F A S C I C U L I M A T H E M A T I C I

Nr 49 2012

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STATISTICAL CONVERGENCE OF SEQUENCES OF SETS

Abstract. The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The three such extensions that we will consider in this paper are those of Kuratowski, Wijsman and Hausdorff. We shall define statistical convergence for sequences of sets and establish some basic theorems, thereby obtaining generalizations of the corresponding results for statistical convergence of sequences of points.

Key words: statistical convergence, sequence of sets, Kuratowski convergence, Wijsman convergence, Hausdorff convergence, almost convergence.

AMS Mathematics Subject Classification: 40A05, 46E25.

1. Introduction and background

The natural density of a set K of positive integers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n.

Statistical convergence of sequences of points was introduced by Fast [8]. In [17] Schoenberg established some basic properties of statistical convergence and also studied the concept as a summability method.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case we write $st - \lim x_k = L$. $\lim x_k = L$ implies $st - \lim x_k = L$, so statistical convergence may be considered as a regular summability method. This was observed by Schoenberg [17] along with the fact that the statistical limit is a linear functional on some sequence spaces.

If $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that x_k satisfies P for almost all k, and we abbreviate this by "a.a. k.". In [11], Fridy proved that if x is a statistically convergent sequence then there is a convergent sequence y such that $x_k = y_k$ a.a.k.

The concepts of statistical limit superior and statistical limit inferior were introduced by Fridy and Orhan [12]: For a real number sequence $x = (x_k)$ let B_x denote the set

$$B_x := \{b \in \mathbb{R} : \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k > b\}| \neq 0\},$$

and, similarly,

$$A_x := \{a \in \mathbb{R} : \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : x_k < a\}| \neq 0\}.$$

If x is a real number sequence, then the statistical limit superior of x is defined by

$$st - \limsup x := \begin{cases} \sup B_x, & B_x \neq \emptyset \\ -\infty, & B_x = \emptyset. \end{cases}$$

Similarly, the statistical limit inferior of x is defined by

$$st - \liminf x := \begin{cases} \inf A_x, & A_x \neq \emptyset \\ \infty, & A_x = \emptyset. \end{cases}$$

Limit of sequences of sets have been introduced by Painleve in 1902, as is reported by his student Zoretti. They have been popularized by Kuratowski in his famous book *Topologie* and thus, often called Kuratowski limit of sequences. Although set convergence, introduced by Painleve , has a long mathematical history, it is only during the last three decades that it has started to be viewed as a major tool for dealing with approximations in optimization, systems of equations and related objects.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

Let $\{A_k\}$ be a sequence of sets in a metric space (X, ρ). Define the lower limit and upper limit of the sequence $\{A_k\}$ as follows:

$$\liminf A_k := \{x \in X : \exists(a_k) \subset (A_k), a_k \to x\}$$

and

$$\limsup A_k := \{x \in X : \exists(k_l)\exists(a_{k_l}) \subset (A_{k_l}) a_{k_l} \to x\}$$

where (k_l) denotes an increasing sequence of natural numbers and represents the index set for a subsequence.

The lower limit of a sequence of subsets $\{A_k\}$ is the set of limits of sequences of elements $a_k \in A_k$ and the upper limit is the set of cluster points of such sequences. Lower and upper limits are obviously closed. Clearly, $\liminf A_k \subset \limsup A_k$ and the upper limit and lower limit of subsets $\{A_k\}$ and of their closures $\{\overline{A_k}\}$ do coincide, since $d(x, A_k) = d(x, \overline{A_k})$.

A subset A of X is said to be the limit or the set limit of the sequence $\{A_k\}$ if

$$A = \liminf A_k = \limsup A_k = \lim A_k.$$

Alternatively, in the literature, convergence in this sense is called Painleve-Kuratowski convergence, topological convergence, or closed convergence(see, for example, [1],[2],[3], [4],[5]) Any decreasing sequence of subsets of A_k has a limit, which is the intersection of their closures:

If $A_n \subset A_m$ when $n \geq m$, then

$$\lim A_k = \bigcap_{k \geq 0} \overline{A_k}.$$

An upper limit may be empty(no sequence of elements $a_k \in A_k$ has a cluster point). Concerning sequences of singleton $\{a_k\}$, the set limit, when it is exists, is either empty (the sequence of elements a_k is not converging), or is a singleton made of the limit of the sequence (see [1]).

