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SOBOLEV SPACES "WITH MIXED FUNCTIONS"

Abstract. This paper describes some generalization of modular function spaces $L_{\varphi,\psi}$ defined by a modular $I_{\varphi,\psi}(f) = \int_a^b \varphi\left(x, \int_c^d \psi(y, f(x,y))\, dy\right) dx$, ([3]). The next part of this paper focuses on using of spaces, defined previously, to introduce Sobolev spaces as a vector subspace of the generalized space $L_{\varphi,\psi}$. Some selected properties of these spaces are presented.

Key words: modular space, Sobolev space.

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1. Basic notions

Let us denote $T = (a,b) \times (c,d) \subset R^2$, $-\infty < c < d < +\infty$ and let $L(T)$ be the space of Lebesgue integrable real functions on T, with equality almost everywhere. Let real functions $\varphi : (a,b) \times R \to [0,+\infty)$ and $\psi : (c,d) \times R \to [0,+\infty)$ satisfy the following conditions:

1. φ and ψ are measurable functions of the first variable for every fixed value of the second one;
2. $\varphi(t,u)$ and $\psi(t,u)$ are even, convex and continuous at zero with respect to the second variable, $\varphi(t,0) = \psi(t,0) = 0$, $\varphi(t,u) > 0$ and $\psi(t,u) > 0$ if $u \neq 0$ for a.e. t.
3. $\int_a^b \varphi(t,u)\, dt < \infty$, $\int_c^d \psi(t,u)\, dt < \infty$ for every u.

2. Selected properties of the space $L_{\varphi,\psi}$

For any function $f \in L(T)$ we define a functional

$$I_{\varphi,\psi}(f) = \int_a^b \varphi\left(x, \int_c^d \psi(y, f(x,y))\, dy\right) dx \tag{1}$$

and we denote by $L_{\varphi,\psi}(T)$ the vector space of all functions f from $L(T)$ such that $I_{\varphi.\psi}(\lambda f) < \infty$ for some $\lambda > 0$, ([2]). By $E_{\varphi,\psi}(T)$ we denote the vector space of all finite elements of $L(T)$ i.e. such that $I_{\varphi.\psi}(\lambda f) < \infty$ for every $\lambda > 0$. The functional $I_{\varphi,\psi}$ is a convex modular in $L(T)$, hence

$$\|f\|_{\varphi,\psi} = inf\left\{u > 0 : \; I_{\varphi,\psi}\left(\frac{f}{u}\right) \leq 1\right\}$$

is norm in $L_{\varphi,\psi}(T)$. Convergence $f_n \to f$ in the sense of this norm is equivalent to the condition

$$I_{\varphi,\psi}(\lambda(f_n - f)) \to 0, \;\; n \to \infty \tag{2}$$

for every $\lambda > 0$. If (2) holds only for some $\lambda > 0$, we say that the sequence f_n is convergent to f in the sense of the modular $I_{\varphi,\psi}$.

Theorem 1. *The space $L_{\varphi,\psi}(T)$ is complete with respect to the modular $I_{\varphi,\psi}$. Moreover, $L_{\varphi,\psi}(T)$ is also complete in the sense of the norm $\|\cdot\|_{\varphi,\psi}$.*

Proof. Let (f_n)be a Cauchy sequence in the sense of $I_{\varphi,\psi}$in $L_{\varphi,\psi}(T)$. Then (f_n) is also a Cauchy sequence in measure. Thus there exists a measurable function f such that (f_n) is convergent in measure to f. So (f_n) contains a subsequence (f_{n_k}) convergent to f almost everywhere in T. Hence, for fixed n and a.e. $y \in (c,d)$ we have $\psi(y, \lambda(f_n(x,y) - f_{n_k}(x,y))) \to \psi(\lambda(f_n(x,y) - f(x,y)))$ for a.e. $x \in (a,b)$ as $k \to \infty$, for $\lambda > 0$. Applying Fatou lemma with respect to the variable y and then with respect to the variable x we obtain

