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OSCILLATION PROPERTIES OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

Oscillatory and asymptotic behaviour of the solutions of a class of nonlinear first order neutral delay differential equations with positive and negative coefficients of the form

$$(E_1) \qquad \frac{d}{dt}[y(t) + p(t)y(t-\tau)] + f_1(t)G_1(y(t-\sigma_1)) - f_2(t)G_2(y(t-\sigma_2)) = g(t)$$

and

$$(E_2) \qquad \frac{d}{dt}[y(t) + p(t)y(t-\tau)] + f_1(t)G_1(y(t-\sigma_1)) - f_2(t)G_2(y(t-\sigma_2)) = 0$$

are studied under various ranges of $p(t)$. Sufficient conditions are obtained for the existence of positive bounded solution of (E_1).

Key words: oscillation, nonoscillation, neutral differential equation, asymptotic behaviour.

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1. Introduction

Consider the following nonlinear delay differential equation

$$(1) \qquad \frac{d}{dt}[y(t) + p(t)y(t-\tau)] + f_1(t)G_1(y(t-\sigma_1)) - f_2(t)G_2(y(t-\sigma_2)) = g(t)$$

where $G_i \in C(R, R)$ with $xG_i(x) > 0$, $x \neq 0$ for $i = 1, 2$, G_i is nondecreasing, $p, g \in C([0,\infty), R)$, $f_i \in C([0,\infty), [0,\infty))$, $i = 1, 2$ and $\tau > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ are constants.

Recently, there has been an increasing interest in the study of the oscillatory and asymptotic behaviour of solutions of the following special form of Eq. (1)

$$\frac{d}{dt}[y(t) - R(t)y(t-r)] + P(t)y(t-\tau_1) - Q(t)y(t-\sigma_2) = g(t) \tag{2}$$

for $t \geq 0$, where $P, Q, R \in C([t_0, \infty), R^+)$, $g \in C([t_0, \infty), R)$, $r \in (0, \infty)$, $\tau, \sigma \in R^+$ and $\tau \geq \sigma$. See for example ([1], [3], [5] - [12]) and the references cited there in. In [2], [7] - [11], authors have discussed the oscillation properties of Eq. (2) with $\sigma_1 \geq \sigma_2$ or $\sigma_1 \leq \sigma_2$ and $R(t) \geq 0$. The following example

$$\begin{aligned} \frac{d}{dt}[y(t) + e^{-\pi}(1 + e^{-t})y(t-\tau)] &+ e^{(t-6\pi)}y^3(t-2\pi) \\ &- 2^{(t+20\pi)}y^5(t-4\pi)) = g(t), \end{aligned} \tag{3}$$

where $g(t) = (2; \sin t + \sin 3t - \cos t)e^{-2t} - e^{-6t} \ \sin^5 t$ suggests that the above works can not be applied to (3) which has an oscillatory solution $y(t) = e^{-t}\sin t$. Hence it seems that Eq. (1) may admit oscillatory solutions.

The object of this work is to study the oscillatory behaviour of solutions of Eq. (1) under various ranges of $p(t)$. Its associated homogeneous equation

$$\frac{d}{dt}[y(t)+p(t)y(t-\tau)]+f_1(t)G_1(y(t-\sigma_1))-f_2(t)G_2(y(t-\sigma_2)) = 0, \ t \geq 0. \tag{4}$$

$t \geq 0$ is also considered, where every solution or every bounded solution oscillates or tends to zero as $t \to \infty$. Unlike the work in [7], [8] and [10], an attempt is made here to establish sufficient conditions under which every solution/every bounded solution of Eq. (1)/Eq. (4) oscillates/oscillates or tends to zero as $t \to \infty$. Of course, the impact of forcing term is considered. Keeping in view of the influence of forcing functions, this work is separated for forced and unforced equations.

By a solution of Eq. (1)/Eq. (4) we understand a function $y \in C([-\rho, \infty), R)$ such that $(y(t)+p(t)y(t-\tau))$ is once continuously differentiable and (1) or (4) is satisfied for $t \geq 0$, where $\rho = \max\{\tau, \sigma_1, \sigma_2\}$ and $\sup\{|y(t)| : t \geq t_0\} > 0$ for every $t_0 \geq 0$. A solution of Eq. (1)/Eq. (4) is said to be oscillatory if it has arbitary large zeros; otherwise it is called nonoscillatory.

2. Oscillation properties of Eq. (1)

Sufficient conditions are obtained for oscillation of solutions of the Eq. (1). We need the following conditions for our use in the sequel:

(H_1) $\quad \int_0^\infty f_1(t)dt = \infty, \qquad \int_0^\infty f_2(t)dt < \infty;$

(H_2) There exists $\lambda > 0$ such that $G_1(u) + G_1(v) \geq \lambda G_1(u+v)$ for $u > 0$, $v > 0$;

(H_3) $G_1(u)G_1(v) \geq G_1(uv)$ for u and $v \in R$;

(H_4) $G_i(-u) = -G_i(u), \ \ u \in R, \ \ i = 1, 2;$

(H_5) There exists $F \in C([0,\infty), R)$ such that F(t) changes sign with $-\infty < \liminf\limits_{t\to\infty} F(t) < 0 < \limsup\limits_{t\to\infty} F(t) < \infty$ and $F'(t) = g(t)$;