For the sequence $\{A_k\}$ of non-empty subsets A_k of X, we have (see, for example, [19])

$$\liminf A_k := \{x \in X : \lim_{k \to \infty} d(x, A_k) = 0\}$$

and

$$\limsup A_k := \{x \in X : \liminf_{k \to \infty} d(x, A_k) = 0\}.$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d_k(x) = d(x)$$

for each $x \in X$, where $d_k, d : X \to \mathbb{R}^+$ are defined as $d(x) := d(x, A)$ and $d_k(x) := d(x, A_k)$. In this case we write $W - \lim A_k = A$ (see [20],[21]).

As an example, consider the following sequence of circles in the (x, y)-plane: $A_k = \{(x, y) : x^2 + y^2 - 2ky = 0\}$. As $k \to \infty$ the sequence is Wijsman convergent to the $x-$axis $A = \{(x, y) : \quad y = 0\}$.

Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is bounded if $\sup_k |d(x, A_k)| < \infty$ for each $x \in X$.

We now define Cauchy Wijsman sequences.

Definition 1. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman Cauchy if $d_k(x)$ is a Cauchy sequence; i.e., if for $\epsilon > 0$ and for each $x \in X$, there is a positive integer k_0 such that for all $m, n > k_0$, $|d_n(x) - d_m(x)| < \epsilon$.*

If the pointwise convergence $d(x, A_k) \to d(x, A)$ is replaced by uniform convergence, then Hausdorff convergence is obtained, which has been known for a long time.

Let (X, ρ) be a metric space. A sequence A_k of closed subsets of X is said to be Hausdorff convergent to a closed subset A of X if

$$\lim_{k\to\infty} \sup_{x\in X} |d_k(x) - d(x)| = 0.$$

In this case we write $A = H - \lim A_k$. Hausdorff and Wijsman definitions of convergence of sequences of sets require that the sets be closed, since otherwise the limit sets need not be well-defined (see[13]).

It is easy to see that, in any metric space X, Hausdorff convergence $\Rightarrow$ Wijsman convergence $\Rightarrow$ Kuratowski convergence. Kuratowski convergence need not imply pointwise convergence of distance functions, and even when pointwise convergence occurs to a finite limit, the limit need not be a distance function (see [3]).

2. Statistical convergence of sequences of sets

In this section, we introduce Kuratowski, Wijsman and Hausdorff statistical convergences of sequences of sets.

Let (X, ρ) be a metric space. For the sequence $\{A_k\}$ of non-empty closed subsets A_k of X, define the the statistical lower limit and statistical upper limit of $\{A_k\}$ as follows:

$$st - \liminf A_k := \{x \in X : \ \exists (a_k) \subset (A_k), \ \ st - \lim a_k = x\}$$

and

$$st - \limsup A_k := \{x \in X : \exists (k_l) \ \exists (a_{k_l}) \subset (A_{k_l}) \ \ st - \lim a_{k_l} = x\}.$$

Definition 2. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Kuratowski statistically convergent to A if*

$$st - \limsup A_k = st - \liminf A_k = A.$$

In this case we write $st - \lim A_k = A$.

The lower statistical limit of a sequence of subsets $\{A_k\}$ is the set of statistical limits of sequences of elements $a_k \in A_k$ and the upper statistical limit is the set of statistical cluster points of such sequences.

Definition 3. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; i.e., for each $\epsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \ |d(x, A_k) - d(x, A)| \geq \epsilon\}| = 0,$$

i.e.,

$$|d(x, A_k) - d(x, A)| < \epsilon \quad a.a.k. \ . \tag{1}$$

In this case we write $st - \lim_W A_k = A$.

It is clear that if the inequality in (1) holds for all but finitely many k, then $W - \lim A_k = A$. It follows that $W - \lim A_k = A$ implies $st - \lim_W A_k = A$.

