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COMMON FIXED POINTS OF CONDITIONALLY COMMUTING MAPPINGS SATISFYING WEAKER CONTINUITY CONDITIONS

Abstract. The aim of the present paper is to obtain some new common fixed point theorems for a pair of Lipschitzian type self-mappings satisfying a minimal commutativity and weaker continuity conditions. In the setting of our results we establish a situation in which a pair of mappings may possess common fixed points as well as coincidence points which may not be common fixed points. Our results generalize several fixed point theorems.

Key words: commuting maps, conditional reciprocal continuity, weak reciprocal continuity, absorbing maps, fixed point theorems.

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1. Introduction

Sessa [11] obtained the first weaker version of commutativity by introducing the notion of weak commutativity.

Definition 1 ([11]). *Two self-maps f and g of a metric space (X, d) are called weakly commuting if $d(fgx, gfx) \leq d(fx, gx)$ for all x in X.*

In 1986, Jungck [2] generalized the notion of weakly commuting maps by introducing the concept of compatible maps.

Definition 2 ([2]). *Two self-maps f and g of a metric space (X, d) are called compatible iff $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.*

The study of common fixed points has developed around compatible maps and its weaker forms and it has become an area of vigorous research activity. However, the study of noncompatible mappings is equally interesting and Pant [6, 7] has initiated some work along these lines. Interestingly enough, the best examples of noncompatible maps are found among pairs of mappings which are discontinuous at their common fixed point [6].

The definition of compatibility implies that the mappings f and g will be noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either non zero or nonexistent.

In 2002, Aamri and Moutawakil [1] have introduced a new property, namely the $(E.A.)$ property which is more general than noncompatible mappings.

Definition 3 ([1]). *A pair (f, g) of self-mappings of a metric space (X, d) is said to satisfy the property $(E.A.)$ if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t, \quad \text{for some } t \in X.$$

If two maps are noncompatible they satisfy the $(E.A.)$ property. The converse, however, is not necessarily true [1].

Definition 4 ([1]). *Two self-maps f and g are called pointwise $R-$weakly commuting on X if given x in X there exists $R \geq 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$. It is well known now that pointwise $R-$weak commutativity* [4] *is*

(i) equivalent to commutativity at coincidence points and in the setting of metric spaces this notion is equivalent to weak compatibility and

(ii) a necessary, hence minimal, condition for the existence of common fixed points of contractive type mappings.

In 2010, Pant et al [10] have introduced the notion of conditionally commuting mappings which is the weakest form of the commutativity known so far.

Definition 5 ([10]). *Two self-mappings f and g of a metric space (X, d) are called conditionally commuting if they commute on a nonempty subset of the set of coincidence points whenever the set of their coincidences is nonempty.*

From the definition itself it is clear that if two maps are pointwise $R-$weakly commuting then they are necessarily conditionally commuting, however, as shown in Example 2 conditionally commuting mappings are not necessarily pointwise $R-$weakly commuting.

In earlier works, Pant [4, 5] introduced the concept of reciprocal continuity and obtained the first results that established a situation in which a collection of mappings has a fixed point which is a point of discontinuity for all the mappings.

Definition 6 ([4], [5]). *Two self-mappings f and g of a metric space (X, d) are called reciprocally continuous iff* $\lim_n fgx_n = ft$ *and* $\lim_n gfx_n =$

gt, whenever $\{x_n\}$ *is a sequence such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some* t *in* X.

If f and g are both continuous then they are obviously reciprocally continuous but the converse is not true [4, 5]. The notion of reciprocal continuity is mainly applicable to compatible mapping satisfying contractive conditions. To widen the scope of the study of fixed points from the class of compatible mappings satisfying contractive conditions to a wider class including compatible as well as noncompatible mappings satisfying contractive, nonexpansive or Lipschitz type condition Pant et al [8] generalized the notion of reciprocal continuity by introducing the new concept of weak reciprocal continuity.

