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ON THE CONVERGENCE OF AN ITERATIVE METHOD FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN INTERMEDIATE SENSE

Abstract. In this paper, we introduced an iterative method for approximating a fixed point of asymptotically non-expansive mappings in the intermediate sense in a uniformly convex Banach space. We establish some strong and weak convergence theorems.

Key words: Asymptotically nonexpansive mapping in the intermediate sense, condition (A), uniformly convex, modified Noor iteration.

AMS Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Let X be a normed space, Cbe a nonempty subset of X and let $T : C \to C$ be a given mapping. Then $T$is said to be asymptotically [4] if there exists a sequence $\{k_n\}$, $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and each $n \geq 1$. The weaker definition [9] requires that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

for each $x, y \in C$ and that T^N be continuous for some $N \geq 1$.

Bruck et al.[1] gave a definition which is somewhere between these two. T is called asymptotically nonexpansive mapping in the intermediate sense [1] provided $T$is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

T is said to be uniformly $L-$Lipschitzian if there exist a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and all $n \geq 1$.

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically in the intermediate sense. It is known [16] that if $C$is a nonempty closed bounded subset of a uniformly convex Banach space and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense that $F(T) \neq \varphi$.

Example 1 ([8])**.** Let $X = R$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and $|k| < 1$. For each $x \in C$ define

$$T(x) = \begin{cases} kx \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is asymptotically nonexpansive mapping in the intermediate sense, but is not a Lipschitzian mapping but $T^n x \to 0$ uniformly so that it is not asymptotically nonexpansive mapping.

Iterative methods for asymptotically nonexpansive mappings in Banach spaces including Mann and Ishikawa iteration process have been further studied by various authors to solve the nonlinear operator equations as well as variational inequalities in Hilbert spaces and Banach spaces. Noor [14,15] introduced three step iterative methods and study the approximate solution of variational inequalities in Hilbert spaces by using the technique of updating the solution and the auxiliary principle. Glowinski and Le Tallec [3] used three step iterative scheme to solve the elastoviscoplasticity problem, liquid crystal theory and eigenvalue problems. In 1998, Haubruge, Nguyen and Strocliot [5] studied the convergence analysis of three scheme of [3] and obtain new splitting type algorithm for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. Recently Xu and Noor [26] introduced and studied a three step iteration with errors scheme to approximate fixed points of asymptotically nonexpansive mappings of asymptotically in Bach spaces. Cho et al.[2] extended the worf of Xu and Noor to three step iteration scheme with errors and gave weak & strong convergence theorems for asymptotically mappings. Moreover, Suantai [22] gave weak and strong convergence theorem for a new three step iterative scheme which can be viewed as an extension for three step and two step iterative schemes of Glowinski and Le Tallec [3], Noor [14], Xu and Noor [26], Ishikawa [7].

Inspired and motivated by these works we introduced three step iterative scheme for asymptotically nonexpansive mappings in the intermediate sense as follows:

Let X be a normed space, Cbe a nonempty convex subset of X and $T : C \to C$ be a given mapping. Then for a given $x_1 \in C$ compute the sequence $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by the iterative scheme of asymptotically nonexpansive mappings.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\beta_n-\gamma_n)x_n+\alpha_n T^n y_n+\beta_n T^n z_n+\gamma_n T^n x_n \\ y_n &= (1-b_n-c_n)x_n+b_n T^n z_n+c_n T^n x_n \\ z_n &= (1-a_n)x_n+a_n T^n x_n \qquad n\geq 1 \end{aligned} \tag{1}$$

where $\{a_n\},\{b_n\},\{c_n\},\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{b_n+c_n\},\{\alpha_n+\beta_n+\gamma_n\}$ are appropriate sequences in $[0,1]$. The iterative scheme (1) is called the modified Noor iteration scheme [21]. Noor iterations include the Mann & Ishikawa iterations as special cases. If $c_n=\beta_n=\gamma_n$ then (1) reduces to Noor iteration defined by Xu and Noor [26]:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n+\alpha_n T^n y_n \\ y_n &= (1-b_n)x_n+b_n T^n z_n \\ z_n &= (1-a_n)x_n+a_n T^n x_n \qquad n\geq 1 \end{aligned} \tag{2}$$

where $\{a_n\},\{b_n\},\{\alpha_n\}$are appropriate sequences in $[0,1]$.

For $a_n=c_n=\beta_n=\gamma_n=0$ then (1) reduces to the usual Ishikawa iterative scheme.

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n+\alpha_n T^n y_n \\ y_n &= (1-b_n)x_n+b_n T^n x_n, \qquad n\geq 1 \end{aligned} \tag{3}$$

Where $\{b_n\},\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

For $a_n=b_n=c_n=\beta_n=\gamma_n=0$ then (1) reduces to the usual Mann iteration.