For example, let $X = \mathbb{R}$ and $\{A_k\}$ be following sequence:

$$A_k := \begin{cases} \{x \in \mathbb{R} : \ 2 \leq x \leq k\}, & \text{if } k \geq 2 \text{ and } k \text{is a square integer,} \\ \{1\}, & \text{otherwise.} \end{cases}$$

This sequence is not Wijsman convergent. But since

$$\frac{1}{n}|\{k \leq n : \ |d(x, A_k) - d(x, \{1\})| \geq \epsilon\}| \leq \frac{\sqrt{n}}{n},$$

this sequence is Wijsman statistically convergent to set $A = \{1\}$.

As another example, let $X = \mathbb{R}^2$ and $\{A_k\}$ be following sequence:

$$A_k := \begin{cases} \{(x, y) \in \mathbb{R}^2 : \ (x-1)^2 + y^2 = \frac{1}{k}\}, & \text{if } k \text{ is a square integer,} \\ \{(0, 0)\}, & \text{otherwise.} \end{cases}$$

This sequence also is Wijsman statistically convergent to set $A = \{(0, 0)\}$ but it is not Wijsman convergent.

Definition 4. *Let (X, ρ) be a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is Hausdorff statistically convergent to a closed subset A of X if for each $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \sup_{x\in X} |d(x, A_k) - d(x, A)| \geq \epsilon\}| = 0,$$

i.e.,

$$\sup_{x\in X} |d(x, A_k) - d(x, A)| < \epsilon \quad a.a.k.$$

in this case we write $A = st_H - \lim A_k$.

In most convergence theories it is desirable to have a criterion that can be used to verify convergence without using the value of the limit. For this purpose we introduce the statistical analog of the Cauchy convergence criterion.

Definition 5. *Let (X,ρ) be a metric space. For any non-empty closed subsets $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically Cauchy if for each $\epsilon > 0$ there exists a number $N(= N(\epsilon))$ such that for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |d(x,A_k) - d(x,A_N)| \geq \epsilon\}| = 0.$$

Theorem 1. *Let (X,ρ) be a metric space. The following statements are equivalent:*

(i) $\{A_k\}$ is a Wijsman statistically convergent sequence;

(ii) $\{A_k\}$ is a Wijsman statistically Cauchy sequence;

(iii) $\{A_k\}$ is a sequence for which there is a Wijsman convergent sequence $\{B_k\}$ such that $A_k = B_k$ a.a.k.

Proof. Suppose $st - \lim_W A_k = A$ and $\epsilon > 0$. Then $|d(x,A_k) - d(x,A)| < \frac{\epsilon}{2}$ *a.a.k.*, if N is chosen so that $|d(x,A_N) - d(x,A)| < \frac{\epsilon}{2}$, then we have $|d(x,A_k) - d(x,A_N)| \leq |d(x,A_k) - d(x,A)| + |d(x,A_N) - d(x,A)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ *a.a.k*. Hence, $\{A_k\}$ is a Wijsman statistically Cauchy sequence.

Next, assume that (ii) is true and choose N so that the interval $J = [d(x,A_N) - 1, d(x,A_N) + 1]$ contains $d(x,A_k)$ *a.a.k*. Now apply (ii) to choose N_2 so that $J' = [d(x,A_{N_2}) - \frac{1}{2}, d(x,A_{N_2}) + \frac{1}{2}]$ contains $d(x,A_k)$ *a.a.k*. We assert that $J_1 = J \cap J'$ contains $d(x,A_k)$ *a.a.k*. For

$$\begin{aligned}\{k \leq n : d(x,A_k) \notin J \cap J'\} &= \{k \leq n : d(x,A_k) \notin J\} \\ &\quad \cup \{k \leq n : d(x,A_k) \notin J'\},\end{aligned}$$

so

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : d(x,A_k) \notin J \cap J'\}| &\leq \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : d(x,A_k) \notin J\}| \\ &\quad + \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : d(x,A_k) \notin J'\}| = 0.\end{aligned}$$