Definition 7 ([8]). *Two self-mappings* f *and* g *of a metric space* (X, d) *are called weakly reciprocally continuous iff* $\lim_n fgx_n = ft$ *or* $\lim_n gfx_n = gt$, *whenever* $\{x_n\}$ *is a sequence such that* $\lim_n fx_n = \lim_n gx_n = t$ *for some* t *in* X.

If f and g are either continuous or reciprocally continuous then they are obviously weakly reciprocally continuous but, as shown in Example 1 the converse is not true.

Most recently Pant and Bisht [9] further generalized the notion of reciprocal continuity by introducing the new notion of conditional reciprocal continuity which turns out to be a necessary condition for the existence of a common fixed point.

Definition 8 ([9]). *Two self-mappings* f *and* g *of a metric space* (X, d) *are called conditionally reciprocally continuous (CRC) iff whenever the set of sequences* $\{x_n\}$ *satisfying* $\lim_n fx_n = \lim_n gx_n$ *is nonempty, there exists a sequence* $\{y_n\}$ *satisfying* $\lim_n fy_n = \lim_n gy_n = t(say)$ *for some* t *in* X *such that* $\lim_n fgy_n = ft$ *and* $\lim_n gfy_n = gt$.

If f and g are either continuous or reciprocally continuous then they are obviously conditionally reciprocally continuous but, as shown in Example 1 the converse is not true. It may be observed that the aspects of weak reciprocal continuity and conditional reciprocal continuity are independent from each other [9].

Definition 9 ([3]). *Let* f *and* g *be two self-maps of a metric space* (X, d) *then* f *is called* $g-$*absorbing if there exists some positive real number* R *such that* $d(gx, gfx) \leq Rd(fx, gx)$ *for all* x *in* X. *Similarly* g *will be called* $f-$*absorbing if there exists some positive real number* R *such that* $d(fx, fgx) \leq Rd(fx, gx)$ *for all* x *in* X.

It is well known that the absorbing maps are neither a subclass of compatible maps nor a subclass of noncompatible maps [3].

The principal areas of applications of the notion of conditionally commuting include nonexpansive mapping pairs and mappings satisfying Lipschitz type conditions. In this paper we prove some common fixed point theorems for a pair of selfmaps satisfying a minimal commutativity condition and weaker continuity conditions. Our results also demonstrate the usefulness of the notion of the absorbing maps in fixed point considerations.

2. Main results

Theorem 1. *Let f and g be conditionally commuting self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq kd(gx, gy)$, $k \geq 0$.

If f and g satisfy the property $(E.A.)$. Suppose f and g are weakly reciprocally continuous and g is $f-$absorbing or f is $g-$absorbing then f and g have a common fixed point.

Proof. Since f and g satisfy the property $(E.A.)$, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X. Since $fX \subseteq gX$, for each x_n there exists y_n in X such that $fx_n = gy_n$. Thus $fx_n \to t$, $gx_n \to t$ and $gy_n \to t$ as $n \to \infty$. By virtue of this and using (ii) we obtain $fy_n \to t$. Therefore, we have

$$fx_n = gy_n \to t, gx_n \to t, \qquad fy_n \to t. \tag{1}$$

Suppose that g is $f-$absorbing. Then $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$ and $d(fy_n, fgy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$fgx_n \to t, \qquad fgy_n (= ffx_n) \to t. \tag{2}$$

Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. By virtue of (ii) we get $d(ffx_n, ft) \leq kd(gfx_n, gt)$. On letting $n \to \infty$, we get $ffx_n \to ft$. In view of (2) this yields $t = ft$. Since $fX \subseteq gX$ there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. Making $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Conditional commutativity of f and g implies that f and g commute at u or there exists a coincidence point v of f and g at which f and g commute. Suppose f and g commute at the coincidence point v. Then $fv = gv$ and $fgv = gfv$. Also $fgv = ffv = gfv = ggv$. Since g is $f-$absorbing $d(fv, fgv) \leq Rd(fv, gv)$. This yields $fv = fgv$. Hence $fv = ffv = gfv$ and fv is a common fixed point of f and g.

