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ALTERING DISTANCES, FIXED POINT FOR OCCASIONALLY HYBRID MAPPINGS AND APPLICATIONS

Abstract. The purpose of this paper is to prove a general fixed point theorem by altering distance for two pairs of hybrid occasionally weakly compatible mappings and to reduce the study of fixed points of pairs of mappings satisfying a contractive condition of integral type at the study of fixed point in symmetric spaces by altering distance satisfying an implicit relation.

Key words: symmetric space, occasionally weakly compatible, hybrid mappings, common fixed point, implicit relation, altering distance, integral type.

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1. Introduction

Let A and S be self mappings of a metric space (X, d). Jungck [11] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$. We denote by $C(A, S)$ the set of all coincidence points of A and S. In [21], Pant defined A and S to be pointwise R - weakly commuting if for each $x \in X$ there exists $R > 0$ such that $d(SAx, ASx) \leq Rd(Ax, Sx)$. It is proved in [22] that pointwise R - weakly commuting is equivalent with the commuting at coincidence points.

Definition 1. *A and S are said to be weakly compatible [12] if $ASu = SAu$ for $u \in C(A, S)$.*

Definition 2. *A and S are said to be occasionally weakly compatible (owc) [5] if $ASu = SAu$ for some $u \in C(A, S)$.*

Remark 1. If A and S are weakly compatible and $C(A, S) \neq \varnothing$ then A and S are owc, but the converse in not true (see Example [5]).

Some fixed point theorems for owc mappings are proved in [3], [4], [15] and in other papers.

Let X be a nonempty set. A symmetric on X is a nonnegative real valued function D on $X \times X$ such that:
(i) $D(x,y) = 0$ if and only if $x = y$;
(ii) $D(x,y) = D(y,x)$ for all $x, y \in X$.

Let (X,d) be a metric (symmetric) space and $B(X)$ the set of all nonempty bounded subset of X. As in [8], [9] we define the functions $\delta(A,B)$ and $D(A,B)$, where $A, B \in B(X)$:

$$\begin{aligned} D(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ \delta(A,B) &= \sup\{d(a,b) : a \in A, b \in B\}. \end{aligned}$$

If A consists of a single point "a" we write $\delta(A,B) = \delta(a,B)$.

If B consists also of a single point "b" we write $\delta(A,B) = d(a,b)$.

If follows immediately from definition of δ that:

$$\delta(A,B) = \delta(B,A), \ \ \forall A, B \in B(X).$$

If $\delta(A,B) = 0$ then $A = B = \{a\}$.

Definition 3. *Let $f : (X,d) \to (X,d)$ and $F : X \to B(X)$.*

1) *A point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. We denote by $C(f,F)$ the set of all coincidence points of f and F.*

2) *A point $x \in X$ is a fixed point of F if $x \in Fx$.*

Definition 4. *Let (X,d) be a metric space. A sequence $\{A_n\}$ of nonempty sets of X is said to be convergent to a set A of X [8], [9] if*

(i) *each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$ for all $n \in \mathbb{N}$*

(ii) *for arbitrary $\varepsilon > 0$, there exists an integer $m > 0$ such that $A_n \subset A_\varepsilon$ for $n > m$, where A_ε denote the set of all point $x \in X$ for which there exists a point $a \in X$, depending on x, such that $d(a,x) < \varepsilon$.*

A is said to be the limit of the sequence $\{A_n\}$.

Definition 5. *The mappings $f : X \to X$ and $F : X \to B(X)$ are δ - compatible [13] if $\lim_{n\to\infty} \delta(Ffx_n, fFx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X)$, $fx_n \to t$, $Ffx_n \to t$ for some $t \in X$.*

Definition 6. *The pair $f : X \to X$ and $F : X \to B(X)$ is weakly compatible [14] if for each $x \in C(f,F)$, $fFx = Ffx$.*

If the pair (f,F) is δ-compatible then (f,F) is weakly compatible, but the converse is not true [14].

Definition 7. *The hybrid pair $f : X \to X$ and $F : X \to B(X)$ is occasionally weakly compatible [1] if there exists $x \in C(f, F)$ such that $fFx \subset Ffx$.*

Remark 2. Every weakly compatible hybrid mappings are occasionally weakly compatible. The converse is not true (see Example 1.3 [1]).

The following theorems are proved in [1].

Theorem 1. *Let f, g be self maps of a metric space (X, d) and F, G be maps of X into $B(X)$ such that (f, F) and (g, G) are owc. If*

$$\delta(Fx, Gy) < \max\{d(fx, gy), D(fx, Fx), D(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)\}$$

for all $x, y \in X$ for which $fx \neq gy$, then f, g, F and G have a unique common fixed point.

