R-r))^{\rho_R(q)}$ is a q-proximate type of a GBASP $F^{(\alpha,\beta)}$.*

**Proof.** Let

$$S_{\rho_R}(q,r) = \frac{\log(\eta^{-1}\log^{[q-2]} M(r,F^{(\alpha,\beta)}))}{\log(R/(R-r))^{\rho_R(q)}}. \tag{15}$$

Since $\log^{[q-1]} M(r,F^{(\alpha,\beta)})$ is positive, continuous and increasing function of $r$ for $R>r>r_0>0$, which is differentiable in adjacent open intervals, it follows that $S_{\rho_R}(q,r)$ satisfies $(c)$. Existence of limit in (9) implies that $F^{(\alpha,\beta)}$

is of perfectly regular growth and moreover, $S_{\rho_R}(q,r) \to T_R(q,r)$ *as* $r \to R$. Differentiating (15), we obtain

$$\frac{S'_{\rho_R}(q,r)}{S_{\rho_R}(q,r)} = \frac{M'(r,F^{(\alpha,\beta)})}{\log(\eta^{-1}\log^{[q-2]} M(r,F^{(\alpha,\beta)}))\Delta_{[q-2]}M(r,F^{(\alpha,\beta)})} - \frac{\rho_R(q)}{(R-r)}$$

or

$$\frac{(R-r)S_{\rho_R}(q,r)}{S_{\rho_R}(q,r)} \simeq \frac{w(r,F^{(\alpha,\beta)})M(r,F^{(\alpha,\beta)})(R-r)}{r\Delta_{[q-1]}M(r,F^{(\alpha,\beta)})} - \rho_R(q)$$

Proceeding to limits as $r \to R$ and taking (1.4)into account, we get

$$\frac{(R-r)S'_{\rho_R}(q,r)}{S_{\rho_R}(q,r)} \to 0 \quad \text{as} \quad r \to R.$$

Thus $S_{\rho_R(q,r)}$ satisfies the condition $(c)$.

From (15), $(d)$ is easily obtained. In this way all the conditions for $S_{\rho_R}(q,r)$ to be a $q$-proximate type of GBASP $F^{(\alpha,\beta)}$ are satisfied and hence the proof of the theorem is completed. ■

**Definition 3.** *A real valued GBASP* $F^{(\alpha,\beta)}$ *is said to be of irregular growth if* $0 < \lambda_R(q) \neq \rho_R(q) < \infty$.

### 3.2. Lower $q$-proximate type

**Lemma 3.** *The lower q-type of a real valued GBASP* $F^{(\alpha,\beta)}$ *of irregular growth is zero.*

**Proof.** If $F^{(\alpha,\beta)}$ is of irregular growth then $\rho_R(q) > \lambda_R(q) > 0$. We have

$$\liminf_{R\to r} \frac{\log^{[q]} M(r,F^{(\alpha,\beta)})}{\log(R/(R-r))} = \lambda_R(q). \tag{16}$$

Since $M(r,F^{(\alpha,\beta)}) \to \infty$ *as* $r \to R, \log^{+}$ may be replaced by log. For given $\varepsilon > 0$ and $r > r_0(\varepsilon)$,

$$\log^{[q]} M(r,F^{(\alpha,\beta)}) < (\lambda_R(q)-\varepsilon)\log(R/(R-r))$$

or

$$\log^{[q-1]} M(r,F^{(\alpha,\beta)}) < (R/(R-r))^{\lambda_R(q)-\varepsilon} \tag{17}$$

whereas for a sequence of values of $r \to \infty$,

$$\log^{[q-1]} M(r,F^{(\alpha\beta)}) < (R/(R-r))^{\lambda_R(q)+\varepsilon} \tag{18}$$

Dividing (17),(18) by $(R/(R-r))^{\rho_R}(q)$ and passing to limits to the argument shows that

$$\liminf_{r\to R}\frac{\log^{[q-1]}M(r,F^{(\alpha,\beta)})}{(R/(R-r))^{\rho_R(q)}}=0.$$

Hence the proof is completed. ∎

From Lemma 3, we conclude that the case $t_R(q)>0$ is only limited to the study of GBASP $F^{\alpha,\beta}$ of regular $q$-growth. In such case we can define similarly lower $q$-proximate type. But to be more general, let $\lambda_R(q)$ be such that

$$\liminf_{r\to R}\frac{\log^{[q-1]}M(r,F^{(\alpha,\beta)})}{(R/(R-r))^{\lambda_R(q)}}=t_{\lambda_R}(q), \tag{19}$$

$t_{\lambda_R}(q)$ is said to be the $\lambda_R(q)$-type of GBASP $F^{(\alpha,\beta)}$. If $\rho_R(q)=\lambda_R(q)$, then $t_{\lambda_R}(q)$ is the same as lower $q$-type $t_R(q)$. There exist GBASP $F^{(\alpha,\beta)}$ for which $t_{\lambda_R}(q)$ is nonzero and finite. For such GBASP $F^{(\alpha,\beta)}$ we define $\lambda_R(q)-$proximate type as follows:

**Definition 4.** *A real valued positive function $t_{\lambda_R}(q,r)$ is said to be a $\lambda_R(q)$-proximate type of GBASP $F^{(\alpha,\beta)}$ having index - $q$, $q$-order $\rho_R(q)$, lower $q$-order $\lambda_R(q)$ and $\lambda_R(q)-$type $t_{\lambda_R}(q)(0<t_{\lambda_R}(q)<\infty)$ if for given $\eta>0, t_{\lambda_R}(q,r)$ satisfies the following properties:*

*(i) $t_{\lambda_R}(q,r)$ is continuous and piecewise differentiable for $r\to r_0$,*