Next Suppose that $fgx_n \to ft$. In view of (2) this yields $t = ft$. Since $fX \subseteq gX$ there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. Making $n \to \infty$, we get $fu = t$. Thus

$fu = gu$. This, in view of conditional commutativity and $f-$absorbing property of g implies that f and g have a common fixed point.

Now suppose that f is $g-$absorbing. Then $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$ and $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$gfx_n (= ggy_n) \to t, \qquad gfy_n \to t. \tag{3}$$

Weak reciprocal continuity of f and g implies that $fgy_n \to ft$ or $gfy_n \to gt$. Let $gfy_n \to gt$. In view of (3) this yields $t = gt$. By virtue of (ii) we get $d(fx_n, ft) \leq kd(gx_n, gt)$. On letting $n \to \infty$, we obtain $t = ft$. Hence $t = ft = gt$ and t is a common fixed point of f and g.

Next suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some u in X. Therefore, $fgy_n \to ft = gu$. Using (ii) and in view of (3), we get $d(fy_n, fgy_n) \leq kd(gy_n, ggy_n)$. On letting $n \to \infty$, we get $t = gu$. Again, by virtue of (ii), we obtain $d(fy_n, fu) \leq kd(gy_n, gu)$. Making $n \to \infty$, we get $t = fu$. Hence $fu = gu$. Conditional commutativity of f and g implies that f and g commute at u or there exists a coincidence point v of f and g at which f and g commute. Suppose f and g commute at the coincidence point v. Then $fv = gv$ and $fgv = gfv$. Also $fgv = ffv = gfv = ggv$. Since f is $g-$absorbing $d(gv, gfv) \leq Rd(fv, gv)$. This yields $gv = gfv$. Hence $fv = ffv = gfv$ and fv is a common fixed point of f and g. This completes the proof of the theorem. ■

We now give examples to illustrate Theorem 1.

Example 1. Let $X = [2, 15]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows

$$fx = 2 \quad \text{if} \quad x = 2 \quad \text{or} \quad x > 5, \quad fx = 6 \quad \text{if} \quad 2 < x \leq 5,$$

$$g2 = 2, \quad gx = 6, \quad \text{if} \quad 2 < x \leq 5, \quad gx = \frac{(x+5)}{5} \quad \text{if} \quad x > 5.$$

Then f and g satisfy all the conditions of Theorem 1 and have infinitely many coincedence points in the interval $[2, 5]$ and a common fixed point at $x = 2$. It can be verified in this example that f and g satisfy the condition (ii) for $k = 3$. The mappings f and g are conditionally commuting maps since they commute at their coincidence point $x = 2$. Furthermore, f is $g-$absorbing with $R = \frac{29}{18}$. It can also be noted that f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 2$ and either $x_n = 2$ for each n from some place onwards, or $x_n = 5 + \epsilon_n$ where $\epsilon_n \to 0$ as $n \to \infty$. If $x_n = 2$ for each n from some place onwards, $fgx_n \to 2 = f2$ and $gfx_n \to 2 = g2$. If $x_n = 5 + \epsilon_n$ then $fx_n \to 2$, $gx_n = (2 + \frac{\epsilon_n}{3}) \to 2$, $fgx_n = f(2 + \frac{\epsilon_n}{3}) \to 6 \neq f2$

and $gfx_n \to 2 = g2$. Thus $\lim_n gfx_n = g2$ but $\lim_n fgx_n \neq f2$. Hence f and g are weakly reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings but conditionally reciprocally continuous. To see that f and g satisfy property $(E.A.)$, let us consider a sequence $\{z_n\}$ in X such that $z_n = 5 + \frac{1}{n}$. Then $fz_n \to 2$, $gz_n \to 2$. Therefore, f and g satisfy property $(E.A.)$.

