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SOME PROPERTIES OF T-METRIC SPACES AND A COMMON FIXED POINT THEOREM

Abstract. In this paper, we introduce the new definitions of T-metric space and give some properties of it. Also, we prove a common fixed point theorem for for four mappings under the condition of weakly compatible in complete T-metric spaces. A lot of fixed point theorems on ordinary metric space are special case of our main result, since every ordinary metric space is also T-metric space.

Key words: T-metric space, contractive mapping, fixed point.

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1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, many authors have extended, generalized and improved this theorem in different ways (See for details, [1], [4], [5], [6]). Also recently, fixed and common fixed point results in different types of spaces have been developed. For example, ultra metric spaces [14], fuzzy metric spaces [9] and uniform spaces [13]. In this paper we introduce the new definitions of T-metric space and give some properties of it. After then, we prove a common fixed point theorem for four mappings under the condition of weakly compatible in complete T-metric spaces. We begin this paper by giving the definition of ultra metric space.

Definition 1 ([14]). *Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:*

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} \quad \forall x, y, z \in X$$

then d is called an ultra metric on X and the pair (X, d) is called an ultra metric space. An ultra metric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection.

Rao and Kishore [11] proved the following:

Theorem 1. *Let (X, d) be a spherically complete ultra metric space. If f and F are self maps on X satisfying $F(X) \subseteq f(X)$,*

$$d(Fx, Fy) < \max\{d(fx, fy), d(fx, Fx), d(fy, Fy)\} \quad \forall x, y \in X,\ x \neq y$$

then there exists $z \in X$ such that $fz = Fz$. Further if f and F are coincidentally commuting at z then z is the unique common fixed point of f and F.

In the following, we introduce a new binary operation which is a probable modification of the definition of ordinary metric. In Section 2, we give the definition of T-metric and some properties of it. In Section 3, we prove a common fixed point theorem for four weakly compatible maps in complete T-metric spaces satisfying a new contractive type condition.

2. T-metric spaces

In what follows, $\mathbb{N}$ is the set of all natural numbers and $\mathbb{R}^+$ is the set of all nonnegative real numbers.

Let $\diamond : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a binary operation satisfying the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a$ for all $a \in \mathbb{R}^+$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in \mathbb{R}^+$.

Five typical examples of $\diamond$ are:

$a \diamond_1 b = \max\{a, b\}$, $a \diamond_2 b = \sqrt{a^2 + b^2}$, $a \diamond_3 b = a + b$, $a \diamond_4 b = ab + a + b$ and $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$ for each $a, b \in \mathbb{R}^+$. It is easy to see that:

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b \leq \min\{a \diamond_4 b, a \diamond_5 b\}.$$

Lemma 1. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous, onto and increasing map. If defined $a \diamond b = f^{-1}(f(a) + f(b))$ for every $a, b \in \mathbb{R}^+$, then $\diamond$ is a binary operation.*

Proof. It is easy to see that $\diamond$ is an increasing in both items, commutative, associative and continuous satisfying $a \diamond 0 = f^{-1}(f(a) + f(0)) = f^{-1}(f(a)) = a$ for all $a \in [0, \infty)$. ∎

Example 1. If function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $f(x) = e^x - 1$, then it is easy to see that f is a continuous, onto and increasing function. Also, for every $a, b \in \mathbb{R}^+$ we have $a \diamond b = \ln(e^a + e^b - 1)$ is a binary operation.

Lemma 2. *Let $\diamond$ be a binary operation on $\mathbb{R}^+$ satisfying the conditions (i)-(iv).*
(a) *If $r, r' \geq 0$, then $\max\{r, r'\} \leq r \diamond r'$.*
(b) *If $0 < \delta < r$, then there exist a $0 < \delta' < r$ such that $\delta' \diamond \delta < r$.*
(c) *For every $\epsilon > 0$ there exist a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$.*

Proof. (a) Since $r' \geq 0$ by properties (iii) and (iv) of binary operation $\diamond$ we have $r \diamond r' \geq r \diamond 0 = r$. Similarly we have $r \diamond r' \geq r'$.