*(ii) $t_{\lambda_R}(q,r)\to t_{\lambda_R}(q)$ as $r\to R$,*

*(iii) $\frac{(R-r)t'_{\lambda_R}(q,r)}{t_{\lambda_R}(q,r)}\to 0$ as $r\to R$, where $t'_{\lambda_R}(q,r)$ is either the left or right hand derivative at points where these are different, and*

*(iv) $\liminf_{r\to R}\frac{\log^{[q-2]}M(r,F^{(\alpha,\beta)})}{exp\{(R/(R-r))^{\lambda_R(q)}t_{\lambda_R}(q,r)\}}=\eta$.*

Now we have

**Theorem 2.** *For every GBASP $F^{(\alpha,\beta)}$ of $q$-order $\rho_R(q)$, lower $q$-order $\lambda_R(q)$ and $\lambda_R(q)-$type $t_{\lambda_R}(q)(0<t_{\lambda_R}(q)<\infty)$, there exists a lower $q$-proximate type $t_{\lambda_R}(q,r)$ satisfying (i) through (iv).*

**Proof.** This theorem can be proved in a similar manner as Theorem 1 for the case $T_R(q,r)$, so we omit the proof. ∎

**Theorem 3.** *Let a real valued GBASP $F^{(\alpha,\beta)}$ regular in open hypersphere $\sum_R^{(\alpha,\beta)}$ and having $q$-order $\rho_R(q)$, lower $q$-order $\lambda_R(q)(0\le\lambda_R(q)\le\rho_R(q)<\infty)$, $q$-type $T_R(q)$ and lower $q$-type $t_R(q)$. Then*

$$\begin{matrix} c/t_R(q) \\ d/t_R(q)\end{matrix}\le\lim_{r\to R}\frac{(R-r)S'_{\rho_R}(q,r)}{S_{\rho_R}(q,r)}+\rho_R(q)\le\frac{c}{t_R(q)}, \tag{20}$$

*where*

$$\lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{w(r, F^{(\alpha,\beta)})RM(r, F^{(\alpha,\beta)})}{r(R/(R-r))^{\rho_R(q)+1}\Delta_{[q-2]}M(r, F^{(\alpha,\beta)})} = \begin{matrix} c \\ d \end{matrix}. \tag{21}$$

*Moreover, if $F^{(\alpha,\beta)}$ is of irregular growth then*

$$-\infty \le \lim_{r\to R} \frac{(R-r)S_{\lambda_R}(q,r)}{S_{\lambda_R}(q,r)} \le \frac{d}{t_{\lambda_R}(q,r)} - \lambda_R(q), \tag{22}$$

*where $S_{\lambda_R}(q,r)$ is a function in (1.15) corresponding to $\lambda_R(q)$ and $t_{\lambda_R}(q)$ is the $\lambda_R(q)-$type of $F^{(\alpha,\beta)}$.*

**Proof.** By (3) and the definition of $q$-type $T_R(q)$ and lower $q$-type $t_R(q)$ we observe that

$$\lim_{r \to R} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[q-2]}(\int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x} dx)}{(R/(R-r))^{\rho_R(q)}} = \begin{matrix} T_R(q) \\ t_R(q) \end{matrix}, \tag{23}$$

similarly, for GBASP $F^{(\alpha,\beta)}$ of irregular growth,

$$\liminf_{r\to R} \frac{\log^{[q-2]}(\int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x} dx)}{(R/(R-r))^{\lambda_R(q)}} = t_{\lambda_R}(q). \tag{24}$$

Fix $r_0 \in [0, \infty]$ such that $\eta = \log^{[q-1]} M(r_0, F^{(\alpha,\beta)})$. Hence

$$\lim_{r\to R} \log(\eta^{-1} \log^{[q-2]} M(r, F^{(\alpha,\beta)})) = \log^{[q-2]} \int_{r_0}^{r} \frac{w(x, F^{(\alpha,\beta)})}{x} dx.$$

On differentiating (15), we have

$$\begin{aligned} \frac{S'_{\rho_R}(q,r)}{S_{\rho_R}(q,r)} &= \frac{M'(r, F^{(\alpha,\beta)})}{\log(\eta^{-1} \log^{[q-2]} M(r, F^{(\alpha,\beta)})\Delta_{[q-2]}M(r, F^{(\alpha,\beta)}))} - \frac{\rho_R(q)}{R-r} \\ &= \frac{w(r, F^{(\alpha,\beta)})M(r, F^{(\alpha,\beta)})}{r \log^{[q-2]}(\int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x} dx)\Delta_{[q-2]}M(r, F^{(\alpha,\beta)})} - \frac{\rho_R(q)}{(R-r)} \end{aligned}$$

$$\begin{aligned} &\frac{(R-r)S'_{\rho_R}(q,r)}{S_{\rho_R}(q,r)} + \rho_R(q) \\ &\qquad = \frac{w(r, F^{(\alpha,\beta)})R}{r(R/(R-r))\Delta_{[q-2]}M(r, F^{(\alpha,\beta)}) \log^{[q-2]} \int_{r_0}^{r} \frac{w(r,F^{(\alpha,\beta)})}{x} dx}. \end{aligned}$$

Proceeding to limits in above and using (21), (23) and (24),we get (20).

In case $\rho_R(q) > \lambda_R(q)$, we have

$$S_{\lambda_R}(q,r) = \frac{\log(\eta^{-1}\log^{[q-2]} M(r, F^{(\alpha,\beta)}))}{(R/(R-r))^{\lambda_R(q)}} = \frac{\log^{[q-2]}\int_{r_0}^{r} \frac{w(x,F^{(\alpha,\beta)})}{x}dx}{(R/(R-r))^{\lambda_R(q)}}.$$

Similarly using (24),we get (20). ■

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