Example 2. Let $X = [0, 1]$ and d be the usual metric on X. Define self-mappings f and g on X as follows

$$\begin{aligned} f(x) &= \frac{1}{2} - \left|\frac{1}{2} - x\right|, \\ g(x) &= \frac{2}{3}(1 - x). \end{aligned}$$

Then f and g satisfy all the conditions of the above theorem and have two coincidence points $x = 1, \frac{2}{5}$ and a common fixed point $x = \frac{2}{5}$. It may be verified in this example that $f(X) = [0, \frac{1}{2}]$, $g(X) = [0, \frac{2}{3}]$ and $fX \subseteq gX$. Furthermore f and g are conditionally commuting since they commute at their coincidence point $x = \frac{2}{5}$. To see that f and g satisfy property $(E.A.)$, let us consider a sequence $\{z_n\}$ in X such that $z_n = 1 - \frac{1}{n}$. Then $fz_n \to 0$, $gz_n \to 0$. Hence, f and g satisfy property $(E.A.)$. It may also be verified that f and g are not pointwise $R-$weakly commuting as they do not commute at the coincidence point $x = 1$ since $f(g(x)) = 0, g(f(x)) = \frac{2}{3}$. It is also easy to verify that f and g satisfy the Lipschitz type condition $d(fx, fy) \leq \frac{3}{2}d(gx, gy)$ together with $f-$absorbing condition $d(fx, fgx) \leq d(fx, gx)$ for all x. It can also be noted that f and g are weakly reciprocally continuous since both f and g are continuous.

In Examples 1 and 2, f and g are not pointwise $R-$weakly commuting as they do not commute at the set of coincidence points $(2, 5]$ and coincidence point $x = 1$ respectively. We now give an example of pointwise $R-$weakly commuting maps satisfying Theorem 1.

Example 3. Let $X = [0, 1]$ and d be the usual metric on X. Define self-mappings f and g on X as follows

$$\begin{aligned} f(x) &= \frac{1}{2} - \left|\frac{1}{2} - x\right|, \\ g(x) &= \frac{2}{3} \text{ fractional part of } (1 - x). \end{aligned}$$

Then f and g satisfy all the conditions of the above theorem and have three coincidence points $x = 0, \frac{2}{5}, 1$ and two common fixed points $x = 0, \frac{2}{5}$. It may be verified in this example that $f(X) = [0, \frac{1}{2}]$, $g(X) = [0, \frac{2}{3})$ and

$fX \subseteq gX$. Also, f and g are pointwise R-weakly commuting maps, hence also conditionally commuting, since they commute at each of their coincidence points viz. $x = 0, \frac{2}{5}, 1$. To see that f and g satisfy property $(E.A.)$, let us consider a sequence $\{z_n\}$ in X such that $z_n = 1 - \frac{1}{n}$. Then $fz_n \to 0$, $gz_n \to 0$. Hence, f and g satisfy property $(E.A.)$. It is also easy to verify that f and g satisfy the Lipschitz type condition $d(fx, fy) \leq \frac{3}{2}d(gx, gy)$ together with $f-$absorbing condition $d(fx, fgx) \leq d(fx, gx)$ for all x. It can also be noted that f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 0$ and either $x_n = 0$ for each n or $x_n \to 1$. If $x_n = 0$ for each n, $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0 = f0$ and $gfx_n \to 0 = g0$. If $x_n \to 1$ then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0 = f0$ and $gfx_n \to 2/3 \neq g0$. Thus $\lim_n fgx_n = f0$ but $\lim_n gfx_n \neq g0$. Hence f and g are weakly reciprocally continuous.