(b) Let $0 < \delta < r$. Suppose for every $\delta' > 0$ we have $\delta' \diamond \delta \geq r$. In particular if take $\delta' = \frac{1}{n}$ then we have $\frac{1}{n} \diamond \delta \geq r$. Thus, this implies that $0 \diamond \delta \geq r$ as $n \to \infty$, which is a contradiction. Hence by part (i) of this lemma we get $\delta' \leq \delta' \diamond \delta < r$.

(c) Let $\epsilon > 0$. Suppose for every $\delta > 0$, we have $\delta \diamond \delta \geq \epsilon$. For $\delta = \frac{1}{n}$ we have $\frac{1}{n} \diamond \frac{1}{n} \geq \epsilon$, hence as $n \to \infty$ we get $0 \geq \epsilon$, which is a contradiction. ■

Now we introduce the new concept of T-metric.

Definition 2. *Let X be a nonempty set. A T-metric on X is a function $T : X^2 \to \mathbb{R}$ that satisfies the following conditions: for each $x, y, z \in X$*
(a) *$T(x, y) \geq 0$ and$T(x, y) = 0$ if and only if $x = y$,*
(b) *$T(x, y) = T(y, x)$,*
(c) *$T(x, y) \leq T(x, z) \diamond T(y, z)$.*
The 3-tuple $(X, T, \diamond)$ is called a T-metric space.

Example 2. (i) Every ordinary metric d is a T-metric with $a \diamond b = a + b$.
(ii) Every ultra metric d is a T-metric with $a \diamond b = \max\{a, b\}$.
(iii) Let $X = \mathbb{R}$ and $T(x, y) = \sqrt{|x - y|}$ for every $x, y \in \mathbb{R}$. If we take $a \diamond b = \sqrt{a^2 + b^2}$, then we have

$$\begin{aligned} T(x, y) &= \sqrt{|x - y|} \\ &\leq \sqrt{|x - z| + |z - y|} \\ &= \sqrt{\sqrt{|x - z|}^2 + \sqrt{|z - y|}^2} \\ &= T(x, z) \diamond T(z, y). \end{aligned}$$

Therefore the function T is a T-metric on X.

(iv) Let $X = \mathbb{R}$ and $T(x, y) = (x - y)^2$ for every $x, y \in \mathbb{R}$. If we take $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, then we have

$$\begin{aligned} T(x, y) &= (x - y)^2 = |x - y|^2 \\ &\leq (|x - z| + |z - y|)^2 \\ &= (\sqrt{|x - z|^2} + \sqrt{|z - y|^2})^2 \\ &= T(x, z) \diamond T(z, y). \end{aligned}$$

Therefore the function T is a T-metric on X.

Remark 1. For fixed $0 \le \alpha \le \frac{\pi}{4}$ if there exist β, γ such that

$$0 \le \alpha \le \beta + \gamma < \frac{\pi}{2},$$

then
$$\tan \alpha \le \tan \beta + \tan \gamma + \tan \beta \tan \gamma.$$

Example 3. Let $X = [0,1]$ and $T(x,y) = \tan(\frac{\pi}{4}|x-y|)$ for every $x, y \in X$. If we take $a \diamond b = a + b + ab$, then by Remark 1 we have

$$\begin{aligned} T(x,y) &= \tan(\frac{\pi}{4}|x-y|) \\ &\le \tan(\frac{\pi}{4}|x-z|) + \tan(\frac{\pi}{4}|z-y|) + \tan(\frac{\pi}{4}|x-z|)\tan(\frac{\pi}{4}|z-y|) \\ &= T(x,z) \diamond T(z,y). \end{aligned}$$

Therefore the function T is a T-metric on X.