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n x_n, \qquad n\geq 1 \tag{4}$$

Where $\{\alpha_n\}$ are appropriate sequences in $[0,1]$.

The purpose of this paper is to establish several strong and weal convergence theorems using (1) for asymptotically nonexpansive mappings in the intermediate sense in the uniformly convex Banach space.

2. Preliminaries

To prove our main results we recall some well results and definitions.

Definition 1 ([16]). *A Banach space X is said to satisfy Opial's condition if $x_n\to x$ and $x\neq y$ imply* $\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|$.

Definition 2 ([1]). *A Banach space X is said to satisfy $\tau-$Opial's condition if for every bounded sequence $\{x_n\} \in X$ that $\tau-$converges to $x \in X$ then*

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for every $x \neq y$, where τ is a Hausdorff linear topology on X.

A Banach space X has the uniform $\tau-$Opial property if for each $c > 0$ there exist $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ such that $\{x_n\}$ is $\tau-$convergent to 0 and $1 \leq \limsup_{n\to\infty} \|x_n\| < \infty$, $\|x\| \geq c$ imply that $\limsup_{n\to\infty} \|x_n - x\| \geq 1 + r$.

Cleary uniform $\tau-$Opial property implies $\tau-$Opial's condition. Note that a uniformly convex space which has the $\tau-$Opial condition necessarily has the uniform τ-Opial property, τ is a Hausdorff linear topology on X.

Definition 3 ([21]). *Let $\{x_n\}$ be a sequence in C. A mapping $T : C \to C$ with nonempty fixed point set $F(T)$ in C is said to satisfy condition (A) with respect to the sequence $\{x_n\}$ if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $f(d(x_n, F(T))) \leq \|x_n - Tx_n\|$ for all $n \geq 1$.*

Lemma 1 ([23]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n \quad \text{for all } \; n = 1, 2, \cdots$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then

(*a*) $\lim_{n\to\infty} a_n$ *exists;*

(*b*) $\lim_{n\to\infty} a_n = 0$ whenever $\liminf_{n\to\infty} a_n = 0$.

Lemma 2 ([24]). *Let $p > 1$, $r > 0$ be two fixed numbers. Then a Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in X : \|x\| \leq r\}$ and $\lambda \in [0, 1]$, where $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 3 ([2]). *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|)$$

for all $x, y \in B_r$ *and* $\lambda, \beta, \gamma \in [0,1]$, *with* $\lambda + \beta + \gamma = 1$.

From Lemma 3 we easily get

Lemma 4. *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. *Then there exists a continuous, strictly increasing and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ *such that*

$$\begin{aligned}\|\lambda x + \beta y + \gamma z\|^2 \ \leq\ & \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ & - \frac{1}{2}\gamma(\lambda g(\|x - z\|) + \beta g(\|y - z\|))\end{aligned}$$

for all $x, y \in B_r$ *and* $\lambda, \beta, \gamma \in [0,1]$, *with* $\lambda + \beta + \gamma = 1$.

Lemma 5 ([13])**.** *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. *Then there exists a continuous, strictly increasing and convex function* $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ *such that*

$$\begin{aligned}\|\lambda x + \mu y + \xi z + \nu w\|^2 \ \leq\ & \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2 \\ & - \frac{1}{3}\nu(\lambda g(\|x - w\|) + \mu g(\|y - w\|) + \xi g(\|z - w\|)).\end{aligned}$$

For every $x, y, z, w \in B_r$ *and* $\lambda, \mu, \xi, \nu \in [0,1]$, *with* $\lambda + \mu + \xi + \nu = 1$.

Lemma 6 ([1])**.** *Suppose a Banach space* X *has the uniformly* τ*-Opial's condition.* C *is a norm bounded sequentially* τ*-compact subset of a* X *and* $T : C \to C$ *is asymptotically nonexpansive in the weak sense. If* $\{y_n\}$ *is a sequence in* C *such that* $\lim_{n\to\infty} \|y_n - z\|$ *exists for each fixed point* z *of* T *and if* $\{y_n - T^k y_n\}$ *is* τ*-convergent to 0 for each* $k \in N$, *then* $\{y_n\}$ *is* τ*-convergent to a fixed point of* T.

3. Main results

In this section we prove our main results. For we begin with the following lemmas.

Lemma 7. *Let* X *be a uniformly convex Banach space and let* C *be a nonempty closed, bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive mapping in the intermediate sense.*
Put $d_n = \sup\limits_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0$, $\forall n \geq 1$ *so that* $\sum_{n=1}^{\infty} d_n < \infty$.

Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n + \gamma_n$ *and* $b_n + c_n$ *in* $[0,1]$ *for all* $n \geq 1$. *For a given* $x_1 \in C$, *let the sequence* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (1).*

(i) *If* q *is a fixed point of* T, *then* $\lim\limits_{n\to\infty} \|x_n - q\|$ *exists*;

(ii) *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$ *then* $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$;

(iii) *If* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$ *then* $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$;

(iv) *If* $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$ *or*

$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$ *and*

$\limsup_{n\to\infty} (b_n + c_n) < 1$ *then* $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$.

Proof. Existence of fixed point of T follows from [9].

So, $F(T) \neq \varphi$. Let $x^* \in F(T)$. Choose a real number $r > 0$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma 2, there exists a continuous, strictly increasing and convex function $g_1 : [0, \infty) \to [0, \infty)$, $g_1(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - w_2(\lambda)g_1(\|x - y\|) \tag{5}$$

for all $x, y \in B_r = \{x \in X : \|x\| \le r\}$ and $\lambda \in [0, 1]$, where $w_2(\lambda) = \lambda(1-\lambda)^2 + \lambda^2(1-\lambda)$. From (5) and (1), we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|(1-a_n)(x_n - x^*) + a_n(T^n x_n - x^*)\|^2 \\ &\le a_n\|T^n x_n - x^*\|^2 + (1-a_n)\|x_n - x^*\|^2 \\ &\quad - w_2(a_n)g_1(\|T^n x_n - x_n\|) \\ &\le a_n(\|x_n - x^*\| + d_n)^2 + (1-a_n)\|x_n - x^*\|^2 \\ &\quad - w_2(a_n)g_1(\|T^n x_n - x_n\|) \\ &\le \|x_n - x^*\|^2 + 2d_n a_n\|x_n - x^*\| + a_n d_n^2. \end{aligned} \tag{6}$$

By Lemma 4, there exists a continuous strictly increasing and convex function $g_2 : [0, \infty) \to [0, \infty)$, $g_2(0) = 0$ such that

$$\begin{aligned} \|\lambda x + \beta y + \gamma z\|^2 &\le \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 \\ &\quad - \frac{1}{2}\gamma(\lambda g_2(\|x - z\|) + \beta g_2(\|y - z\|)) \end{aligned} \tag{7}$$

for all $x, y, z \in B_r$ and all $\lambda, \beta, \gamma \in [0, 1]$ with $\lambda + \beta + \gamma = 1$. It follows from (6) and (1) that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|b_n(T^n z_n - x^*) + (1-b_n-c_n)(x_n - x^*) + c_n(T^n x_n - x^*)\|^2 \\ &\le b_n\|T^n z_n - x^*\|^2 + (1-b_n-c_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2 \\ &\quad - \frac{1}{2}(1-b_n-c_n)(b_n g_2(T^n z_n - x_n) + c_n g_2(T^n x_n - x_n) \end{aligned} \tag{8}$$

$$
\begin{aligned}
&\leq\ b_n(\|z_n-x^*\|+d_n)^2+(1-b_n-c_n)\|x_n-x^*\|^2+c_n(\|x_n-x^*\|+d_n)^2\\
&\quad -\frac{1}{2}b_n(1-b_n-c_n)g_2(\|T^nz_n-x_n\|)\\
&=\ b_n\|z_n-x^*\|^2+2b_nd_n\|z_n-x^*\|+b_nd_n^2\\
&\quad +(1-b_n-c_n)\|x_n-x^*\|^2+c_n\|x_n-x^*\|^2\\
&\quad +2c_nd_n\|x_n-x^*\|+c_nd_n^2-\frac{1}{2}b_n(1-b_n-c_n)g_2(\|T^nz_n-x_n\|)
\end{aligned}
$$

By Lemma 5, there exist a continuous strictly increasing and convex function $g_3:[0,\infty)\to[0,\infty)$, with $g_3(0)=0$ such that

$$
\begin{aligned}
\text{(9)}\qquad \|\lambda x+\mu y+\xi z+\nu w\|^2 \ \leq\ & \lambda\|x\|^2\mu\|y\|^2+\xi\|z\|^2+\nu\|w\|^2\\
&-\frac{1}{3}\nu(\lambda g_3(\|x-w\|)+\mu g_3(\|y-w\|)\\
&+\xi g_3(\|z-w\|))
\end{aligned}
$$

for all $x,y,z,w\in B_r$ and all $\lambda,\mu,\xi,\nu\in[0,1]$ with $\lambda+\mu+\xi+\nu=1$. It follows from (1) and using (6), (7), (8), (9), we get