Therefore J_1 is a closed interval of length less than or equal 1 that contains $d(x,A_k)$ *a.a.k*. Now we proceed to choose N_3 so that $J'' = [d(x,A_{N_3}) - \frac{1}{4}, d(x,A_{N_3}) + \frac{1}{4}]$ contains $d(x,A_k)$ *a.a.k*, and by the proceeding argument $J_2 = J_1 \cap J''$contains $d(x,A_k)$ *a.a.k*, and J_2 has length less than or equal to $\frac{1}{2}$. Continuing this process, by induction we construct a sequence (J_m) of closed intervals such that, for each m, $J_{m+1} \subseteq J_m$, the length of J_m is not greater than 2^{1-m}, and $d(x, A_k) \in J_m \quad a.a.k.$ By the Nested Intervals Theorem there is a number η equal to $\cap_{m=1}^{\infty} J_m$. Using the fact that $d(x, A_k) \in J_m$ $a.a.k$ we choose an increasing positive integer sequence $\{T_m\}$ such that

$$\frac{1}{n}|\{k \leq n : d(x, A_k) \notin J_m\}| < 1/m \tag{2}$$

if $n > T_m$. Now define a subsequence $C = (C_k)$, consisting of all terms of (A_k) such that $k > T_1$ and if $T_m < k \leq T_{m+1}$, then $d(x, A_k) \notin J_m$.

Next define the sequence (B_k) by

$$B_k := \begin{cases} \{\eta\}, & \text{if } A_k \text{ is a term of } C, \\ A_k, & \text{otherwise.} \end{cases}$$

Then $\lim B_k = \{\eta\}$; for if $\epsilon > 1/m > 0$ and $k > T_m$, then either A_k, which means $B_k = \{\eta\}$, or $B_k = A_k \in J_m$ and $|d(x, B_k) - d(x, \{\eta\})| \leq$ length of $J_m \leq 2^{1-m}$. We also assert that $A_k = B_k$ $a.a.k.$ To verify this we observe that if $T_m < k < T_{m+1}$, then

$$\{k \leq n : d(x, A_k) \neq d(x, B_k)\} \subseteq \{k \leq n : d(x, A_k) \notin J_m\}.$$

So by (2)

$$\frac{1}{n}|\{k \leq n : d(x, B_k) \neq d(x, A_k)\}| \leq \frac{1}{n}|\{k \leq n : d(x, A_k) \notin J_m\}| < \frac{1}{m}.$$

Hence the limit as $n \to \infty$ is 0 and, $A_k = B_k$ $a.a.k.$ Therefore (ii) implies (iii). Finally, assume that (iii) holds; say $A_k = B_k$ $a.a.k$ and $\lim B_k = \{\eta\}$. Let $\epsilon > 0$, then for each n,

$$\begin{aligned} &\{k \leq n : |d(x, A_k) - d(x, \{\eta\})| \geq \epsilon\} \subseteq \\ &\quad \{k \leq n : d(x, B_k) \neq d(x, A_k)\} \cup \{k \leq n : |d(x, B_k) - d(x, \{\eta\})| > \epsilon\}. \end{aligned}$$

Since $\lim B_k = \{\eta\}$, the latter set contains a fixed number of elements, say, $l = l(\epsilon)$. Therefore

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, \{\eta\})| \geq \epsilon\}| \\ &\qquad \leq \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : d(x, B_k) \neq d(x, A_k)\}| + \lim_{n\to\infty} \frac{l}{n} = 0, \end{aligned}$$

because $A_k = B_k$ $a.a.k.$ So (i) holds and proof is complete. ∎

In the following theorems, we give a Tauberian condition for Wijsman and Hausdorff statistical convergences.

Theorem 2. *Let (X, ρ) be a metric space. If $\{A_k\}$ is a sequence such that $st-\lim_W A_k = A$ and $\triangle d_k(x) = O(\frac{1}{k})$ for each $x \in X$, then $W-\lim A_k = A$ where $\triangle d_k(x) := d_{k+1}(x) - d_k(x)$.*

Proof. Assume that $\{A_k\}$ is Wijsman statistically convergent to A. Then $st - \lim_W A_k = A$ and we can choose a sequence B_k such that $W - \lim B_k = A$ and $A_k = B_k$ *a.a.k.* For each k, write $k = m(k) + p(k)$, where $m(k) = max\{i \leq k : \quad A_i = B_i\}$. If the set $\{i \leq k : A_i = B_i\}$ is empty, take $m(k) = -1$. This can occur for at most a finite number of k. We assert that