Let $(X, T, \diamond)$ be a T-metric space. For $r > 0$ define

$$B_T(x,r) = \{y \in X : T(x,y) < r\}.$$

Definition 3. *Let $(X, T, \diamond)$ be a T-metric space $r > 0$ and $A \subset X$.*

(*a*) *The set $B_T(x,r) = \{y \in X : T(x,y) < r\}$ is called an open ball centered at x and radius r.*

(*b*) *If for every $x \in A$ there exists $r > 0$ such that $B_T(x,r) \subset A$, then the subset A is called open subset of X.*

(*c*) *The subset A of X is said to be T-bounded if there exists $r > 0$ such that $T(x,y) < r$ for all $x, y \in A$.*

(*d*) *A sequence $\{x_n\}$ in X converges to x if $T(x_n, x) \to 0$ as $n \to \infty$ and write $\lim_{n\to\infty} x_n = x$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x) < \epsilon$ for all $n \ge n_0$, then $\{x_n\}$ converges to x.*

(*e*) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $T(x_n, x_m) < \epsilon$ for all $n, m \ge n_0$.*

(*f*) *The T-metric space $(X, T, \diamond)$ is said to be complete if every Cauchy sequence is convergent.*

Let τ be the set of all open subset of X, then τ is a topology on X (induced by the T-metric T).

Lemma 3. *Let $(X, T, \diamond)$ be a T-metric space. If $r > 0$, then the open ball $B_T(x,r)$ with center $x \in X$ and radius r is an open set.*

Proof. Let $y \in B_T(x,r)$, hence $T(x,y) < r$. If we set $T(x,y) = \delta$ then by Lemma 2 there exists $\delta' > 0$ such that $\delta' \diamond \delta < r$. Now, we prove that $B_T(y, \delta') \subseteq B_T(x,r)$. Let $z \in B_T(y, \delta')$, then by triangular inequality we have

$$T(x,z) \le T(x,y) \diamond T(y,z) < \delta \diamond \delta' < r.$$

Hence $B_T(y,\delta') \subseteq B_T(x,r)$. That is $B_T(x,r)$ is an open set. ■

Lemma 4. *Let $(X,T,\diamond)$ be a T-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $x_n \to y$. For every $\epsilon > 0$ by Lemma 2 we can choose a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Now, since $\{x_n\}$ converges to x and y, for this δ there exists $n_1 \in \mathbb{N}$ such that $T(x_n,x) < \delta$ for all $n \geq n_1$ and there exists $n_2 \in \mathbb{N}$ such that $T(x_n,y) < \delta$ for all $n \geq n_2$. If set $n_0 = \max\{n_1,n_2\}$, then for all $n \geq n_0$ by triangular inequality we have

$$T(x,y) \leq T(x,x_n) \diamond T(x_n,y) < \delta \diamond \delta < \epsilon.$$

Hence $T(x,y) = 0$ and so $x = y$. ■

Lemma 5. *Let $(X,T,\diamond)$ be a T-metric space. Then every convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Proof. For every $\epsilon > 0$ by Lemma 2 we can choose a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Since $x_n \to x$ there exists $n_0 \in \mathbb{N}$ such that $T(x_n,x) < \delta$ for all $n \geq n_0$. Thus for all $n,m \geq n_0$ by triangular inequality we have

$$T(x_n,x_m) \leq T(x_n,x) \diamond T(x,x_m) < \delta \diamond \delta < \epsilon.$$

Hence sequence $\{x_n\}$ is a Cauchy sequence. ■

Definition 4. *Let $(X,T,\diamond)$ be a T-metric space. T is said to be continuous, if*

$$\lim_{n\to\infty} T(x_n,y_n) = T(x,y),$$

whenever

$$\lim_{n\to\infty} T(x_n,x) = \lim_{n\to\infty} T(y_n,y) = 0.$$

Lemma 6. *Let $(X,T,\diamond)$ be a T-metric space. Then T is a continuous function.*

Proof. Let $\lim_{n\to\infty} T(x_n,x) = \lim_{n\to\infty} T(y_n,y) = 0$, then by triangular inequality we have

$$T(x_n,y_n) \leq T(x_n,x) \diamond T(x,y) \diamond T(y,y_n).$$

Hence we have

$$\lim_{n\to\infty} \sup T(x_n,y_n) \leq T(x,y).$$

Similarly, we have

$$T(x,y) \leq T(x,x_n) \diamond T(x_n,y_n) \diamond T(y_n,y)$$

and so

$$T(x,y) \leq \lim_{n\to\infty} \inf T(x_n, y_n).$$

Therefore we have

$$\lim_{n\to\infty} T(x_n, y_n) = T(x,y).$$

■

3. Fixed point result

In this section we give some fixed point results on T-metric spaces. In these results we use an implicit relation for contractive condition. Implicit relation technique on metric space have been used in many articles (See [2], [3], [7], [10], [12]).