(H_6) There exists $F \in C([0,\infty), R)$ such that F(t) changes sign with $\liminf\limits_{t\to\infty} F(t) = -\infty, \limsup\limits_{t\to\infty} F(t) = \infty$ and $F'(t) = g(t)$;

(H_7) $F^+(t) = \max\{F(t), 0\}$, and $F^-(t) = \max\{-F(t), 0\}$;

(H_8) $\int\limits_T^\infty Q(t)|G_1(F^+(t-\sigma_1) - \alpha)|dt = \infty,$

$$\int\limits_T^\infty Q(t)|G_1(F^-(t-\sigma_1) - \alpha)|dt = \infty,$$

where $Q(t) = \min\{f_1(t), \ f_1(t-\tau)\}, \ \ t \geq \tau;$

(H_9) $\int\limits_T^\infty f_1(t)|G_1(F^+(t-\sigma_1) - \alpha)|dt = \infty,$

$$\int\limits_T^\infty f_1(t)|G_1(F^-(t-\sigma_1) - \alpha)|dt = \infty,$$

Theorem 1. *Let $p(t) \geq 0$. If (H_1), (H_4), and (H_6) hold, then (1) is oscillatory.*

Proof. Suppose for contrary that $y(t)$ is a nonoscillatory solution of Eq.(1). Then there exists $t_0 \geq 0$ such that $y(t) > 0$ or < 0 for $t \geq t_0$. Assume that $y(t) > 0$ for $t \geq t_0$. Setting

$$z(t) = y(t) + p(t)y(t-\tau) \quad \text{and} \quad K(t) = \int\limits_t^\infty f_2(s)G_2(y(s-\sigma_2))ds, \tag{5}$$

Eq. (1) can be written as

$$\frac{d}{dt}[z(t) + K(t)] + f_1(t)G_1(y(t-\sigma_1)) = g(t).$$

Using (H_3) and for $w(t) = z(t) + K(t) - F(t)$, further Eq. (1) yields that

$$w'(t) + f_1(t)G_1(y(t - \sigma_1)) = 0. \tag{6}$$

Consequently, $w'(t) \leq 0$ for $t \geq t_1 \geq t_0 + \rho$. Hence we have $w(t) < 0$ or > 0 for $t \geq t_1$. If $w(t) < 0$ for $t \geq t_1$ then $z(t) + K(t) < F(t)$, implies that $F(t) > \sigma$, for $t \geq t_1$, a contradiction. Hence $w(t) > 0$ for $t \geq t_1$, that is, $z(t) + K(t) > F(t)$. On the other hand, $\lim_{t\to\infty} w(t)$ exits and $K'(t) < 0$ implies that $\lim_{t\to\infty} z(t) = \lim_{t\to\infty}(w(t) - K(t))$ exists and hence

$$\limsup_{t\to\infty} F(t) < \limsup_{t\to\infty}(z(t) + K(t)) \leq \limsup_{t\to\infty} z(t) + \limsup_{t\to\infty} K(t) < \infty,$$

a contradiction.

Let $y(t) < 0$ for $t \geq t_0$. Setting $x(t) = -y(t)$, Eq. (1) becomes

$$\frac{d}{dt}[x(t) + p(t)x(t-\tau)] + f_1(t)G_1(x(t-\sigma_1)) - f_2(t)G_2(x(t-\sigma_2)) = \tilde{g}(t). \tag{7}$$

where $\tilde{g}(t) = -g(t)$. If we set $\tilde{F}(t) = -F(t)$, then $\limsup_{t\to\infty} \tilde{F}(t) = -\infty$ and $\liminf_{t\to\infty} \tilde{F}(t) = +\infty$ and hence $\tilde{F}'(t) = \tilde{g}(t)$. Following to the above procedure we have contradictions in this case also. Thus the proof of the theorem is complete. ■

Theorem 2. *Let $0 \leq p(t) \leq p < +\infty$. If (H_1)-(H_5), (H_7) and (H_8) hold, then (1) is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of Eq.(1) such that $y(t) > 0$ for $t \geq t_0$. Setting as in (5) we get (6). Hence $w'(t) \leq 0$ implies that $w(t)$ is non-increasing for $t \geq t_1 \geq t_0 + \rho$. If $w(t) < 0$ for $t \geq t_1$, then $0 < z(t) + K(t) < F(t)$, which is a contradiction. Hence $w(t) > 0$ for $t \geq t_1$ and $\lim_{t\to\infty} w(t)$ exists. Using (6) we obtain

$$\begin{aligned} w'(t) \;&+\; G_1(p)w'(t-\tau) + f_1(t)G_1(y(t-\sigma_1)) \\ &+\; G_1(p)f_1(t-\tau)\, G_1(y(t-\sigma_1-\tau)) = 0. \end{aligned}$$

Consequently,

$$w'(t) + G_1(p)w'(t-\tau) + \lambda Q(t)G_1(z(t-\sigma_1)) \leq 0,$$

due to (H_2) and (H_3), where $z(t) = y(t) + p(t)y(t-\tau) \leq y(t) + py(t-\tau)$ $\lim_{t\to\infty} K(t)$ exists. Hence there exists $\alpha \in (0,1)$ such that $K(t) \leq \alpha$ for $t \geq t^*$. Ultimately, $w(t) > 0$ becomes $z(t) + \alpha \geq F(t)$ and hence $z(t) + \alpha \geq$

$\max\{0, F(t)\}$, that is, $z(t) \geq F^+(t) - \alpha$, for $t \geq t_2 > \max\{t, t^*\}$. Thus in view of the last inequality, we obtain

$$\lambda Q(t) G_1(F^+(t-\sigma_1) - \alpha) \leq -\{w'(t) + G_1(p) w'(t-\tau)\} \tag{8}$$

for $t \geq t_2$. Integrating (8) from t_2 to ∞ we obtain,

$$\lambda \int_{t_2}^{\infty} Q(t) |G_1(F^+(t-\sigma_1) - \alpha)| dt < \infty.$$

a contradiction to (H_8).