$$\lim_k \frac{p(k)}{m(k)} = 0. \tag{3}$$

For, if $\frac{p(k)}{m(k)} > \epsilon > 0$, then

$$\frac{1}{k}|\{i \leq k : A_i \neq B_i\}| \leq \frac{1}{m(k)+p(k)} p(k) \leq \frac{p(k)}{\frac{p(k)}{\epsilon} + p(k)} = \frac{\epsilon}{1+\epsilon}$$

so if $\frac{p(k)}{m(k)} \geq \epsilon$ for infinitely many k, we have a contradiction to $A_k = B_k$ *a.a.k.* Thus (1) holds. Since $\triangle d_k(x) = O(\frac{1}{k})$ there is a constant K such that $|\triangle d_k(x)| \leq \frac{K}{k}$ for all k and for each $x \in X$. Therefore

$$\begin{aligned} |d(x, B_{m(k)}) - d(x, A_k)| &= |d(x, A_{m(k)}) - d(x, A_{m(k)+p(k)})| \\ &\leq \sum_{i=m(k)}^{m(k)+p(k)-1} |\triangle d_i(x)| \leq \frac{p(k)K}{m(k)}. \end{aligned}$$

By (3), the last expression tends to 0 as $k \to \infty$, and since $W - \lim B_k = A$, we conclude that $W - \lim A_k = A$. ■

Theorem 3. *Let (X, ρ) be a metric space. If $\{A_k\}$ is a sequence such that $st - \lim_H A_k = A$ and $sup_{x \in X} \triangle d_k(x) = O(\frac{1}{k})$, then $H - \lim A_k = A$.*

The proof is similar to the proof of Theorem 2.

Theorem 4. *Let (X, ρ) be a metric space and $\{A_k\}$ be a sequence of non-empty closed subsets of X. If $\{A_k\}$ is Wijsman statistically convergent, then $\{A_k\}$ is Kuratowski statistically convergent.*

Proof. We need only show $st - \limsup A_k \subset st - \liminf A_k$. Fix $x \in st - \limsup A_k$ and $\epsilon > 0$. Since a Wijsman statistically convergent sequence is Wijsman statistically Cauchy, choose N so that $|d(x, A_k) - d(x, A_N)| < \frac{\epsilon}{2}$ *a.a.k.* and $d(x, A_N) < \frac{\epsilon}{2}$. We have $d(x, A_k) \leq d(x, A_N) + |d(x, A_k) - d(x, A_N)| < \epsilon$. *a.a.k.* By the definition we get $x \in st - \liminf A_k$ and this completes the proof. ■

The following theorem follows immediately from Definitions 3 and 4.

Theorem 5. *Let (X,ρ) be a metric space and $\{A_k\}$ be a sequence of non-empty closed subsets of X. If $\{A_k\}$ is Hausdorff statistically convergent, then $\{A_k\}$ is Wijsman statistically convergent.*

3. Strongly summable set sequences

In this section we introduce Kuratowski Cesaro summable, Wijsman summable and Wijsman strongly summable sequences of sets and give the relation between Wijsman statistically convergent and Wijsman strongly summable sequences of sets.

Let (X,ρ) be a metric space. For the sequence $\{A_k\}$ of non-empty subsets A_k of X, we have define the lower Cesaro limit and upper Cesaro limit as follows:

$$(C,1)-\liminf A_k := \{x \in X : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = 0\}$$

and

$$(C,1)-\limsup A_k := \{x \in X : \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = 0\}.$$

Definition 6. *Let (X,ρ) be a metric space. For any non-empty subsets $A_k \subseteq X$, we say that $\{A_k\}$ is Kuratowski Cesaro summable to if*

$$(C,1)-\liminf A_k = (C,1)-\limsup A_k.$$

Definition 7. *Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman Cesaro summable to A if $\{d(x,A_k)\}$ is Cesaro summable to $d(x,A)$; i.e., for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(x,A_k) = d(x,A).$$

Definition 8. *Let (X,ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly Cesaro summable to A if $\{d(x,A_k)\}$ strongly summable to $d(x,A)$; i.e., for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x,A_k) - d(x,A)| = 0.$$

Definition 9. *Let (X,ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly p-Cesaro summable* *to A if $\{d(x, A_k)\}$ strongly p-summable to $d(x, A)$; i.e., for each p positive real number and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p = 0.$$