$$\begin{aligned}
&\int_a^b \varphi\left(x, \int_c^d \psi(y, \lambda(f_n(x,y) - f(x,y)))\,dy\right)dx \\
&\qquad = \int_a^b \varphi\left(x, \int_c^d \lim_{k\to\infty} \psi(y, \lambda(f_n(x,y) - f_{n_k}(x,y)))\,dy\right)dx \\
&\qquad \leq \liminf_{k\to\infty} I_{\varphi,\psi}(\lambda(f_n - f_{n_k})) \leq \varepsilon
\end{aligned}$$

for sufficiently large n. Thus $I_{\varphi,\psi}(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for some $\lambda > 0$. From the inequality

$$I_{\varphi,\psi}\left(\frac{1}{2}\lambda f\right) \leq I_{\varphi,\psi}(\lambda(f_n - f)) + I_{\varphi,\psi}(\lambda f_n)$$

we conclude that $f \in L_{\varphi,\psi}(T)$. ■

Let $S(T)$ be the set of all simple functions from $L(T)$ and let $L^\infty(T)$ be the set of essentially bounded functions from $L(T)$. Then $S(T) \subset L^\infty(T)$. Let us denote $K = \text{supess}_{(x,y)\in T} \mid f(x,y) \mid$ for $f \in L^\infty(T)$. Then $I_{\varphi,\psi}(\lambda f) < \infty$ for every $\lambda > 0$. Thus $L^\infty(T) \subset E_{\varphi,\psi}(T)$.

Lemma 1. *The set $S(T)$ of simple functions on T is dense in $L_{\varphi,\psi}(T)$ in the sense of the modular $I_{\varphi,\psi}$. Moreover, $S(T)$ is dense in $E_{\varphi,\psi}(T)$ in the sense of the norm.*

Proof. Let $f \in L_{\varphi,\psi}(T)$, $f \geq 0$, and let $\lambda > 0$ be a constant such that $I_{\varphi,\psi}(\lambda f) < \infty$. Let (f_n) be a non-decreasing sequence of nonnegative simple functions such that $f_n \to f$ on T. Then

$$f(x,y) \geq f(x,y) - f_n(x,y)$$

for arbitrary n and every $(x,y) \in T$. Hence

$$\psi(y, \lambda f(x,y)) \geq \psi(y, \lambda(f(x,y) - f_n(x,y))) \to 0 \quad \text{as} \quad n \to \infty$$

for any $\lambda > 0$ and $(x,y) \in T$. Since $f \in L_{\varphi,\psi}(T)$, we have $\int_c^d \psi(y, \lambda f(x,y))\, dy < \infty$ for a.e. $x \in (a,b))$ and for sufficiently small $\lambda > 0$. By the dominated convergence theorem we obtain

$$\int_c^d \psi(y, \lambda(f(x,y) - f_n(x,y)))\, dy \to 0 \quad \text{as} \quad n \to \infty$$

for a.e. $x \in (a,b)$. Using continuity of φ with respect to the second variable, we have

$$\varphi\left(x, \int_c^d \psi(y, \lambda(f(x,y) - f_n(x,y)))\, dy\right) \to 0 \quad \text{as} \quad n \to \infty$$

almost everywhere in (a,b). Moreover

$$\varphi\left(x, \int_c^d \psi(y, \lambda(f(x,y) - f_n(x,y)))\, dy\right) \leq \varphi\left(x, \int_c^d \psi(y, \lambda f(x,y))\, dy\right)$$

and $\int_a^b \varphi\left(x, \int_c^d \psi(y, \lambda f(x,y))\, dy\right) dx < \infty$ for sufficiently small $\lambda > 0$. Applying the dominated convergence theorem again, we obtain $I_{\varphi,\psi}(\lambda(f_n - f)) \to 0$ as $n \to \infty$ for small $\lambda > 0$. Thus (f_n) is convergent to f in the sense of the modular $I_{\varphi,\psi}$. If $f \in L_{\varphi,\psi}(T)$ is arbitrary, we may split f into positive and negative parts and apply the above result. Arguing in the like manner it is shown that $S(T)$ is dense also in $E_{\varphi,\psi}(T)$ in the sense of the norm. ■

Let $S_0(T)$ be the set of all simple functions of the form $g(x,y) = \sum_{i=1}^n b_i \chi_{A_i}(x,y)$, where b_i are rational numbers and χ_{A_i} are the characteristic functions of the measurable sets $A_i \subset T$.