If $y(t) < 0$ for $t \geq t_0$, then we set $x(t) = -y(t)$ to obtain $x(t) > 0$ for $t \geq t_0$ and hence using Eq. (7), we obtain similar contradiction. This completes the proof of the theorem. ■

Theorem 3. $-1 < -p \leq p(t) \leq 0$. *Assume that* (H_1), (H_4), (H_5), (H_7), (H_9) *and the following conditions*

$$\int_{\sigma_1}^{\infty} f_1(t) G_1\left(\tfrac{1}{p} F^-(t+\tau-\sigma_1)\right) dt = \infty, \tag{H_{10}}$$

$$\int_{\sigma_1}^{\infty} f_1(t) G_1\left(\tfrac{1}{p} F^+(t+\tau-\sigma_1)\right) dt = \infty \tag{H_{11}}$$

hold. Then Eq. (1) is oscillatory.

Proof. Let $y(t)$ be a nonoscillatory solution of (1) such that $y(t) > 0$ for $t \geq t_0$. Setting as in (5) we get (6). Hence $w'(t) \leq 0$. Consequently, $w(t)$ is non-increasing for $t \geq t_1 \geq t_0 + \rho$. Since $K(t) \leq \alpha$, $0 < \alpha < 1$, then $w(t) > 0$ implies that $z(t) + K(t) > F(t)$, that is, $y(t) + K(t) \geq z(t) + K(t) > F(t)$ and hence $y(t) > F^+(t) - \alpha$ for $t \geq t_1$. Integrating Eq. (6) from t_2 to ∞, we get.

$$\int_{t_2}^{\infty} f_1(t) |G_1(F^+(t-\sigma_1) - \alpha)| dt < \infty, \quad \text{for} \quad t_2 > t_1$$

because $\lim_{t\to\infty} w(t)$ exists. Following to Theorem 2 and using (H_7) we have a contradiction to (H_9). Ultimately, $w(t) < 0$ for $t \geq t_1$. Then $z(t) + K(t) < F(t)$ for $t \geq t_1$. If $z(t) > 0$, then $F(t) > 0$ for $t \geq t_1$, a contradiction. Thus $z(t) < 0$ $t \geq t_1$, which implies that $y(t) < y(t-\tau)$ for $t \geq t_2 \geq t_1$, that is, $y(t)$ is bounded for $t \geq t_2$. Consequently, $\lim_{t\to\infty} w(t)$ exists. Since $z(t) + K(t) < 0$, then

$$-p\, y(t-\tau) < z(t) + K(t) < F(t) \quad \text{and} \quad -p y(t-\tau) < \min\{0, F(t)\} \tag{9}$$

for $t \geq t_2$, implies that $y(t-\sigma_1) > (1/p)F^-(t+\tau-\sigma_1)$ for $t \geq t_3 > t_2$. Integrating Eq. (6) from t_3 to ∞, we obtain a contradiction to (H_{10}). ■

The case $y(t) < 0$ for $t \geq t_0$ can similarly be dealt with. Hence the theorem is proved

Theorem 4. *Let $-\infty < -p \leq p(t) \leq -1$. If all the conditions of Theorem 3 hold, then every bounded solution of (1) oscillates.*

Proof. The proof of the Theorem can be followed from Theorem 3 and hence the details are omitted. Thus the proof of the theorem is complete. ■

Remark 1. In the proof of Theorems 3 and 4, we consider the equation (7) for the case $y(t) < 0$, for $t \geq t_0$. Indeed, $\tilde{F}^+(t) = F^-(t)$ and $\tilde{F}^-(t) = F^+(t)$ hold.

Theorem 5. *Let $-\infty < -p \leq p(t) \leq -1$ and $\tau \geq \sigma_1$. Suppose that (H_1), (H_4), (H_5), (H_7), and (H_9) and the following condtions*

(H_{12}) $\frac{G_1(x_1)}{(x_1)^\beta} \geq \frac{G_1(x_2)}{(x_2)^\beta}, \quad x_1 \geq x_2 > 0, \;\; \beta \geq 1,$

(H_{13}) $\int_{\sigma_1}^{\infty} f_1(t) \frac{G_1\left(\frac{1}{p}F^-(t+\tau-\sigma_1)\right)}{[F^-(t+\tau-\sigma_1)]^\beta} dt = \infty,$

(H_{14}) $\int_{\sigma_1}^{\infty} f_1(t) \frac{G_1\left(\frac{1}{p}F^+(t+\tau-\sigma_1)\right)}{[F^+(t+\tau-\sigma_1)]^\beta} dt = \infty,$

hold. Then every solution of (1) oscillates.