Theorem 6. *Let (X, ρ) be a metric space and p be a positive real number. Then, for any non-empty closed subsets A, $A_k \subseteq X$*

(*a*) *$\{A_k\}$ is Wijsman statistically convergent to A if it is Wijsman strongly p-Cesaro summable to A,*

(*b*) *If $\{A_k\}$ is bounded and Wijsman statistically convergent to A then it is Wijsman statistically convergent to A.*

Proof. (*a*) For any $\{A_k\}$, fix an $\epsilon > 0$. Then

$$\sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p \geq \epsilon|\{k \leq n : |d(x, A_k) - d(x, A)|^p \geq \epsilon\}|,$$

and it follows that if $\{A_k\}$ is Wijsman strongly p-Cesaro summable to A then $\{A_k\}$ is Wijsman statistically convergent to A.

(*b*) Let $\{A_k\}$ be bounded and Wijsman statistically convergent to A. Since $\{A_k\}$ is bounded, set $\sup_k |d(x, A_k)| + d(x, A) = M$. Let $\epsilon > 0$ be given and select N_ϵ such that

$$\frac{1}{n}|\{k \leq n : |d(x, A_k) - d(x, A)| \geq (\frac{\epsilon}{2})^{1/p}\}| < \frac{\epsilon}{2M^p}$$

for all $n > N_\epsilon$ and set $L_n = \{k \leq n : |d(x, A_k) - d(x, A)| \geq (\frac{\epsilon}{2})^{1/p}\}$.

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p &= \frac{1}{n}(\sum_{k\in L_n} |d(x, A_k) - d(x, A)|^p \\ &\quad + \sum_{k\leq n; k\notin L_n} |d(x, A_k) - d(x, A)|^p) \\ &< \frac{1}{n}\frac{n\epsilon}{2M^p}M^p + \frac{1}{n}\frac{n\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, $\{A_k\}$ is Wijsman strongly p-Cesaro summable to A. ■

4. Strongly almost convergent set sequences

The idea of almost convergence of sequences of points was introduced by Lorentz [15]: the sequence $x = (x_k)$ is said to be almost convergent to L if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{k+i} = L$$

uniformly in i.

Maddox [16] and (independently) Freedman et al. [10] introduced the notion of strong almost convergence of sequences of points: the sequence $x = (x_k)$ is said to be strongly almost convergent to L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+i} - L| = 0$$

uniformly in i.

Let ℓ_∞, c, f and $[f]$, respectively, denote the sets of all bounded, convergent, almost convergent and strongly almost convergent sequences. It is known [16] that

$$c \subset f \subset [f] \subset \ell_\infty.$$

In this section we introduce the concepts of Wijsman almost convergence, Wijsman strongly almost convergence and Wijsman almost statistical convergence for sequences of sets and give the relation between Wijsman almost statistically convergent and Wijsman strongly almost convergent sequences of sets. Kuratowski almost convergent and Hausdorff almost convergent sequences can be defined in a similar manner.

Definition 10. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman almost convergent to A if for each $\epsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_{k+i}) = d(x, A).$$

uniformly in i.

Definition 11. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $\epsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0.$$

uniformly in i.

Let L_∞, C, F and $[F]$, respectively, denote the sets of all Wijsman bounded, Wijsman convergent, Wijsman almost convergent and Wijsman strongly almost convergent sequences of sets. It is easy to see that

$$C \subset F \subset [F] \subset L_\infty.$$

Definition 12. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly p-almost convergent to A if for each $\epsilon > 0$, p positive real number and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)|^p = 0$$

uniformly in i.

Definition 13. *Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman almost statistically convergent to A if for each $\epsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x, A_{k+i}) - d(x, A)| \ge \epsilon\}| = 0,$$

uniformly in i.

Theorem 7. *Let (X, ρ) be a metric space and p be a positive real number. Then, for any non-empty closed subsets A, $A_k \subseteq X$*

(a) $\{A_k\}$ is Wijsman almost statistically convergent to A if it is Wijsman strongly p-almost convergent to A,

(b) If $\{A_k\}$ is bounded and Wijsman almost statistically convergent to A, then it is Wijsman strongly p-almost convergent to A.

The proof is similar to the proof of Theorem 6.

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Received on 06.12.2011 and, in revised form, on 06.02.2012.