$$
\begin{aligned}
\text{(10)}\quad \|x_{n+1}-x^*\|^2 &= \|\alpha_n(T^ny_n-x^*)+(1-\alpha_n-\beta_n-\gamma_n)(x_n-x^*)\\
&\quad +\beta_n(T^nz_n-x^*)+\gamma_n(T^nx_n-x^*)\|^2\\
&\leq \alpha_n\|T^ny_n-x^*\|^2+\beta_n\|T^nz_n-x^*\|^2+\gamma_n\|T^nx_n-x^*\|^2\\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-x^*\|^2-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\\
&\quad\ \times\{\alpha_ng_3(\|T^ny_n-x_n\|)+\beta_ng_3(\|T^nz_n-x_n\|)+\gamma_ng_3(\|T^nx_n-x_n\|)\}\\
&\leq \alpha_n(\|y_n-x^*\|+d_n)^2+\beta_n(\|z_n-x^*\|+d_n)^2+\gamma_n(\|x_n-x^*\|+d_n)^2\\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-x^*\|^2-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\\
&\quad \times\{\alpha_ng_3(\|T^ny_n-x_n\|)+\beta_ng_3(\|T^nz_n-x_n\|)+\gamma_ng_3(\|T^nx_n-x_n\|)\}\\
&\leq \alpha_n[(\|y_n-x^*\|^2+2d_n\|y_n-x^*\|+d_n^2)]+\beta_n[(\|z_n-x^*\|^2\\
&\quad +2d_n\|z_n-x^*\|+d_n^2]+\gamma_n[(\|x_n-x^*\|^2+2d_n\|x_n-x^*\|+d_n^2)]\\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-x^*\|^2-\frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\\
&\quad \times\{\alpha_ng_3(\|T^ny_n-x_n\|)+\beta_ng_3(\|T^nz_n-x_n\|)+\gamma_ng_3(\|T^nx_n-x_n\|)\}\\
&\leq \Big\{b_n\|z_n-x^*\|^2+2b_nd_n\|z_n-x^*\|+b_nd_n^2+(1-b_n-c_n)\|x_n-x^*\|^2\\
&\quad +c_n\|x_n-x^*\|^2+2c_nd_n\|x_n-x^*\|+c_nd_n^2\\
&\quad -\frac{1}{2}b_n(1-b_n-c_n)g_2(T^nz_n-x_n)\Big\}\\
&\quad +2\alpha_nd_n\{(1-b_n-c_n)\|x_n-x^*\|+b_n\|x_n-x^*\|\\
&\quad +a_nb_nd_n+c_n\|x_n-x^*\|+c_nd_n\}+\alpha_nd_n^2+\beta_n\{\|x_n-x^*\|^2\\
&\quad +2a_nd_n\|x_n-x^*\|+a_nd_n^2\}+2\beta_nd_n\{\|x_n-x^*\|+a_nd_n\}\\
&\quad +\beta_nd_n^2+\gamma_n\|x_n-x^*\|^2+\gamma_nd_n^2+2\gamma_nd_n\|x_n-x^*\|\\
&\quad +(1-\alpha_n-\beta_n-\gamma_n)\|x_n-x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\{\alpha_n g_3(\|T^n y_n - x_n\|) \\
& + \beta_n g_3(\|T^n z_n - x_n\|) + \gamma_n g_3(\|T^n x_n - x_n\|)\} \\
\leq & \Big\{ b_n\|x_n - x^*\|^2 + 2a_n b_n d_n\|x_n - x^*\| + a_n b_n d_n^2 + 2b_n d_n\|x_n - x^*\| \\
& + b_n d_n^2 + (1-b_n-c_n)\|x_n - x^*\|^2 + c_n\|x_n - x^*\|^2 \\
& + 2c_n d_n\|x_n - x^*\| + c_n d_n^2 - \frac{1}{2} b_n(1-b_n-c_n) g_2(T^n z_n - x_n) \Big\} \\
& + 2\alpha_n(1-b_n-c_n)d_n\|x_n - x^*\| + 2\alpha_n b_n d_n\|x_n - x^*\| \\
& + 2a_n\alpha_n b_n d_n^2 + 2\alpha_n c_n d_n\|x_n - x^*\| + 2\alpha_n c_n d_n^2 + 2\alpha_n d_n^2 + \beta_n\|x_n - x^*\|^2 \\
& + 2a_n\beta_n d_n\|x_n - x^*\| + 2a_n\beta_n d_n^2 + 2\beta_n d_n\|x_n - x^*\| + 2a_n\beta_n d_n^2 + \beta_n d_n^2 \\
& + \gamma_n\|x_n - x^*\|^2 + \gamma_n d_n^2 + 2\gamma_n d_n\|x_n - x^*\| + (1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\|^2 \\
& - \frac{1}{2} b_n(1-b_n-c_n) g_2(T^n z_n - x_n) - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n) \\
& \times \{\alpha_n g_3(\|T^n y_n - x_n\|) + \beta_n g_3(\|T^n z_n - x_n\|) + \gamma_n g_3(\|T^n x_n - x_n\|)\} \\
\leq & \|x_n - x^*\|^2 + [2a_n b_n\alpha_n + 2\alpha_n b_n + 2\alpha_n c_n + 2a_n b_n + 2\alpha_n + 2\beta_n + 2\gamma_n] \\
& \times d_n\|x_n - x^*\| + [3a_n b_n\alpha_n d_n + 3\alpha_n c_n d_n + 3a_n\beta_n d_n + \alpha_n b_n d_n \\
& + (\alpha_n+\beta_n+\gamma_n)d_n]d_n \leq \|x_n - x^*\|^2 \\
& + \Big\{ [2a_n b_n\alpha_n + 2\alpha_n(b_n+c_n) + 2a_n b_n + 2\alpha_n + 2\beta_n + 2\gamma_n]\|x_n - x^*\| \\
& + [3a_n b_n\alpha_n d_n + 3\alpha_n c_n d_n + 3a_n\beta_n d_n + \alpha_n b_n d_n + (\alpha_n+\beta_n+\gamma_n)d_n] \Big\} d_n.
\end{aligned}
$$