Proof. Proceeding as in the proof of the Theorem 3, we concluded that $w(t) < 0$ and $z(t) < 0$, for $t \geq t_1$. Consequently, $z(t) > p(t)y(t-\tau)$ and $z(t) + K(t) > K(t) + p(t)y(t-\tau) > p(t)y(t-\tau)$ implies that $w(t) > p(t)y(t-\tau) - F(t)$ for $t \geq t_2 > t_1$, that is, $w(t) - p(t)y(t-\tau) > -F(t)$. Due to (H_5), $w(t) - p(t)y(t-\tau) > 0$ and hence $w(t) - p(t)y(t-\tau) > F^-(t)$,

$$w(t) > p(t)y(t-\tau) - F(t) \geq -py(t-\tau) + F^-(t) > -py(t-\tau), \quad t \geq t_2 \geq t_1.$$

Since $w(t)$ is decreasing and $\tau \geq \sigma_1$, then for $t \geq t_3 > t_2$,

$$-w(t) \leq -w(t+\tau-\sigma_1) < py(t-\sigma_1).$$

Hence

$$(10) \quad -\frac{d}{dt}\left[(-w(t))^{(1-\beta)}\right] = (\beta-1)\frac{f_1(t)G_1(y(t-\sigma_1))}{[-w(t)]^\beta}$$
$$\geq (\beta-1)p^{-\beta} f_1(t) y^{-\beta}(t-\sigma_1) G_1(y(t-\sigma_1)).$$

Using (H_{12}), (9) and then integrating (10) from t_3 to ∞ we get

$$\int_{t_3+\sigma_1}^{\infty} f_1(t) \frac{G_1\left(\frac{1}{p}F^+(t+\tau-\sigma_1)\right)}{[F^+(t+\tau-\sigma_1)]^\beta} dt < \infty,$$

a contradiction. Rest of the proof follows from Theorem 3. Thus the theorem is proved. ■

Remark 2. It seems that the solution in Theorem 4 is bounded which makes Eq. (1) oscillatory. However, Theorem 5 holds for any solution. The conditions (H_{10}), (H_{11}), (H_{13}), and (H_{14}) are not comparable and hence Theorem 4 and Theorem 5 are different. We note that Theorem 5 is restricted to super linear G_1 but in Theorem 4, G_1 could be linear, sub linear or super linear.

Theorem 6. *Let* $-\infty < -p_1 \le p(t) \le p_2 < \infty$, $p_1 > 0$. *Let the conditions* (H_1) - (H_5), (H_7), (H_8), *and*

$$(H_{15}) \qquad \int_T^\infty f_1(t)|G_1\left(F^+(t+\tau-\sigma_1)-\alpha\right)|dt = \infty$$
$$\int_T^\infty f_1(t)|G_1\left(F^-(t+\tau-\sigma_1)-\alpha\right)|dt = \infty$$

hold. Then every solutions of (1) oscillates.

The proof of the Theorem can be followed from Theorem 2 and Theorem 3 and hence the details are omitted.

Example 1. Consider

$$(11) \qquad \frac{d}{dt}[y(t)+(1+e^{-t})y(t-\pi/2)] + (\sqrt{2})y(t-\pi/4) - (\sqrt{2}e^{-t})y(t-7\pi/4) = 2\sin t.$$

Here $Q(t) \equiv \sqrt{2}$; $g(t) = 2\sin t$. If we set $F(t) = -2\cos t$, then $F'(t) = g(t)$, $F^+(t) = -2\cos t$, $2n\pi + \pi/2 \le t \le 2n\pi + 3\pi/2$, $F^+(t) = 0$, otherwise. Also, $F^-(t) = 2\cos t$, $2n\pi + 3\pi/2 \le t \le 2n\pi + 5\pi/2$, $F^-(t) = 0$, otherwise.

$$\begin{aligned} F^+(t-\pi/4) &= -2\cos(t-\pi/4), \;\; 2n\pi+\pi/4+\pi/2 \le t \le 2n\pi+\pi/4+3\pi/2, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

$$\begin{aligned} F^-(t-\pi/4) &= 2\cos(t-\pi/4), \;\; 2n\pi+\pi/4+3\pi/2 \le t \le 2n\pi+\pi/4+5\pi/2, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Choose $\alpha = \frac{1}{10\sqrt{2}}$. It is easy to verify that (H_8) hold.

Hence the conditions of Theorem 2 are satisfied. Indeed $y(t) = \sin t$ is an oscillatory solution of Eq. (11).

3. Oscillation properties of Eq. (4)

This section deals with the oscillatory and asymptotic behaviour of solutions of Eq. (4). Here G_i, $i = 1, 2$ could be linear, sub linear or super linear.