Definition 5. *Let $\mathbb{R}^+$ be the set of all non-negative real numbers and let $\mathcal{H}$ be the set of all continuous functions $H : (\mathbb{R}^+)^5 \to \mathbb{R}$ satisfying the following conditions:*

H_1 : *$H(t_1, \cdots, t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, \cdots, t_5$.*

H_2 : *there exists $h \in (0,1)$ such that*

$$H(u,v,v,u,v \diamond u) \leq 0 \quad or \quad H(u,v,u,v,v \diamond u) \leq 0$$

implies $u \leq hv$.

H_3 : *$H(u,0,0,u,u) > 0, H(u,0,u,0,u) > 0$ and $H(u,u,0,0,u \diamond u) > 0$, for all $u > 0$.*

Now, we give some examples.

Example 4. Let $a \diamond b = a + b$ for all $a, b \in [0,\infty)$ and $H(t_1, \cdots, t_5) = t_1 - \alpha \max\{t_2, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$.

H_1 : Obvious.

H_2 : Let $u > 0$ and

$$\begin{aligned} H(u,v,v,u,v \diamond u) &= H(u,v,v,u,v+u) \\ &= u - \alpha \max\{u,v\} - \beta(v+u) \leq 0. \end{aligned}$$

Thus $u \leq \max\{(\alpha+\beta)u + \beta v, (\alpha+\beta)v + \beta u\}$. Now if $u \geq v$, then $u \leq (\alpha+\beta)u+\beta v \leq (\alpha+2\beta)u$, a contradiction. Thus $u < v$ and $u \leq (\alpha+\beta)v+\beta u$ and so $u \leq \frac{\alpha+\beta}{1-\beta}v$. Similarly, let $u > 0$ and

$$\begin{aligned} H(u,v,u,v,v \diamond u) &= H(u,v,u,v,v+u) \\ &= u - \alpha \max\{u,v\} - \beta(v+u) \leq 0, \end{aligned}$$

then we have $u \leq \frac{\alpha+\beta}{1-\beta}v$. If $u = 0$, then $u \leq \frac{\alpha+\beta}{1-\beta}v$. Thus H_2 is satisfying with $h = \frac{\alpha+\beta}{1-\beta} < 1$.

$H_3: H(u,0,0,u,u) = H(u,0,u,0,u) = u(1-\alpha-\beta) > 0$ and

$$H(u,u,0,0,u\diamond u) = H(u,u,0,0,u+u) = u(1-\alpha-2\beta) > 0,$$

for all $u > 0$. Therefore $H \in \mathcal{H}$.

Example 5. Let $a\diamond b = (\sqrt{a}+\sqrt{b})^2$ for all $a, b \in [0,\infty)$ and $H(t_1,\cdots,t_5) = t_1 - m\max\{t_2,t_3,t_4,\frac{1}{4}t_5\}$, where $0 \le m < 1$.

H_1 : Obvious.

H_2 : Let $u > 0$ and

$$\begin{aligned} H(u,v,v,u,v\diamond u) &= H(u,v,v,u,\frac{1}{4}(\sqrt{v}+\sqrt{u})^2) \\ &= u - m\max\{u,v,\frac{1}{4}(\sqrt{v}+\sqrt{u})^2\} \le 0. \end{aligned}$$

Thus $u \le m\max\{u,v\}$. Now if $u \ge v$, then $u \le mu$, a contradiction. Thus $u < v$ and $u \le mv$. Similarly, let $u > 0$ and

$$H(u,v,u,v,v\diamond u) = H(u,v,u,v,\frac{1}{4}(\sqrt{v}+\sqrt{u})^2) = u - m\max\{u,v\} \le 0,$$

then we have $u \le mv$. If $u = 0$, then $u \le mv$. Thus H_2 is satisfying with $h = m < 1$.

$H_3: H(u,0,0,u,u) = H(u,0,u,0,u) = H(u,u,0,0,u\diamond u) = u(1-m) > 0$, for all $u > 0$. Therefore $H \in \mathcal{H}$.