Since $\{d_n\}$ and K are bounded, so there exist a constant $M > 0$ such that

$$
\begin{aligned}
(11) \qquad & [2a_n b_n\alpha_n + 2\alpha_n(b_n+c_n) + 2a_n b_n + 2\alpha_n + 2\beta_n + 2\gamma_n]\|x_n - x^*\| \\
& + [3a_n b_n\alpha_n d_n + 3\alpha_n c_n d_n + 3a_n\beta_n d_n + \alpha_n b_n d_n \\
& + (\alpha_n+\beta_n+\gamma_n)d_n] \leq M
\end{aligned}
$$

From (10) and (11), we get

$$
\begin{aligned}
(12) \quad \|x_{n+1} - x^*\|^2 \leq & \|x_n - x^*\|^2 + Md_n \\
& - \frac{1}{2}\alpha_n b_n(1-b_n-c_n) g_2(\|T^n z_n - x_n\|) \\
& - \frac{1}{3}(1-\alpha_n-\beta_n-\gamma_n)\{\alpha_n g_3(\|T^n y_n - x_n\|) \\
& + \beta_n g_3(\|T^n z_n - x_n\|) + \gamma_n g_3(\|T^n x_n - x_n\|)\}
\end{aligned}
$$

Thus it follows from (12) that

$$
\begin{aligned}
(13) \qquad & \alpha_n(1-\alpha_n-\beta_n-\gamma_n) g_3(\|T^n y_n - x_n\|) \\
& \leq 3(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + Md_n)
\end{aligned}
$$

$$
\begin{aligned}
(14) \qquad & \beta_n(1-\alpha_n-\beta_n-\gamma_n) g_3(\|T^n z_n - x_n\|) \\
& \leq 3(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + Md_n)
\end{aligned}
$$

$$\text{(15)} \qquad \gamma_n(1-\alpha_n-\beta_n-\gamma_n)g_3(\|T^n x_n - x_n\|) \\ \leq 3(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + Md_n)$$

$$\text{(16)} \qquad \alpha_n b_n(1-b_n-c_n)g_2(\|T^n z_n - x_n\|) \\ \leq 2(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + Md_n).$$

(i) If $q \in F(T)$, by taking $x^* = q$ in (12), we get

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + Md_n$$

Since $\sum_{n=1}^{\infty} d_n < \infty$, so from Lemma 1, we get $\lim_{n\to\infty} \|x_n - q\|$ exist.

(ii) If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(1-\alpha_n-\beta_n) < 1$, then there exists $n_0 \in N$ and $\eta, \eta' \in (0,1)$ such that $0 < \eta < \alpha_n$ and $\alpha_n + \beta_n + \gamma_n < \eta' < 1$.

From (13),

$$\eta(1-\eta')g_3(\|T^n y_n - x_n\|) \leq 3(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + Md_n) \ \ \forall\, n \geq n_0.$$

Thus from the above relation we have

$$\eta(1-\eta')\sum_{n=n_0}^{\infty} g_3(\|T^n y_n - x_n\|) \leq 3(\|x_{n_0} - x^*\|^2 + M\sum_{n=n_0}^{\infty} d_n) < \infty.$$

This implies that $\sum_{n=n_0}^{\infty} g_3(\|T^n y_n - x_n\|) < \infty$ which implies that $\lim\limits_{n\to\infty} g_3(\|T^n y_n - x_n\|) = 0$.