Theorem 7. *Let $0 \le p(t) \le p < +\infty$. Assume that (H_1)-(H_4) hold. If*

$$(H_{16}) \qquad \int_{\tau}^{\infty} Q(t)dt = +\infty,$$

where $Q(t)$ is same as in (H_8), then every solution of Eq. (4) either oscillates or tends to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of Eq. (4) such that $y(t) > 0$ for $t \ge t_0$. The case $y(t) < 0$ for $t \ge t_0$ is similar. Setting as in (5) equation (4) can be written as

$$T'(t) + f_1(t)G_1(y(t - \sigma_1)) = 0, \tag{12}$$

where $T(t) = z(t) + K(t)$. Using (12), we get

$$T'(t)+G_1(p)T'(t-\tau)+f_1(t)G_1(y(t-\sigma_1))+G_1(p)f_1(t-\tau)G_1(y(t-\sigma_1-\tau)) = 0,$$

that is,

$$T'(t) + G_1(p)T'(t - \tau) + \lambda Q(t)G_1(z(t - \sigma_1)) \le 0$$

due to (H_2), (H_3) and $z(t) \le y(t) + py(t - \tau)$. From (12), it follows that $T'(t) \le 0$ for $t \ge t_1 > t_0 + \rho$. Since $T(t) > 0$, then $\lim_{t\to\infty} T(t)$ exists. Consequntly, $\lim_{t\to\infty} K(t)$ exists implies that $\lim_{t\to\infty} z(t)$ exists. If $\lim_{t\to\infty} z(t) = 0$, then ultimately $\lim_{t\to\infty} y(t) = 0$. Assume that $\lim_{t\to\infty} z(t) = \alpha > 0$. Hence there exists $\beta > 0$ and $t^* > 0$ such that $z(t) \ge \beta$ for $t \ge t^*$. Thus the last inequality becomes

$$T'(t) + G_1(p)T'(t - \tau) + \lambda Q(t)G_1(\beta) \le 0 \tag{13}$$

for $t \ge t_2 > \max\{t_1, t^*\}$. Integrating (13) from t_2 to ∞, we get a contradiction to (H_{16}). This completes the proof of the theorem. ■

Theorem 8. *Let $-1 < -p \le p(t) \le 0$. If (H_1) and (H_4) hold, then every solution of (4) either ocillates or tends to zero as $t \to \infty$.*

Proof. Proceeding as in the proof of Theorem 7, we get (12) and hence $T'(t) \leq 0$ for $t \geq t_1 > t_0 + \rho$, $T(t)$ is decreasing. If $T(t) > 0$, then $\lim_{t\to\infty} T(t)$ exists. Consequently, $\lim_{t\to\infty} z(t)$ exists. Let $z(t) > 0$ for $t \geq t_2 \geq t_1$. We claim that $y(t)$ is bounded. If not, there exists a sequence of points $\{t_n\}$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$ and

$$y(t_n) = \max\{y(t) : t_2 \leq t \leq t_n\}$$

Hence

$$z(t_n) \geq y(t_n) - py(t_n - \tau) \geq (1 - p)\, y(t_n)$$

implies that $z(t_n) \to \infty$ as $n \to \infty$, a contradiction. So our claim holds. If $z(t) < 0$ for $t \geq t_2 \geq t_1$, then $y(t) < y(t - \tau)$ implies that $y(t)$ is bounded. Let $\lim_{t\to\infty} z(t) = 0$. Then

$$\begin{aligned} 0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}[y(t) + p(t)y(t-\tau)] \\ &\geq \limsup_{t\to\infty}[y(t) - py(t-\tau)] \\ &\geq \limsup_{t\to\infty}[y(t)] - \liminf_{t\to\infty}[py(t-\tau)] \\ &= (1-p)\limsup_{t\to\infty} y(t) \end{aligned}$$

implies that $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. Assume that $\lim_{t\to\infty} z(t) = \beta$, $0 < |\beta| < \infty$. If $\beta > 0$, there exists $\gamma > 0$ and $t^* > 0$ such that $z(t) \geq \gamma$ for $t \geq t^*$. Consequenty, $y(t) \geq z(t) \geq \gamma$ for $t \geq t_3 > \max\{t_2, t^*\}$ implies that

$$\int_{t_3}^{\infty} f_1(t)dt < +\infty$$

due to the equation (12), a contradiction. Let $-\infty < \beta < 0$. Then there exists $\gamma < 0$ such that $z(t) \leq \gamma$ for $t \geq t^*$. Since $z(t) < 0$, then $z(t) > p(t)y(t-\tau)$, that is,

$$y(t - \sigma_1) \geq (-1/p)z(t + \tau - \sigma_1) \geq (-\gamma/p),$$

for $t \geq t_3 > \max\{t_2, t^*\}$. Integrating (13) from t_3 to ∞, we get a contradiction to (H_1).

Next, we suppose that $T(t) < 0$, for $t \geq t_2 \geq t_1$. Ultimately, $z(t) < 0$ for $t \geq t_2$, that is, $y(t) < y(t - \tau)$ implis $y(t)$ is bounded and hence $z(t)$ is bounded. Proceeding as above we get a contradiction to (H_1).

Thus the proof of the theorem is complete. ■

Theorem 9. *Let $-\infty < -p \leq p(t) \leq -p_1 < -1$. If (H_1), and (H_4) hold, then evry bounded solution of (4) either oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be a bounded nonoscillatory solution of (4) such that $y(t) > 0$ for $t \geq t_0$. Proceeding as in Theorem 8, if $z(t) > 0$ for $t \geq t_2 \geq t_1$, then $y(t) + p(t)y(t-\tau) > 0$ implies that $y(t) > -p(t)y(t-\tau)$, that is $y(t) > y(t-\tau)$ for $t \geq t_2$. It follows that

$$y(t + n\tau) > y(t_2), \qquad n = 0, 1, 2, 3, \cdots,$$

that is, $\liminf_{t\to\infty} y(t) > 0$. So there exists a constant $M > 0$, such that $\liminf_{t\to\infty} y(t) > M$. Consequently, integration of (12) from t_2 to ∞ gives a contradiction to (H_1). The case $z(t) < 0$ for $t \geq t_2$ follows from the proof of the Theorem 8. When $\lim_{t\to\infty} z(t) = 0$, we note that