As a direct consequence of the above theorem we get the following:

Corollary 1. *Let f and g be conditionally commuting noncompatible self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq kd(gx, gy), \quad k \geq 0.$

Suppose f and g are weakly reciprocally continuous and g is $f-$absorbing or f is $g-$absorbing then f and g have a common fixed point.

If we put $k = 1$ in Theorem 2.1, we get a common fixed point theorem for nonexpansive type mapping pairs. We state it as follows:

Corollary 2. *Let f and g be conditionally commuting self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq d(gx, gy).$

If f and g satisfy the property $(E.A.)$. Suppose f and g are weakly reciprocally continuous and g is $f-$absorbing or f is $g-$absorbing then f and g have a common fixed point.

The next theorem demonstrates the applicability of conditional commutativity and conditional reciprocal continuity in diverse settings by establishing the existence of common fixed point under Lipschitz type condition.

Theorem 2. *Let f and g be conditionally commuting self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq kd(gx, gy), \quad k \geq 0.$

If f and g satisfy the property $(E.A.)$. Suppose f and g are conditionally *reciprocally continuous and g is f-absorbing or f is g-absorbing then f and g have a common fixed point.*

Proof. Since f and g satisfy the property $(E.A.)$, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X. Since f and g are conditionally reciprocally continuous and $fx_n \to t, gx_n \to t$ there exists a sequence $\{y_n\}$ satisfying $\lim_n fy_n = \lim_n gy_n = u$ such that $\lim_n fgy_n = fu$ and $\lim_n gfy_n = gu$. Since $fX \subseteq gX$, for each y_n there exists z_n in X such that $fy_n = gz_n$. Thus $fy_n \to u$, $gy_n \to u$ and $gz_n \to u$ as $n \to \infty$. By virtue of this and using (ii) we obtain $fz_n \to u$. Therefore, we have

$$fy_n = gz_n \to u, \quad gy_n \to u, \quad fz_n \to u. \tag{4}$$

Suppose that g is $f-$absorbing. Then $d(fy_n, fgy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, this inequality yields $fgy_n \to u$. Hence $u = fu$. Since $fX \subseteq gX$ there exists v in X such that $u = fu = gv$.

Now using (ii), we obtain $d(fy_n, fv) \leq kd(gy_n, gv)$. Making $n \to \infty$, we get $fv = u$. Thus $fv = gv$. Conditional commutativity of f and g implies that f and g commute at v or there exists a coincidence point w of f and g at which f and g commute. Suppose f and g commute at the coincidence point w. Then $fw = gw$ and $fgw = gfw$. Also $fgw = ffw = gfw = ggw$. Since g is $f-$absorbing $d(fw, fgw) \leq Rd(fw, gw)$. This yields $fw = fgw$. Hence $fw = ffw = gfw$ and fw is a common fixed point of f and g.

Now suppose that f is $g-$absorbing. Then $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, this inequality yields $gfy_n \to u$. Hence $u = gu$. Using (ii) we get $d(fz_n, fu) \leq kd(fz_n, gu)$. Making $n \to \infty$, we get $fu = u$. Hence $u = fu = gu$ and u is a common fixed point of f and g. This completes the proof of the theorem. ■

Examples 1, 2 and 3 illustrate the above theorem also.

The following corollary directly follows from Theorem 2.

Corollary 3. *Let f and g be conditionally commuting noncompatible self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq kd(gx, gy), \quad k \geq 0.$

Suppose f and g are conditionally reciprocally continuous and g is $f-$absorbing or f is $g-$absorbing then f and g have a common fixed point.

By choosing $k = 1$ in Theorem 2, we get a common fixed point theorem for nonexpansive type mapping pairs. We state it as follows:

Corollary 4. *Let f and g be conditionally commuting self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$

(ii) $d(fx, fy) \leq d(gx, gy)$.
If f and g satisfy the property (E.A.). Suppose f and g are conditionally reciprocally continuous and g is $f-$absorbing or f is $g-$absorbing then f and g have a common fixed point.

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