Since g_3 is strictly increasing and continuous at zero with $g\,(0) = 0$ so we have

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0$$

(iii) Similarly if $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty}(1-\alpha_n-\beta_n) < 1$. Then from (14) and using the same process as in (ii) we can show that

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$$

(iv) Finally, if $0 < \liminf\limits_{n\to\infty} \gamma_n \leq \limsup\limits_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$. Then from (15) we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

On the other hand, using (16) we can also prove that $\lim\limits_{n\to\infty} \|T^n x_n - x_n\| = 0$ as follows:

Consider

$$\|T^n x_n - x_n\| \leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\ \leq \|x_n - y_n\| + d_n + \|T^n y_n - x_n\|.$$

Again $\|y_n - x_n\| \leq b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\|$.

Thus

$$(17)\ \|T^n x_n - x_n\| \leq \|x_n - y_n\| + d_n + \|T^n y_n - x_n\|$$
$$\leq b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + d_n + \|T^n y_n - x_n\|.$$

Let $\{m_j\}$ be a subsequence of $\{n\}$. If $\liminf_{j\to\infty} b_{m_j} > 0$, then from (16), we get

$$\lim_{j\to\infty} \|T^{m_j} z_{m_j} - x_{m_j}\| = 0.$$

Again, since $0 < \liminf_{n\to\infty} \alpha_n \leq \lim\sup_{n\to\infty} (\alpha_n + \beta_n + \gamma_n) < 1$, then from (ii) we get

$$\lim_{j\to\infty} \left\|T^{m_j} y_{m_j} - x_{m_j}\right\| = 0.$$

So from (17), we get

$$\lim_{j\to\infty} (1 - c_{m_j})\|T^{m_j} x_{m_j} - x_{m_j}\| = 0.$$

Thus we have

$$\lim_{j\to\infty} \left\|T^{m_j} x_{m_j} - x_{m_j}\right\| = 0.$$

Again, if $\liminf_{j\to\infty} b_{m_j} = 0$, then there exist a subsequence $\{b_{n_k}\}$ of $\{b_{m_j}\}$ such that

$$\lim_{k\to\infty} b_{n_k} = 0.$$

From (ii) and (17), we get

$$\lim_{k\to\infty} (1 - c_{n_k})\|T^{n_k} x_{n_k} - x_{n_k}\| = 0.$$

Thus we have

$$\lim_{k\to\infty} \|T^{n_k} x_{n_k} - x_{n_k}\| = 0.$$

Hence

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$$

■

Theorem 1. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be an asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0$, for all $n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1) with $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$,*

$\{\gamma_n\}$ be real sequences in [0,1] such that $\alpha_n+\beta_n+\gamma_n$ and b_n+c_n are in $[0,1]$ for all $n\geq 1$ and

(i) $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n)<1$ or

$0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}(\alpha_n+\beta_n+\gamma_n)<1$ and

(ii) $\limsup_{n\to\infty}(b_n+c_n)<1$

If T satisfies condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. By Lemma 7 (iv) we have $\lim_{n\to\infty}\|T^nx_n-x_n\|=0$. Now

$$\begin{aligned}(18)\ \|T^nz_n-x_n\| &\leq \|T^nz_n-T^nx_n\|+\|T^nx_n-x_n\|\\ &\leq \|z_n-x_n\|+\|T^nx_n-x_n\|+d_n\\ &= a_n\|T^nx_n-x_n\|+d_n+\|T^nx_n-x_n\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Again

$$\begin{aligned}(19)\qquad \|T^ny_n-x_n\| &\leq \|T^ny_n-T^nx_n\|+\|T^nx_n-x_n\|\\ &\leq \|y_n-x_n\|+\|T^nx_n-x_n\|+d_n\\ &= b_n\|T^nz_n-x_n\|+c_n\|T^nx_n-x_n\|\\ &\quad +d_n+\|T^nx_n-x_n\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Now

$$\|x_{n+1}-x_n\|\leq\alpha_n\|T^ny_n-x_n\|+\beta_n\|T^nz_n-x_n\|+\gamma_n\|T^nx_n-x_n\|\to 0 \text{ as } n\to\infty.$$

$$\begin{aligned}(20)\qquad \|x_n-Tx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|\\ &\quad +\|T^{n+1}x_{n+1}-T^{n+1}x_n\|+\|T^{n+1}x_n-Tx_n\|\\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T^{n+1}x_{n+1}\|\\ &\quad +\|x_{n+1}-x_n\|+d_{n+1}+\|T^{n+1}x_n-Tx_n\|\\ &= 2\|x_{n+1}-x_n\|+\|x_{n+1}-T^{n+1}x_{n+1}\|\\ &\quad +d_{n+1}+\|T^{n+1}x_n-Tx_n\|.\end{aligned}$$