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \liminf_{t\to\infty}(y(t) + p(t)y(t-\tau)) \\
&\leq \liminf_{t\to\infty}(y(t) - p_1 y(t-\tau)) \\
&\leq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(-p_1 y(t-\tau)) \\
&= \limsup y(t) - p_1 \limsup_{t\to\infty} y(t-\tau) \\
&= (1-p_1)\limsup_{t\to\infty} y(t)
\end{aligned}$$

that is, $\limsup_{t\to\infty} y(t) = 0$.

Hence $T(t) < 0$ for $t \geq t_1$. Using the fact that $y(t)$ is bounded, it follows that $z(t) < 0$ and $\lim_{t\to\infty} z(t)$ exists. Using the same reasoning as in Theorem 8, we have the necessary contradiction. The case $y(t) < 0$ may similarly be dealt with. Hence the theorem is proved. ■

Theorem 10. *Let $-\infty < -p \leq p(t) \leq -1$. Suppose that (H_1), (H_4), (H_{12}) and the following condition*

(H_{17}) *Suppose that for every sequence $\{\eta_j\} \in (0, \infty), \eta_j \to \infty$ as $j \to \infty$ and for every $k > 0$ such that the intervals $(\eta_j - k,\ \eta_j + k)$, $j = 1, 2, \cdots$ are nonoverlapping, such that*

$$\sum_{i=1}^{\infty} \int_{\eta_i - k}^{\eta_i + k} f_1(t)dt = \infty$$

hold. If $\tau > \sigma_1$, then (4) is oscillatory.

Let $y(t)$ be a nonoscillatory solution of (4) such that $y(t) > 0$ for $t \geq t_0$.

Proof. First half of the proof, that is $T'(t) \leq 0$, $T(t) > 0$, and $z(t) > 0$ for $t \geq t_2$ is same as in Theorem 9. Consider the next half, that is $T'(t) \leq 0$, $T(t) < 0$, and $z(t) < 0$ for $t \geq t_2$. Following to (10), we get

$$(14)\quad -\frac{d}{dt}\left[(-z(t))^{(1-\beta)}\right] = (\beta-1)\frac{f_1(t)G_1(y(\tau-\sigma_1))}{[-z(t)]^{\beta}} \geq (\beta-1)p^{-\beta}f_1(t)y^{-\beta}(t-\sigma_1)G_1(y(t-\sigma_1))$$

We consider two cases, viz: $y(t)$ is bounded and $y(t)$ is unbounded. If the former holds, then $\lim_{t\to\infty} w(t)$ and $\lim_{t\to\infty} z(t)$ exist. Let $\lim_{t\to\infty} z(t) = \beta$, $-\infty < \beta < 0$ and hence integrating (14) from t_3 to ∞, we get a contradiction to (H_1) for $t \geq t_3 > t_2$. Assume that the later holds. Hence there exists a sequence $\{t_n\}$ in $[t_2, \infty)$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$. So for every $M > 0$ there exists $N > 0$ such that $y(t_n) > M$ for $n \geq N$. Hence, there exists $\delta_n > 0$ and $(t_n - \delta_n, t_n + \delta_n)$ such that $y(t) > M$ for $t \in (t_n - \delta_n, t_n + \delta_n)$, $n \geq N$ and $\liminf_{t\to\infty} \delta_n = \delta > 0$. Let $\delta_n > \delta > 0$ for $n \geq N^*$. Then for $N_1 > \max\{N, N^*\}$, we have

$$\int_{t_{N_1}-\delta_{N_1}+\sigma_1}^{\infty} f_1(t)(y(t-\sigma_1))^{-\alpha}G_1(y(t-\sigma_1))dt$$
$$= \sum_{i=N_1}^{\infty}\int_{t_i-\delta_i+\sigma_1}^{t_i+\delta_i+\sigma_1} f_1(t)(y(t-\sigma_1))^{-\alpha}G_1(y(t-\sigma_1))dt$$
$$\geq \frac{G_1(M)}{M^{\alpha}}\sum_{i=N_1}^{\infty}\int_{t_i-\delta_i+\sigma_1}^{t_i+\delta_i+\sigma_1} f_1(t)dt$$
$$\geq \frac{G_1(M)}{M^{\alpha}}\sum_{i=N_1}^{\infty}\int_{t_i+\sigma_1-\delta}^{t_i+\sigma_1+\delta} f_1(t)dt.$$

due to (H_{12}). Integrating (15) from $t_{N_1} - \delta_{N_1} + \sigma_1$ to ∞, we get a contradiction to (H_{17}).

This completes the proof of the theorem. ■

Remark 3. Theorem 9 and 10 are different in their own rights, especially due to G_1.