Lemma 7. *Let $(X,T,\diamond)$ be T-metric space with $a\diamond b \le (\sqrt{a}+\sqrt{b})^2$. If for all $n \in \mathbb{N}$*

$$T(x_{n+1},x_n) \le kT(x_n,x_{n-1})$$

for $0 < k < 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For all $n \in \mathbb{N}$, we have

$$T(x_{n+1},x_n) \le kT(x_n,x_{n-1}) \le \cdots \le k^nT(x_1,x_0).$$

Thus for $m > n$ we have

$$\begin{aligned} T(x_n,x_m) &\le T(x_n,x_{n+1})\diamond T(x_{n+1},x_{n+2})\diamond\cdots\diamond T(x_{m-1},x_m) \\ &\le (\sqrt{T(x_n,x_{n+1})}+\sqrt{T(x_{n+1},x_{n+2})}+\cdots+\sqrt{T(x_{m-1},x_m)})^2 \\ &\le (k^{\frac{n}{2}}\sqrt{T(x_1,x_0)}+k^{\frac{n+1}{2}}\sqrt{T(x_1,x_0)}+\cdots+k^{\frac{m-1}{2}}\sqrt{T(x_1,x_0)})^2 \\ &\le (\sum_{j=n}^{m-1}k^{\frac{j}{2}})^2T(x_1,x_0) = (\frac{k^{\frac{n}{2}}-k^{\frac{m}{2}}}{1-\sqrt{k}})^2T(x_1,x_0) \\ &\le (\frac{k^{\frac{n}{2}}}{1-\sqrt{k}})^2T(x_1,x_0). \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence. ■

In 1998, Jungck and Rhoades [8] introduced the following concept of weak compatibility.

Definition 6. *Let f and F be mappings from a T-metric space $(X, T, \diamond)$ into itself. Then the pair (F, f) is said to be weak compatible if f and F commute at their coincidence points, that is, $fx = Fx$ implies that $fFx = Ffx$.*

Theorem 2. *Let $(X, T, \diamond)$ be a complete T-metric space where $a \diamond b \leq (\sqrt{a} + \sqrt{b})^2$. Let F, G, f and g be four self-mappings of X satisfying the following conditions:*

(i) $F(X) \subseteq g(X)$, $G(X) \subseteq f(X)$ and $f(X)$ or $g(X)$ is a closed subset of X ,

(ii) the pairs (F, f) and (G, g) are weakly compatible,

(iii) there exists $H \in \mathcal{H}$ such that

$$H(T(Fx, Gy), T(fx, gy), T(fx, Fx), T(gy, Gy), \\ T(fx, Gy) \diamond T(gy, Fx)) \leq 0$$

for all x, y in X,

Then there exists a unique $p \in X$ such that $p = fp = gp = Fp = Gp$.

Proof. Let x_0 be an arbitrary point in X. By (i), we choose a point x_1 in X such that $y_0 = gx_1 = Fx_0$. For this point x_1 there exists a point x_2 in X such that $y_1 = fx_2 = Gx_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows

$$y_{2n} = gx_{2n+1} = Fx_{2n}, \qquad y_{2n+1} = fx_{2n+2} = Gx_{2n+1},$$

for $n = 0, 1, 2, \cdots$. We prove that $\{y_n\}$ is a Cauchy sequence. From (iii), we have

$$H\left(T(Fx_{2n}, Gx_{2n+1}), T(fx_{2n}, gx_{2n+1}), T(fx_{2n}, Fx_{2n}), \\ T(gx_{2n+1}, Gx_{2n+1}), T(fx_{2n}, Gx_{2n+1}) \diamond T(gx_{2n+1}, Fx_{2n})\right) \leq 0.$$

Thus we get

$$H\left(T(y_{2n}, y_{2n+1}), T(y_{2n-1}, y_{2n}), T(y_{2n-1}, y_{2n}), T(y_{2n}, y_{2n+1}), \\ T(y_{2n-1}, y_{2n+1}) \diamond T(y_{2n}, y_{2n})\right) \leq 0.$$