As T is uniformly continuous, we have

$$(21)\qquad \lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Since T satisfies condition (A) with respect to the sequence $\{x_n\}$ so there is a nondecreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$f(d(x_n,F(T)))\leq\|x_n-Tx_n\|\to 0 \text{ as } n\to\infty.$$

So $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now by (1) we get

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\| + \alpha_n\|T^n y_n - x^*\| \\ &\quad + \beta_n\|T^n z_n - x^*\| + \gamma_n\|T^n x_n - x^*\| \\ &\le (1-\alpha_n-\beta_n-\gamma_n)\|x_n - x^*\| + \alpha_n(\|y_n - x^*\| + d_n) \\ &\quad + \beta_n(\|z_n - x^*\| + d_n) + \gamma_n(\|x_n - x^*\| + d_n) \\ &\le \|x_n - x^*\| + (\alpha_n b_n + 2\alpha_n + 2\beta_n + \gamma_n)d_n \\ &\le \|x_n - x^*\| + 6d_n\end{aligned}$$

Therefore

$$\begin{aligned}\|x_{n+m} - x^*\| &\le \|x_{n+m-1} - x^*\| + 6d_{n+m-1} \\ &\le \|x_{n+m-2} - x^*\| + 6d_{n+m-2} + 6d_{n+m-1} \\ &\quad \vdots \\ &\le \|x_n - x^*\| + 6\sum_{j=n}^{n+m-1} d_j.\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} d_n < \infty$, for given $\varepsilon > 0$, there exists $N_0 \in N$ such that $d(x_n, F(T)) < \frac{\varepsilon}{2}$, $\sum_{j=n}^{\infty} d_j < \frac{\varepsilon}{24}$, for all $n \ge N_0$. In particular, $d(x_{N_0}, F(T)) < \frac{\varepsilon}{4}$. So there exists $q \in F(T)$ such that $\|x_{N_0} - q\| = d(x_{N_0}, q) < \frac{\varepsilon}{4}$.

From (18) we get

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - q\| + \|x_n - q\| \\ &\le \|x_n - q\| + 6\sum_{j=n}^{n+m-1} d_j + \|x_n - q\| \\ &\le 2\|x_n - q\| + 6\sum_{j=n}^{n+m-1} d_j \\ &\le 2\|x_{N_0} - q\| + 12\sum_{j=N_0}^{n-1} d_j + 6\sum_{j=n}^{n+m-1} d_j \\ &\le 2\|x_{N_0} - q\| + 12\sum_{j=N_0}^{\infty} d_j \\ &< 2\frac{\varepsilon}{4} + 12\frac{\varepsilon}{24} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Hence $\{x_n\}$is a Cauchy sequence in C. So by completeness of C we get there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$. By (17) and continuity of T we get $Tp = p$ that is $p \in F(T)$. This completes the proof of the theorem. ■

For $\gamma_n = 0$ in Theorem 1 we obtain the following result.

Corollary 1. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be an asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, for all $n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1) (with $\gamma_n = 0$) with $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ be real sequences in $[0,1]$ such that $\alpha_n + \beta_n$ and $b_n + c_n$ are in $[0,1]$ for all $n \geq 1$ and*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$

(*ii*) $\limsup_{n\to\infty}(b_n + c_n) < 1$.

If T satisfies condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

For $c_n = \beta_n = \gamma_n = 0$ in Theorem 1 we obtain the following result.

Corollary 2. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be an asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, for all $n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (2) with $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ and*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

For $a_n = c_n = \beta_n = \gamma_n = 0$ in Theorem 1 we obtain the following result.

Corollary 3. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be an asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, for all $n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (3) with $\{b_n\}$,$\{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ and*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T satisfies condition (A) with respect to the sequence $\{x_n\}$, then $\{x_n\}$ converges strongly to a fixed point of T.

Corollary 4. *Let X be a uniformly convex Banach space and C be a nonempty closed bounded and convex subset of X. Let T be an asymptotically nonexpansive self map of C in the intermediate sense. Put $d_n =$*

$\sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, *for all* $n \geq 1$ *so that* $\sum_{n=1}^{\infty} d_n < \infty$. *Let* $\{x_n\}$ *be the sequence defined as in (4) with*$\{\alpha_n\}$ *be real sequences in* $[0,1]$ *for all* $n \geq 1$ *such that* $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$.
If T *satisfies condition* (A) *with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Since every asymptotically nonexpansive mapping is uniformly continuous, we immediately get.

Corollary 5. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed, bounded and convex subset of* X*. Let* T *be asymptotically nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{x_n\}$ *be the sequence defined as in (1) with* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n + \gamma_n$ *and* $b_n + c_n$ *are in* $[0,1]$ *for all* $n \geq 1$ *and*

(i) $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$,

(ii) $\limsup_{n\to\infty}(b_n + c_n) < 1$.