4. Existence of positive solution

Theorem 11. *Let G_i; $i = 1, 2$ be Lipschitzian on the intervals of the form $[a, b]$, $0 < a < b < \infty$. Suppose that $g(t)$ satisfies (H_5). If*

$$\int_0^\infty f_i(t)dt < \infty, \quad i = 1, 2,$$

then Equation (1) *admits a positive bounded solution.*

Proof. Let $-\infty < b_1 \le p(t) \le b_2 < -1$. It is possible to find a positive number T such that

$$M_1 \int_{t=T}^\infty f_1(t)dt < \frac{-b_2}{2(D-b_2)}, \qquad M_2 \int_{t=T}^\infty f_2(t)dt < \frac{-b_2}{2(D-b_2)},$$

where $M_1 = \max\{L_1,\ G_1(K)\}$, $M_2 = \max\{L_2, G_2(K)\}$, $D > \max\{-b_1,\ b_2 + \frac{b_2}{1+b_2}\}$, $K = \frac{2D - b_2(D+1)}{(b_2 - D)(b_2+1)} > 0$, and L_1, L_2 are Lipschitz constants on $\left[\frac{-b_2}{D-b_2}, K\right]$. Let $F(t)$ be such that $\frac{-1}{2(D-b_2)} \le F(t) \le \frac{1}{2(D-b_2)}$.

Let $X = BC([t_0, \infty), R)$ be the Banach space of all bounded real valued continuous functions $x(t)$, $t \ge T$ with supremum norm defined by

$$\|x\| = \sup\{|x(t)| : t \ge T\}.$$

Set

$$S = \{x \in X : \frac{-b_2}{D-b_2} \le x(t) \le K,\ t \ge T\}.$$

For $y \in S$, define

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \quad T \le t \le T+\rho \\ &= -\frac{y(t+\tau)}{p(t+\tau)} - \frac{D(2-b_2)}{p(t+\tau)(D-b_2)} + \frac{1}{p(t+\tau)} \int_{s=t+\tau}^\infty f_1(s)G_1(y(s-\sigma_2))ds \\ &\quad - \frac{1}{p(t+\tau)} \int_{s=t+\tau}^\infty f_2(s)G_2(y(s-\sigma_2))ds + \frac{F(t+\tau)}{p(t+\tau)}, \quad t \ge T+\rho. \end{aligned}$$

For every $y \in S$ and $t \ge T + \rho$,

$$\begin{aligned} Ty(t) &\le \frac{-K}{b_2} - \frac{D(2-b_2)}{b_2(D-b_2)} + \frac{1}{2(D-b_2)} + \frac{1}{2(D-b_2)} \\ &= -\frac{K(D-b_2) + 2D - b_2(D+1)}{b_2(D-b_2)} = K \end{aligned}$$

and

$$\begin{aligned} Ty(t) &\geq -\frac{D(2-b_2)}{b_2(D-b_2)} - \frac{1}{2(D-b_2)} - \frac{1}{2(D-b_2)} \\ &\geq \frac{-b_2}{(D-b_2)} \end{aligned}$$

that is, $Ty \in S$. Immediately, it follows that

$$\begin{aligned} |Ty(t) - Tx(t)| &\leq \frac{1}{|p(t+\tau)|}\left[\|y-x\| - \frac{b_2}{2(D-b_2)}\|y-x\| \right. \\ &\quad \left. - \frac{b_2}{2(D-b_2)}\|y-x\|\right] \\ &\leq \frac{-1}{b_2}(1 - \frac{b_2}{D-b_2})\|y-x\|. \end{aligned}$$

Hence

$$\|Ty - Tx\| \leq \left(\frac{1}{D-b_2} - \frac{1}{b_2}\right)\|y-x\|.$$

Consequently, T is a contraction and has a unique fixed point $y(t)$ in the interval $\left[\frac{-b_2}{D-b_2}, K\right]$. In fact, $y(t)$ is a positive bounded solution of (1).

For the other ranges of $p(t)$, the following informations can be noted:

(i) When $0 \leq p(t) \leq b_1 < 1$, it is possible to choose a positive number $T > 0$ such that $M_1 \int\limits_{t=T}^{\infty} f_1(t)dt < \frac{1-b_1}{10}$, $M_2 \int\limits_{t=T}^{\infty} f_2(t)dt < \frac{1-b_1}{20}$, and choose $F(t)$ such that $-\left(\frac{1-b_1}{20}\right) \leq F(t) \leq \left(\frac{1-b_1}{10}\right)$ and $S = \{x \in X : \frac{1-b_1}{10} \leq x(t) \leq 1,\ t \geq T\}$.

We define

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \qquad T \leq t \leq T+\rho \\ &= -\,p(t)y(t-\tau) + \frac{1+4b_1}{5} + \int\limits_{s=t}^{\infty} f_1(s)G_1(y(s-\sigma_1))ds \\ &\quad - \int\limits_{s=t}^{\infty} f_2(s)G_2(y(s-\sigma_2))ds + F(t), \qquad t \geq T+\rho. \end{aligned}$$

(ii) When $-1 < b_1 \leq p(t) \leq 0$, we choose a positive number T so large that $M_1 \int\limits_{t=T}^{\infty} f_1(t)dt < \frac{1+b_1}{11}$, $M_2 \int\limits_{t=T}^{\infty} f_2(t)dt < \frac{1+b_1}{20}$, and choose $F(t)$ such that $-\left(\frac{1+b_1}{20}\right) \leq F(t) \leq \left(\frac{1+b_1}{10}\right)$ and $S = \{x \in X : \frac{1+b_1}{10} \leq x(t) \leq 1,\ t \geq T\}$.