Lemma 2. *The set $S_0(T)$ is dense in the sense of the modular $I_{\varphi,\psi}$ in $S(T)$.*

Proof. Let $h \in S(T)$, $h(x,y) = \sum_{i=1}^{n} a_i \chi_{B_i}(x,y)$, where $B_i \subset T$ are measurable, pairwise disjoint and $| B_i | < \infty$. We denote $r = \max_{1 \le i \le n} | a_i |$. Let $\lambda > 0$ and $\varepsilon > 0$ be given. By the condition 3 and separability of Lebesgue measure, there exists a sequence (A_n) of sets $A_n \subset T$ such that for every set B_i we may choose a set A_{k_i} in (A_n) in such manner, that

$$\iint_{A_{k_i} \dot{-} B_i} \varphi(x,1)\, \psi(y,\lambda r)\, dxdy < \frac{\varepsilon}{n}.$$

Let us take $B = \bigcup_{i=1}^{n} B_i$ and let $\delta > 0$ be fixed. We choose rational numbers $b_1, b_2, \ldots, b_n$ in such that $| b_i - a_i | < \delta$ and $| b_i | < 2r$ for $i = 1, 2, \ldots, n$. Then

$$| h(x,y) - g(x,y) | \le 2r \sum_{i=1}^{n} | \chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y) | + \delta \chi_B(x,y).$$

Hence

$$\begin{aligned} &\psi\left(y, \frac{1}{4}\lambda\,(h(x,y) - g(x,y))\right) \\ &\qquad \le \psi\left(y, \lambda r \sum_{i=1}^{n} (\chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y))\right) + \psi(y, \lambda\delta\chi_B(x,y)) \\ &\qquad = \sum_{i=1}^{n} \psi(y,\lambda r) \mid \chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y) \mid + \psi(y,\lambda\delta)\, \chi_B(x,y). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_c^d \psi\left(y, \frac{1}{4}\lambda\,(h(x,y) - g(x,y))\right) dy \\ &\qquad \le \sum_{i=1}^{n} \int_c^d \psi(y,\lambda r) \mid \chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y) \mid dy \\ &\qquad\quad + \int_c^d \psi(y,\lambda\delta)\, \chi_B(x,y)\, dy. \end{aligned}$$

By convexity of φ we have

$$\begin{aligned} &\int_a^b \varphi\left(x, \int_c^d \psi\left(y, \frac{1}{4}\lambda\,(h(x,y) - g(x,y))\right) dy\right) dx \\ &\qquad \le \sum_{i=1}^{n} \int_a^b \varphi\left(x, 2\int_c^d \psi(y,\lambda r) \mid \chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y) \mid dy\right) dx \\ &\qquad\quad + \int_a^b \varphi\left(x, 2\int_c^d \psi(y,\lambda r)\, \chi_B(x,y)\, dy\right) dx = \sum_{i=1}^{n} I_i + I. \end{aligned}$$

Let us denote $L = \int_c^d \psi(y, \lambda r)\, dy$. Applying Jensen's inequality, we obtain

$$I_i \leq \frac{1}{L} \int_a^b \int_c^d \varphi\left(x, 2L\left(\chi_{B_i}(x,y) - \chi_{A_{k_i}}(x,y)\right)\right) \psi(y, \lambda r)\, dxdy$$
$$\leq c \iint_{A_{k_i} \dot{-} B_i} \varphi(x,1)\, \psi(y, \lambda r)\, dxdy.$$

It is easy to see that $I < \varepsilon$ for sufficiently small $\delta > 0$. Consequently,

$$I_{\varphi,\psi}\left(\frac{1}{4}\lambda(h-g)\right) \leq \sum_{i=1}^{n} \frac{\varepsilon c}{n} + \varepsilon = \varepsilon(c+1).$$