Theorem 2. *Let f, g be self maps of a metric space (X, d) and F, G be maps of X into $B(X)$ such that (f, F) and (g, G) are owc. If*

$$\delta(Fx, Gy) < h \max\Big\{d(fx, gy), D(fx, Fx), D(gy, Gy), \frac{1}{2}[\delta(fx, Gy) + \delta(gy, Fx)]\Big\}$$

for all $x, y \in X$ for which $fx \neq gy$ and $h \in (0, 1)$,then f, g, F and G have a unique common fixed point.

Theorem 3. *Let f, g be self maps of a metric space (X, d) and F, G be maps of X into $B(X)$ such that (f, F) and (g, G) are owc. If*

$$\delta(Fx, Gy) < ad(fx, gy) + b \max\{D(fx, Fx), D(gy, Gy)\} + c \max\{d(fx, gy), \delta(fx, Gy), \delta(gy, Fx)\}$$

for all $x, y \in X$ for which $fx \neq gy$, where $a, b, c > 0$ and $a + b + c = 1$, then f, g, F and G have a unique common fixed point.

2. Contractive condition of integral type

In [7], Branciari established the following result

Theorem 4. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f : X \to X$ be a mapping such that*

$$(1) \qquad \int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(x,y)} h(t)dt$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with a finite integral) on each compact subset of $[0,\infty)$, such that for $\varepsilon > 0$, $\int_0^\varepsilon h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Quite recently, Kumar et al. [18] extended Theorem 4 for two compatible mappings.

Theorem 5. *Let $f, g : X \to X$ two compatible mappings satisfying the following conditions:*
(i) $f(X) \subset g(X)$,
(ii) g is continuous and

$$\int_0^{d(fx,fy)} h(t)dt \leq c \int_0^{d(gx,gy)} h(t)dt \tag{2}$$

for all $x, y \in X$, $c \in (0,1)$, where $h(t)$ is as in Theorem 4. Then, f and g have a unique common fixed point.

Some fixed point theorem in metric and symmetric spaces for compatible, weakly compatible and occasionally weakly compatible mappings satisfying a contractive condition of integral type are proved in [2], [17], [20], [25], [27], [30] and in other papers.

Let (X, d) be a metric space and $D(x,y) = \int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 4. It is proved in [20], [25] that $D(x,y)$ is a symmetric on X. It is proved in [20], [25] that the study of fixed points for mappings satisfying a contractive condition of integral type is reduced to the study of fixed points in symmetric spaces.

The method is not applicable for hybrid pairs of mappings.

Definition 8. *An altering distance is a mapping $\psi : [0,\infty) \to [0,\infty)$ which satisfies the conditions:*
(i) $\psi(t)$ is increasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Fixed point problem involving altering distances have been studied in [16], [19], [26], [28], [29] and in other papers.

Lemma 1. *The function $\psi(x) = \int\limits_0^x h(t)dt$, where $h(t)$ is as in Theorem 4 is an altering distance.*

Proof. By definitions of ψ and h it follows that $\psi(x)$ is increasing and $\psi(x) = 0$ if and only if $x = 0$. By Lemma 2.5 [20], $\psi(x)$ is continuous. ■

A general fixed point theorem for compatible mappings satisfying an implicit relation is proved in [23]. In [10] the result from [23] is improved relaxing the compatibility to weak compatibility.

The purpose of this paper is to prove a general fixed point theorem by altering distance for two pairs of owc hybrid pairs dued to reduce the study of fixed points of hybrid pairs of mappings satisfying a contractive condition of integral type at the study of fixed points in metric (symmetric) spaces by altering distance satisfying an implicit relation.

3. Implicit relations

Definition 9. *Let $\mathcal{F}_W$ be the set of all functions $\phi(t_1, \dots, t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:*

(ϕ_1): ϕ is nonincreasing in variables t_2, t_5 and t_6,

(ϕ_2): $\phi(t, t, 0, 0, t, t) \geq 0$, $\forall t > 0$.

Example 1. $\phi(t_1, \dots, t_6) = t_1 - \max\{t_2, \dots, t_6\}$.

Example 2. $\phi(t_1, \dots, t_6) = t_1 - h \max\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $h \in (0, 1)$.

Example 3. $\phi(t_1, \dots, t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$, where a, b, $c \geq 0$ and $a + b + c = 1$.

Example 4. $\phi(t_1, \dots, t_6) = t_1^p - at_2^p - (1-a)\max\{t_3^p, t_4^p, (t_3t_4)^{p/2}, (t_5t_6)^{p/2}\}$, where $0 < a < 1$, $p \geq 1$.

Example 5. $\phi(t_1, \dots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, where $a > 0$, $b, c \geq 0$ and $a + c \leq 1$.

Example 6. $\phi(t_1, \dots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c\sqrt{t_5t_6}$, where $a, b, c \geq 0$ and $a + c \leq 1$.

Example 7. $\phi(t_1, \dots, t_6) = t_1 - (1 - \alpha)t_2 - \alpha(t_3t_4 + t_5t_6) - at_2 - (1 - a)\max\{t_3, t_4, (t_5t_6)^{1/2}, (t_3t_6)^{1/2}\}$, where $\alpha \geq 0$ and $0 < a < 1$.