Using H_1 we get

$$H\left(T(y_{2n}, y_{2n+1}), T(y_{2n-1}, y_{2n}), T(y_{2n-1}, y_{2n}), T(y_{2n}, y_{2n+1}), \\ T(y_{2n-1}, y_{2n}) \diamond T(y_{2n}, y_{2n+1})\right) \leq 0.$$

That is

$$H(u, v, v, u, v \diamond u) \le 0,$$

where $u = T(y_{2n}, y_{2n+1})$ and $v = T(y_{2n-1}, y_{2n})$. Hence, from H_2,, there exists $h \in (0, 1)$ such that

$$T(y_{2n}, y_{2n+1}) \le hT(y_{2n-1}, y_{2n}).$$

Similarly, from (iii), we have

$$H\left(T(Fx_{2n+2}, Gx_{2n+1}), T(fx_{2n+2}, gx_{2n+1}), T(fx_{2n+2}, Fx_{2n+2}),\right.$$
$$\left. T(gx_{2n+1}, Gx_{2n+1}), T(fx_{2n+2}, Gx_{2n+1}) \diamond T(gx_{2n+1}, Fx_{2n+2})\right) \le 0.$$

Thus we have

$$H\left(T(y_{2n+2}, y_{2n+1}), T(y_{2n+1}, y_{2n}), T(y_{2n+1}, y_{2n+2}), T(y_{2n}, y_{2n+1}),\right.$$
$$\left. T(y_{2n+1}, y_{2n+1}) \diamond T(y_{2n}, y_{2n+2})\right) \le 0.$$

Using H_1 we have

$$H\left(T(y_{2n+2}, y_{2n+1}), T(y_{2n+1}, y_{2n}), T(y_{2n+1}, y_{2n+2}), T(y_{2n}, y_{2n+1}),\right.$$
$$\left. T(y_{2n}, y_{2n+1}) \diamond T(y_{2n+1}, y_{2n+2})\right) \le 0.$$

That is

$$H(u, v, u, v, v \diamond u) \le 0,$$

where $u = T(y_{2n+2}, y_{2n+1})$ and $v = T(y_{2n+1}, y_{2n})$. Hence, from H_2, we have

$$T(y_{2n+2}, y_{2n+1}) \le hT(y_{2n+1}, y_{2n}).$$

Therefore, we obtain

$$T(y_n, y_{n+1}) \le hT(y_{n-1}, y_n)$$

for all $n = 0, 1, \cdots$. Hence by Lemma 2 the sequence $\{y_n\}$ is Cauchy in X. By completeness X there exist $p \in X$ such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Fx_{2n} = p,$$

and

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = \lim_{n\to\infty} Gx_{2n+1} = p.$$

Suppose that $g(X)$ is closed, then for some $v \in X$ we have $p = gv \in g(X)$. Putting $x = x_{2n}$, $y = v$ in (iii), we get

$$H\left(T(Fx_{2n}, Gv), T(fx_{2n}, gv), T(fx_{2n}, Fx_{2n}), T(gv, Gv),\right.$$
$$\left. T(fx_{2n}, Gv) \diamond T(gv, Fx_{2n})\right) \le 0.$$

Thus, we have

$$H\,(T(y_{2n},Gv),T(y_{2n-1},gv),T(y_{2n-1},y_{2n}),T(gv,Gv),\\ T(y_{2n-1},Gv)\diamond T(gv,y_{2n}))\leq 0.$$

On making $n\to\infty$, we have

$$H(T(p,Gv),T(p,gv),T(p,p),T(p,Gv),T(p,Gv)\diamond T(p,p))\leq 0.$$

Thus we get,

$$H(T(p,Gv),0,0,T(p,Gv),T(p,Gv))\leq 0.$$

That is, $H(u,0,0,u,u)\leq 0$, hence from H_3, we get $u=T(p,Gv)=0$. Hence $Gv=p=gv$. From weak compatibility of (G,g), we have $Ggv=gGv$, hence $Gp=gp$. Putting $x=x_{2n}$, $y=p$ in (iii), we get

$$H\,(T(Fx_{2n},Gp),T(fx_{2n},gp),T(fx_{2n},Fx_{2n}),T(gp,Gp),\\ T(fx_{2n},Gp)\diamond T(gp,Fx_{2n}))\leq 0.$$