This shows that the set $S_0(T)$ is dense in $S(T)$ in the sense of the modular. ■

By Lemma 1 and Lemma 2 we obtain

Theorem 2. *The space $L_{\varphi,\psi}(T)$ is separable in the sense of $I_{\varphi,\psi}$.*

The real functions Φ_1 and Φ_2 defined on a product $(\alpha,\beta) \times R$ satisfy the condition $(\star)$ if there holds the following inequality

$$(\star) \qquad \Phi_1(t,u) \leq c_1 \Phi_2(t, c_2 u) + F(t)$$

for all $u > 0$ and almost every $t \in (\alpha,\beta)$, where F is a nonnegative, integrable function in (α,β) and c_1, c_2 are positive constants.

Theorem 3. *If pairs of functions (φ_1, φ_2) and (ψ_1, ψ_2) satisfy the condition $(\star)$, then $L_{\varphi_2,\psi_2}(T) \subset L_{\varphi_1,\psi_1}(T)$.*

Proof. We have

$$\varphi_1(x,u) \leq K_1 \varphi_2(x, K_2 u) + h(x)$$

for all $u > 0$ and almost every $x \in (a,b)$, where h is a nonnegative and integrable function in (a,b), K_1, $K_2 > 0$. We have also

$$\psi_1(y,u) \leq L_1 \psi_2(y, L_2 u) + g(y)$$

for all $u > 0$ and almost every $y \in (c,d)$, where g is a nonnegative and integrable function in (c,d), L_1, $L_2 > 0$.

Let $f \in L_{\varphi_2,\psi_2}(T)$ and let us denote $\lambda_0 = \frac{\lambda}{2L_1L_2K_2}$, where $\lambda > 0$ is such that $I_{\varphi_2,\psi_2}(\lambda f) < \infty$. We may suppose that $L_1 > 1$and $K_2 > 1$. Then

$$\psi_1(y, \lambda_0 f(x,y)) \leq \frac{1}{2K_2} \psi_2(y, \lambda f(x,y)) + g(y)$$

and

$$\int_c^d \psi_1(y, \lambda_0 f(x,y))\, dy \le \frac{1}{2K_2}\int_c^d \psi_2(y, \lambda f(x,y))\, dy + \int_c^d g(y)\, dy.$$

Hence we obtain

$$\begin{aligned}\varphi_1\left(x, \int_c^d \psi_1(y, \lambda_0 f(x,y))\, dy\right) \ \le\ & \frac{1}{2}K_1\varphi_2\left(x, \int_c^d \psi_2(y, \lambda f(x,y))\, dy\right) \\ & + \frac{1}{2}h(x) + \frac{1}{2}\varphi_1\left(x, 2\int_c^d g(y)\, dy\right)\end{aligned}$$

and

$$\begin{aligned}\int_a^b \varphi_1\left(x, \int_c^d \psi_1(y, \lambda_0 f(x,y))\, dy\right) dx \le K_1 I_{\varphi_2,\psi_2}(\lambda f) \\ + \int_a^b \varphi_1\left(x, 2\int_c^d g(y)\, dy\right) dx + \int_a^b h(x)\, dx < \infty.\end{aligned}$$

This shows that $I_{\varphi_1,\psi_1}(\lambda_0 f) < \infty$ and we conclude that $f \in L_{\varphi_1,\psi_1}(T)$. ■

Corollary. *If pairs of functions (φ_1, φ_2) and (ψ_1, ψ_2) satisfy the condition $(\star)$, the embedding $L_{\varphi_2,\psi_2}(T) \subset L_{\varphi_1,\psi_1}(T)$ is continuous.*