If T *satisfies condition (A) with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y\in C}\|x-y\| < \infty$, so the conclusion follows immediately from Theorem 1. ■

Corollary 6. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed, bounded and convex subset of* X*. Let* T *be asymptotically nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$*. Let* $\{x_n\}$ *be the sequence defined as in (1) (for* $\gamma_n = 0$*) with* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{\alpha_n\}$*,* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *such that* $\alpha_n + \beta_n$ *and* $b_n + c_n$ *are in* $[0,1]$ *for all* $n \geq 1$ *and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$

(ii) $\limsup_{n\to\infty}(b_n + c_n) < 1$

If T *satisfies condition* (A) *with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. The conclusion follows from Corollary 1. ■

Remark 1. Corollary 6 includes Theorem 2.2 and Theorem 2.3 of [26].

For $c_n = \beta_n = \gamma_n = 0$ in Theorem 1 we obtain the following result which improves Theorem 2.1 of [25].

Corollary 7. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed, bounded and convex subset of* X*. Let* T *be asymptotically* *nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be the sequence defined as in (2) with* $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ *be real sequences in [0,1] such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ *and*

(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T *satisfies condition* (A) *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. The conclusion follows from Corollary 2. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y\in C} \|x - y\| < \infty$. ■

For $a_n = c_n = \beta_n = \gamma_n = 0$ in Theorem 1 we obtain the following result.

Corollary 8. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed, bounded and convex subset of* X. *Let* T *be asymptotically nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be the sequence defined as in (3) with* $\{b_n\}$, $\{\alpha_n\}$ *be real sequences in* $[0, 1]$ *such that*

(*i*) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$

(*ii*) $\limsup_{n\to\infty} b_n < 1$.

If T *satisfies condition* (A) *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. The conclusion follows from Corollary 3. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y\in C} \|x - y\| < \infty$. ■

For $a_n = b_n = c_n = \beta_n = \gamma_n = 0$ in Theorem 1 we obtain the following result.

Corollary 9. *Let* X *be a uniformly convex Banach space and* C *be a nonempty closed, bounded and convex subset of* X. *Let* T *be asymptotically nonexpansive self map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{x_n\}$ *be the sequence defined as in (4) with* $\{\alpha_n\}$ *be real sequences in* $[0, 1]$ *such that*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

If T *satisfies condition* (A) *with respect to the sequence* $\{x_n\}$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. The conclusion follows from Corollary 3. Since $\sum_{n=1}^{\infty} d_n = \sum_{n=1}^{\infty}(k_n - 1)diam(C) < \infty$, where $diam(C) = \sup_{x,y\in C} \|x - y\| < \infty$. ■

Remark 2. Corollary 9 extends Theorem 2.6 of [22].

Theorem 2. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (1) with $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n$ and $b_n + c_n$ in $[0,1]$ for all $n \geq 1$ and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \gamma_n) < 1$

(ii) $\limsup_{n\to\infty}(b_n + c_n) < 1$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. From (18) we get $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and so $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$ for all $m \in N$ by the uniform continuity of T. Then on applying Lemma 2 with the τ-topology taken as weak topology and get the conclusion as follows: $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} \|x_n - z\|$ exists for each fixed point $z \in F(T)$. Since $\lim_{n\to\infty} \|x_n - T^m x_n\| = 0$ for all $m \in N$ so $\{x_n - T^m x_n\}$ is weakly convergent to zero for each $m \in N$. So by Lemma 2 we get that $\{x_n\}$ converges weakly to a fixed point of T. ■

For $c_n = \beta_n = \gamma_n = 0$ in Theorem 2 we obtain the following result.

Corollary 10. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (2) with $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ and*

(i) $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1,$

(ii) $\limsup\limits_{n\to\infty} b_n < 1$

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = c_n = \beta_n = \gamma_n = 0$ in Theorem 2 we get the following result.

Corollary 11. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (3) with $\{b_n\}$, $\{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ and*

(i) $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$

(ii) $\limsup\limits_{n\to\infty} b_n < 1$

Then $\{x_n\}$ converges weakly to a fixed point of T.

For $a_n = b_n = c_n = \beta_n = \gamma_n = 0$ in Theorem 2 we get the following result.

Corollary 12. *Let X be a uniformly convex Banach space which satisfies Opial's condition and C be a nonempty closed, bounded and convex subset of X. Let $T : C \to C$ be an asymptotically nonexpansive mapping in the intermediate sense. Put $d_n = \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x-y\|) \vee 0$, $\forall n \geq 1$ so that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{x_n\}$ be the sequence defined as in (4) with $\{\alpha_n\}$ be real sequences in $[0,1]$ for all $n \geq 1$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

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