Thus, we have

$$H\,(T(y_{2n},gp),T(y_{2n-1},gp),T(y_{2n-1},y_{2n}),T(gp,gp),\\ T(y_{2n-1},gp)\diamond T(gp,y_{2n}))\leq 0.$$

On making $n\to\infty$, we get

$$H(T(p,gp),T(p,gp),T(p,p),T(gp,gp),T(p,gp)\diamond T(gp,p))\leq 0.$$

That is, $H(u,u,0,0,u\diamond u)\leq 0$, hence from H_3, we have $u=T(p,gp)=0$. Hence $gp=p$. Therefore, $Gp=p$. Since $Gp\in f(X)$, then there exists $w\in X$ such that $fw=Gp=gp=p$. Now putting $x=w$, $y=p$ in (iii), we get

$$H\,(T(Fw,Gp),T(fw,gp),T(fw,Fw),T(gp,Gp),\\ T(fw,Gp)\diamond T(gp,Fw))\leq 0.$$

and so we have

$$H(T(Fw,p),0,T(p,Fw),0,T(p,Fw))\leq 0.$$

That is, $H(u,0,u,0,u)\leq 0$, hence from H_3, we have $u=T(Fw,p)=0$. Hence $Fw=p=Gp=fw=gp$. Since $Fw=fw$ and the pair (F,f) is weakly compatible, then we obtain $Fp=Ffw=fFw=fp$. Therefore, we obtain $Fp=Gp=fp=gp=p$.

The proof is similar when $f(X)$ is assumed to be closed subset of X.

To see the p is unique, suppose that $q = gq = fq = Fq = Gq$, then from (iii) we have

$$H(T(Fp, Gq), T(fp, gq), T(fp, Fp), T(gq, Gq), T(fp, Gq) \diamond T(gq, Fp)) \leq 0,$$

therefore $H(T(p, q), T(p, q), 0, 0, T(p, q) \diamond T(p, q) \leq 0$, that is $T(p, q) = 0$. It follows that $p = q$. ■

Corollary 1. *Let F, G be self-mappings of a complete T-metric space $(X, T, \diamond)$ where $a \diamond b \leq (\sqrt{a} + \sqrt{b})^2$, such that satisfying:*

(iv) $\quad H(T(Fx, Gy), T(x, y), T(x, Fx), T(y, Gy), T(x, Gy) \diamond T(y, Fx)) \leq 0$

for every x, y in X, where $H \in \mathcal{H}$. Then there exists a unique $p \in X$ such that $p = Fp = Gp$.

Proof. Follows from Theorem 2 with $f = g = I$ dentity mapping. ■

Corollary 2. *Let F, G, f and g be four self-mappings of a complete T-metric space $(X, T, \diamond)$ where $a \diamond b \leq (\sqrt{a} + \sqrt{b})^2$, satisfying:*

(i) $Fx \subseteq g(X)$, $Gx \subseteq f(X)$ for every $x \in X$,

(ii) The pair (F, f) and (G, g) are weakly compatible,

(iii) $T(Fx, Gy) \leq m \max\{T(fx, gy), T(fx, Fx), T(gy, Gy)\}$
$\qquad \times \frac{1}{4}(T(fx, Gy) \diamond T(gy, Fx))$

for every x, y in X, where $0 \leq m < 1$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of X, then there exists a unique $p \in X$ such that $p = fp = gp = Fp = Gp$.

Proof. Follows from Theorem 2 with

$$H(t_1, t_2, t_3, t_4, t_5) = t_1 - m \max\{t_2, t_3, t_4, \frac{1}{4} t_5\}.$$

■

If we combine Theorem 2 with Example 4 we have the following corollary.