Proof. If $f \in L_{\varphi_2,\psi_2}(T)$, then $I_{\varphi_2,\psi_2}\left(\frac{f}{\|f\|_{\varphi_2,\psi_2}}\right) \le 1$. Arguing in analogous manner as in the proof of Theorem 3 we have

$$\begin{aligned}\int_a^b \varphi_1\left(x, \int_c^d \psi_1\left(y, \frac{C_1 f(x,y)}{\| f \|_{\varphi_2,\psi_2}}\right) dy\right) dx \le K_1 I_{\varphi_2,\psi_2}\left(\frac{f}{\| f \|_{\varphi_2,\psi_2}}\right) \\ + \int_a^b \varphi_1\left(x, 2\int_c^d g(y)\, dy\right) dx + \int_a^b h(x)\, dx \le C,\end{aligned}$$

where $C_1 = \frac{1}{2L_1L_2K_2}$ and the constants K_1, K_2, L_1, L_2 are from the condition $(\star)$ for pairs of functions (φ_1, φ_2) and (ψ_1, ψ_2). The inequality

$$\int_a^b \varphi_1\left(x, \int_c^d \psi_1\left(y, \frac{C_1 f(x,y)}{\| f \|_{\varphi_2,\psi_2}}\right) dy\right) dx \le C, \quad \text{where} \quad C \ge 1$$

implies

$$\| f \|_{\varphi_1,\psi_1} \le \frac{C}{C_1} \| f \|_{\varphi_2,\psi_2} .$$

■

3. Concept of Sobolev space “with mixed functions”

Let k be an arbitrary nonnegative integer number and let φ and ψ satisfy the conditions 1 - 3. Denote by X the space of real valued, measurable functions f on T possessing distributional derivatives $D^{\alpha} f$ up to order k belonging to the space $L_{\varphi,\psi}(T)$. Define a functional $I_{\varphi,\psi}^{(k)}$ on X

$$I_{\varphi,\psi}^{(k)}(f) = \sum_{|\alpha|\leq k} \int_a^b \varphi\left(x, \int_c^d \psi\left(y, D^{\alpha} f(x,y)\right) dy\right) dx.$$

The modular space generating by the modular $I_{\varphi,\psi}^{(k)}$ we denote by $W_{\varphi,\psi}^{k}(T)$. The space $W_{\varphi,\psi}^{k}(T)$ we call the Sobolev space “with mixed functions”. Since $I_{\varphi,\psi}^{(k)}$ is a convex modular, so

$$\| f \|_{\varphi,\psi}^{(k)} = inf\left\{\varepsilon > 0 \ : \ I_{\varphi,\psi}^{(k)}\left(\varepsilon^{-1} f\right) \leq 1\right\}$$

is a norm in $W_{\varphi,\psi}^{k}(T)$. Convergence $f_n \to f$ in the sense of the norm $\| \cdot \|_{\varphi,\psi}^{(k)}$ is equivalent to the condition

$$I_{\varphi,\psi}^{(k)}\left(\lambda\left(f_n - f\right)\right) \to 0 \quad \text{as} \quad n \to \infty$$

for every $\lambda > 0$.

Lemma 3. *Let ψ be integrable in (c,d) for every u. If $k_{\varphi} = \inf_{x\in(a,b)} \varphi(x,1) > 0$ and $k_{\psi} = \inf_{y\in(c,d)} \psi(y,1) > 0$, then is true the following inequality*

$$u \leq \frac{1}{k_{\varphi}} \varphi\left(x, \frac{1}{(d-c)\, k_{\psi}} \int_c^d \psi(y,u)\, dy\right)$$

for $u \geq 1$.

Proof. The condition $k_{\psi} > 0$ and continuity of ψ with respect to the second variable imply

$$u \leq \frac{1}{k_{\psi}} \psi(y,u) \tag{3}$$

for $u \geq 1$. Integrating (3) over $y \in (c,d)$ we obtain

$$u \leq \frac{1}{(d-c)\, k_{\psi}} \int_c^d \psi(y,u)\, dy \tag{4}$$

for $u \geq 1$. Moreover, for the function φ we have

$$u \leq \frac{1}{k_{\varphi}} \varphi(x,u) \tag{5}$$

for $u \geq 1$. Applying (4) and (5) we obtain easily that

$$u \leq \frac{1}{k_\varphi}\varphi\left(x, \frac{1}{(d-c)\,k_\psi}\int_c^d \psi\,(y,u)\,dy\right).$$