We define the mapping T by

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \quad T \le t \le T+\rho \\ &= -p(t)y(t-\tau) + \frac{1+b_1}{5} + \int_{s=t}^{\infty} f_1(s)G_1(y(s-\sigma_1))ds \\ &\quad - \int_{s=t}^{\infty} f_2(s)G_2(y(s-\sigma_2))ds + F(t), \quad t \ge T+\rho. \end{aligned}$$

T is a contraction with a contraction constant $\frac{3-17b_1}{20}$.

(*iii*) When $-1 < b_1 \le p(t) \le b_2 < 1$, be such that $b_1 < 0,\ b_2 > 0$ and $b_2 < 1 + 5b_1$, it is possible to choose a positive number T large enough such that $M_1 \int_{t=T}^{\infty} f_1(t)dt < \frac{b_1}{2} + \frac{1-b_2}{10}$, $M_2 \int_{t=T}^{\infty} f_2(t)dt < \frac{1-b_2}{20}$, and choose $F(t)$ such that $-\left(\frac{1-b_2}{20}\right) \le F(t) \le \left(\frac{b_1}{2} + \frac{1-b_2}{10}\right)$ and $S = \{x \in X : \frac{1-b_2}{10} \le x(t) \le 1,\ t \ge T\}$.

We define

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \quad T \le t \le T+\rho \\ &= -p(t)y(t-\tau) + \frac{1+4b_2}{5} + \int_{s=t}^{\infty} f_1(s)G_1(y(s-\sigma_1))ds \\ &\quad - \int_{s=t}^{\infty} f_2(s)G_2(y(s-\sigma_2))ds + F(t), \quad t \ge T+\rho. \end{aligned}$$

T is a contraction with contraction constant $\frac{95b_1+20}{20}$.

(*iv*) Let $p(t) \equiv -1$. Let $0 < b_1 < 1$ be such that $b_1 \ne 1/2$. Let T be sufficiently large such that $M_1 \int_{t=T}^{\infty} f_1(t)dt < \frac{1-2b_1}{20}$, $M_2 \int_{t=T}^{\infty} f_2(t)dt < \frac{1-2b_1}{40}$, and choose $F(t)$ suchthat $-\left(\frac{1-2b_1}{40}\right) \le F(t) \le \left(\frac{1-2b_1}{20}\right)$, and $S = \{x \in X : \frac{1-b_1}{20} \le x(t) \le b_1,\ t \ge T\}$.

We define

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \quad T \le t \le T+\rho \\ &= -y(t-\tau) + \frac{1-b_1}{10} + \int_{s=t}^{\infty} f_1(s)G_1(y(s-\sigma_1))ds \\ &\quad - \int_{s=t}^{\infty} f_2(s)G_2(y(s-\sigma_2))ds + F(t), \quad t \ge T+\rho. \end{aligned}$$

Therefore T is a contraction with a contraction constant $\frac{43-6b_1}{40}$.

(v) When $p(t) \equiv 1$ for all t. Let $-1 < b_1 < 0$ be such that $b_1 \neq -1/2$.

We replace $-b_1$ in the place of b_1, in the earlier settings in (iv).

(vi) When $1 < b_1 \leq p(t) \leq b_2 \leq \frac{1}{2}b_1^2$. Let $T > 0$ be sufficiently large such that $M_1 \int\limits_{t=T}^{\infty} f_1(t)dt < \frac{b_1-1}{8b_1} + \frac{b_1-1}{16b_2}$, $M_2 \int\limits_{t=T}^{\infty} f_2(t)dt < \frac{b_1-1}{16b_2}$, and choose $F(t)$ such that $-\left(\frac{b_1-1}{16b_1b_2}\right) \leq F(t) \leq \left(\frac{b_1-1}{8b_1^2} + \frac{b_1-1}{16b_1b_2}\right)$, and $S = \{x \in X : \frac{b_1-1}{8b_1b_2} \leq x(t) \leq 1,\ t \geq T\}$.

We define $T : S \to S$ by

$$\begin{aligned} Ty(t) &= Ty(T+\rho), \quad T \leq t \leq T+\rho \\ &= -\frac{y(t+\tau)}{p(t+\tau)} + \frac{(2b_1^2+b_1-1)}{4b_1p(t+\tau)} + \frac{1}{p(t+\tau)} \int\limits_{s=t+\tau}^{\infty} f_1(s)G_1(y(s-\sigma_1))ds \\ &\quad -\frac{1}{p(t+\tau)} \int\limits_{s=t+\tau}^{\infty} f_2(s)G_2(y(s-\sigma_2))ds + \frac{F(t+\tau)}{p(t+\tau)}, \quad t \geq T+\rho. \end{aligned}$$

Therefore T is a contraction with a contraction constant $\frac{1}{b_1} + \frac{b_1}{8b_1^2} + \frac{b_1-1}{8b_1b_2}$. ■

Several authors hve investigated the oscillation properties of (2) with and without $g(t)$. But study of (2)/(4) is very rare in the literature. Hence this work may initiate for further study in this area.

5. Summary

In this work, we could succeed to provide oscillation results for the equation (1). However, the results are not satisfactory for Eq. (4). Due to the methods adopted here, Theorems 9 and 10 are different to that of Theorems 7 and 8. It seems that more conditions are required to show that Eq. (4) is oscillatory.

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