Corollary 3. *Let F, G, f and g be four self-mappings of a complete T-metric space $(X, T, \diamond)$ where $a \diamond b = a + b$, satisfying:*

(i) $Fx \subseteq g(X)$, $Gx \subseteq f(X)$ for every $x \in X$,

(ii) The pair (F, f) and (G, g) are weakly compatible,

(iii) $\quad T(Fx, Gy) \leq \alpha \max\{T(fx, gy), T(fx, Fx), T(gy, Gy)\}$
$\qquad + \beta(T(fx, Gy) + T(gy, Fx))$

for every x, y in X, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$. Suppose that one of $g(X)$ or $f(X)$ is a closed subset of X, then there exists a unique $p \in X$ such that $p = fp = gp = Fp = Gp$.

Example 6. Let $X = [0,1]$, $T(x,y) = \sqrt{|x-y|}$ and $a \diamond b = (\sqrt{a}+\sqrt{b})^2$, then $(X, T, \diamond)$ is a complete T-metric space. Define $F, G, f, g : X \to X$ as follows:

$$Fx = \frac{1}{2}, \qquad Gx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ \frac{3}{7}, & x \in (\frac{1}{2}, 1] \end{cases},$$

$$fx = \begin{cases} \frac{1}{2}, & x \in [0, \frac{1}{2}] \\ \frac{x+1}{4}, & x \in (\frac{1}{2}, 1] \end{cases}, \quad gx = \begin{cases} 1-x, & x \in [0, \frac{1}{2}] \\ 0, & x \in (\frac{1}{2}, 1] \end{cases}.$$

It is clear that $F(X) = \{\frac{1}{2}\} \subseteq g(X) = \{0\} \cup [\frac{1}{2}, 1]$, $G(X) = \{\frac{3}{7}, \frac{1}{2}\} \subseteq f(X) = (\frac{3}{8}, \frac{1}{2}]$ and $g(X)$ is closed subset of X. Now we consider the following cases:

Case 1. If $x \in [0, \frac{1}{2}]$ and $y \in [0, \frac{1}{2}]$, then

$$T(Fx, Gy) = 0 \leq \sqrt{\frac{4}{21}} T(fx, gy).$$

Case 2. If $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, then

$$T(Fx, Gy) = \sqrt{\left|\frac{1}{2} - \frac{3}{7}\right|} = \sqrt{\frac{1}{14}} \leq \sqrt{\frac{4}{21}}\sqrt{\frac{1}{2}} = \sqrt{\frac{4}{21}} T(fx, gy).$$

Case 3. If $x \in (\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$, then

$$T(Fx, Gy) = 0 \leq \sqrt{\frac{4}{21}} T(fx, gy).$$

Case 4. If $x \in (\frac{1}{2}, 1]$ and $y \in (\frac{1}{2}, 1]$, then

$$\begin{aligned} T(Fx, Gy) &= \sqrt{\left|\frac{1}{2} - \frac{3}{7}\right|} = \sqrt{\frac{1}{14}} = \sqrt{\frac{4}{21}}\sqrt{\frac{3}{8}} \\ &\leq \sqrt{\frac{4}{21}}\sqrt{\left|\frac{x+1}{4}\right|} \leq \sqrt{\frac{4}{21}} T(fx, gy). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} T(Fx,Gy) &\leq \sqrt{\frac{4}{21}}T(fx,gy) \\ &\leq \sqrt{\frac{4}{21}}\max\Big\{T(fx,gy),T(fx,Fx),T(gy,Gy), \\ &\qquad\qquad \frac{1}{4}(T(fx,Gy)\diamond T(gy,Fx))\Big\} \end{aligned}$$

for all $x,y\in X$. That is, the condition (iii) of Theorem 2 is satisfied with

$$H(t_1,\cdots,t_5)=t_1-\sqrt{\frac{4}{21}}\max\{t_2,t_3,t_4,\frac{1}{4}t_5\}.$$

Also, the coincidence points of F and f are $\frac{1}{2}$ and 1, and it is clear that F and f are commuting at $\frac{1}{2}$ and 1. Similarly, the only coincidence point of G and g is $\frac{1}{2}$, and G and g are commuting at $\frac{1}{2}$. Thus F and f as well as G and g are weakly compatible. Consequently all conditions of Theorem 2 are satisfied and so these mappings have a unique common fixed point on X.

Remark 2. We can obtain several fixed point results on ordinary metric spaces as a special case of Theorem 2.

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