∎

Theorem 4. *Let ψ be integrable in (c,d) for every u. If k_φ and k_ψ are positive, then the space $W^k_{\varphi,\psi}(T)$ is complete with respect to the norm.*

Proof. Let (f_n) be a Cauchy sequence in $W^k_{\varphi,\psi}(T)$. This means that $I^{(k)}_{\varphi,\psi}(\lambda(f_n - f_m)) \to 0$ as $m,n \to \infty$ for every $\lambda > 0$. Then, for every α, $|\,\alpha\,| \leq k$, the sequence $(D^\alpha f_n)$ is a Cauchy sequence in $L_{\varphi,\psi}(T)$. In particular (f_n) is a Cauchy sequence in $L_{\varphi,\psi}(T)$. By completeness of $L_{\varphi,\psi}(T)$ there exists $f_0 \in L_{\varphi,\psi}(T)$ such that (f_n) is convergent to f_0 in the sense of the norm $\|\cdot\|_{\varphi,\psi}$. We will prove that f_n are locally integrable on T. We may suppose k_φ, $k_\psi \leq 1$. Let us denote $p = min\,(1, d-c)$, then

$$\varphi\left(x, \frac{1}{(d-c)\,k_\psi}\int_c^d \psi\left(y, \frac{pk_\psi f_n(x,y)}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}}\right)dy\right) \leq \varphi\left(x, \int_c^d \psi\left(y, \frac{f_n(x,y)}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}}\right)dy\right).$$

Let $B \subset T$ be any compact set and

$$A = \left\{(x,y) \in B :\ \frac{pk_\psi\,|\,f_n(x,y)\,|}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}} \geq 1\right\}.$$

Then, applying Lemma 3, we obtain

$$pk_\psi\frac{|\,f_n(x,y)\,|}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}} \leq \frac{1}{k_\varphi}\varphi\left(x, \frac{1}{(d-c)\,k_\psi}\int_c^d \psi\left(y, \frac{pk_\psi\,|\,f_n(x,y)\,|}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}}\right)dy\right)$$

for $(x,y) \in A$. Hence

$$\text{(6)}\qquad \frac{1}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}}\iint_B |\,f_n(x,y)\,|\,dxdy \leq \frac{d-c}{pk_\varphi k_\psi}\int_a^b \varphi\left(x, \int_c^d \psi\left(y, \frac{f_n(x,y)}{\|\,f_n\,\|^{(k)}_{\varphi,\psi}}\right)dy\right)dx + \frac{1}{pk_\psi}\,|\,B\,|\,.$$

The inequality (6) implies local integrability of f_n in T, $n = 1, 2, \dots$ Hence f_n defines a regular distribution

$$T_{f_n} g = \iint_T f_n(x,y)\, g(x,y)\, dxdy,$$

where $g \in C_0^\infty(T)$. For every $\alpha, \mid \alpha \mid \le k$, we have

$$(7) \qquad \mid T_{D^\alpha f_n} g - T_{D^\alpha f_0} g \mid \le C \iint_K \mid f_n(x,y) - f_0(x,y) \mid dxdy$$

where $C = \max_{(x,y)\in T, |\alpha| \le k} \mid D^\alpha g(x,y) \mid$ and $K \subset T$, K is the support of g. We have from the inequalities (6) and (7)

$$\begin{aligned} &\mid T_{D^\alpha f_n} g - T_{D^\alpha f_0} g \mid \\ &\quad \le \left(C_1 \int_a^b \varphi\left(x, \int_c^d \psi\left(y, \frac{f_n(x,y) - f_0(x,y)}{\| f_n - f_0 \|_{\varphi,\psi}}\right) dy\right) dx + C_2 \mid K \mid \right) \\ &\quad \times \| f_n - f_0 \|_{\varphi,\psi} \le (C_1 + C_2 \mid K \mid) \| f_n - f_0 \|_{\varphi,\psi} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $T_{D^\alpha f_n} \to T_{D^\alpha f_0}$, $n \to \infty$. Since $(D^\alpha f_n)$, $\mid \alpha \mid \le k|$, is the Cauchy sequence in $L_{\varphi,\psi}(T)$, thus there exists $f_\alpha \in L_{\varphi,\psi}(T)$ such that $(T_{D^\alpha f_n})$ is convergent to T_{f_α} as $n \to \infty$. Consequently $f_\alpha = D^\alpha f_0$ for every $\mid \alpha \mid \le k$. Now, we have $f_0 \in W^k_{\varphi,\psi}(T)$. Moreover, $I_{\varphi,\psi}(\lambda(D^\alpha f_n - D^\alpha f_0)) \to 0$ as $n \to \infty$ for every $\lambda > 0$ and $\mid \alpha \mid \le k$. Thus we proved that (f_n) is convergent to f_0 with respect to the norm of $W^k_{\varphi,\psi}(T)$. ■

Let us observe, that arguing in a like manner as in the proof of Theorem 3, we obtain the following theorem.

Theorem 5. *If pairs of functions (φ_1, φ_2) and (ψ_1, ψ_2) satisfy the condition $(\star)$, then $W^k_{\varphi_1,\psi_1}(T) \subset W^k_{\varphi_2,\psi_2}(T)$. The embedding of $W^k_{\varphi_1,\psi_1}(T)$ in $W^k_{\varphi_2,\psi_2}(T)$ is continuous with respect to the norms.*

4. Separability of $W^k_{\varphi,\psi}(T)$

Let $l = \sum_{|\alpha| \le k} 1$ and $L^l_{\varphi,\psi}(T) = \prod_{i=1}^l L_{\varphi,\psi}(T)$. For any $f = (f_i)_{i=1}^l \in L^l_{\varphi,\psi}$ we define

$$\rho(f) = \sum_{i=1}^l \int_a^b \varphi\left(x, \int_c^d \psi(y, f_i(x,y))\, dy\right) dx.$$

Obviously, ρ is a convex modular in $L^l_{\varphi,\psi}(T)$. Let $\| \cdot \|_l$ denote the Luxemburg norm in $L^l_{\varphi,\psi}(T)$. The space $L^l_{\varphi,\psi}(T)$ equipped with this norm is a Banach space.

Suppose that the l indices $\alpha = (\alpha_1, \alpha_2)$ satisfying $\mid \alpha \mid\leq k$ are linearly ordered in a convenient fashion so that with each $f \in W^k_{\varphi,\psi}(T)$ we may associate a well-defined vector Pf in $L^l_{\varphi,\psi}(T)$given by

$$Pf = (D^\alpha f)_{|\alpha|\leq k}.$$

We have $\| f \|^{(k)}_{\varphi,\psi} = \| Pf \|_l$ for any $f \in W^k_{\varphi,\psi}(T)$. So P is an isometric isomorphism of $W^k_{\varphi,\psi}(T)$ onto subspace of $L^l_{\varphi,\psi}(T)$.

Theorem 6. *The space* $W^{(k)}_{\varphi,\psi}$ *is separable in the sense of the modular* $I^{(k)}_{\varphi,\psi}$.

Proof. The space $L^l_{\varphi,\psi}(T)$ is separable in the sense of ρ because $L_{\varphi,\psi}(T)$ is separable in the sense of $I_{\varphi,\psi}$. The operator P is an isometric isomorphism of $W^k_{\varphi,\psi}(T)$ onto $W = P\left(W^k_{\varphi,\psi}\right) \subset L^l_{\varphi,\psi}$. Since $W^k_{\varphi,\psi}(T)$ is complete, $P\left(W^k_{\varphi,\psi}\right)$ is a closed subspace of $L^l_{\varphi,\psi}(T)$. Thus $P\left(W^k_{\varphi,\psi}\right)$ is separable in the sense of ρ, and hence $W^k_{\varphi,\psi}(T)$ is separable in the sense of $I^{(k)}_{\varphi,\psi}$. ■

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