Example 8. $\phi(t_1, \dots, t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $0 < \alpha < 1$, a, $b > 0$ and $a + b < 1$.

Example 9. $\phi(t_1, \dots, t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a > 0$, $b, c \geq 0$ and $a + 2c \leq 1$.

Example 10. $\phi(t_1, ..., t_6) = t_1^3 - t_2^3 - \dfrac{t_3^2 \cdot t_5 + t_4^2 \cdot t_6}{1 + t_3 + t_4}$.

Example 11. $\phi(t_1, \dots, t_6) = t_1^2 - at_2^2 - b\dfrac{\min\{t_5^2, t_6^2\}}{1 + t_3 + t_4}$, where $a > 0$, $b \geq 0$ and $a + b \leq 1$.

Example 12. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\dfrac{t_5 t_6}{1 + t_3 + t_4}$, where $a > 0$, $b \geq 0$ and $a + 2b < 1$.

Example 13. $\phi(t_1, \ldots, t_6) = t_1 - c \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $0 < c \leq 1$, $a \geq 0$, $b \geq 0$ and $a + b < 1$.

Example 14. $\phi(t_1, \ldots, t_6) = t_1 - \varphi(\max\{t_2, t_3, \ldots, t_6\})$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ with $\varphi(t) < t$, $\forall t > 0$.

Example 15. $\phi(t_1, ..., t_6) = t_1 - \max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$

and others.

4. Main results

Definition 10. *A weakly altering distance is a mapping* $\psi : \mathbb{R}_+ \to \mathbb{R}$ *which satisfies:*
$(\psi_1) : \psi$ *is increasing,*
$(\psi_2) : \psi(t) = 0$ *if and only if* $t = 0$.

Theorem 6. *Let* f, g *be self maps of the symmetric space* (X, d) *and* F, G *be maps of* X *into* $B(X)$ *such that the pairs* (f, F) *and* (g, G) *are owc. If*

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(D(fx, Fx)), \psi(D(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))) < 0 \tag{3}$$

for all $x, y \in X$ *for which* $fx \neq gy$, *where* $\psi(t)$ *is an weakly altering distance and* $\phi \in \mathcal{F}_W$, *then* f, g, F *and* G *have a unique common fixed point.*

Proof. Because (f, F) and (g, G) are owc, there exist $x, y \in X$ such that $fx \in Fx$, $gy \in Gy$ and $fFx \subset Ffx$ and $gGy \subset Ggy$. First we prove that $fx = gy$. Suppose that $fx \neq gy$. Then $0 < d(fx, gy) \leq \delta(Fx, Gy)$. By (3) and (ϕ_1) we obtain

$$\phi(\psi(\delta(Fx, Gy)), \psi(\delta(Fx, Gy)), 0, 0, \psi(\delta(Fx, Gy)), \psi(\delta(Fx, Gy))) < 0,$$

a contradiction of (ϕ_2). Next we show that $fx = f^2x$. Suppose that $fx \neq f^2x$. Then $0 < d(fx, f^2x) \leq \delta(fx, fFx) = \delta(gy, fFx) \leq \delta(Ffx, Gy)$. By (3) and (ϕ_1) we have successively

$$\phi(\psi(\delta(Ffx, Gy)), \psi(\delta(f^2x, gy)), 0, 0, \psi(\delta(f^2x, Gy)), \psi(\delta(gy, Ffx))) < 0,$$

$$\phi(\psi(\delta(Ffx, Gy)), \psi(\delta(Ffx, Gy)), 0, 0, \psi(\delta(Ffx, Gy)), \psi(\delta(Ffx, Gy))) < 0,$$

a contradiction of (ϕ_2). Hence $fx = f^2x$. Similarly, $gy = g^2y$. Therefore $fx = f^2x = gy = g^2y = gfx$. Hence fx is a common fixed point of f and g. On the other hand $fx = f^2x \in fFx \subset Ffx$ and fx is a fixed point of F. Similarly, $fx = f^2x = gy = g^2y \in gGy \subset Ggy$ and fx is a fixed point of G. Hence $w = fx$ is a common fixed point of f, g, F and G.

Suppose that $w' \neq w$ is an other common fixed point of f, g, F and G. Because

$$0 < d(w, w') = d(fw, gw') \leq \delta(Fw, Gw'),$$

by (3) and (ϕ_1) we have successively

$$\begin{aligned}\phi(\psi(\delta(Fw, Gw')), \psi(d(fw, gw')), \psi(D(fw, Fw)),\\ \psi(D(gw', Gw')), \psi(\delta(fw, Gw')), \psi(\delta(gw', Fw)) < 0,\end{aligned}$$

$$\phi(\psi(\delta(Fw, Gw')), \psi(\delta(Fw, Gw')), 0, 0, \psi(\delta(Fw, Gw')), \psi(\delta(Fw, Gw')) < 0,$$

a contradiction of (ϕ_2). Therefore, $w = fx$ is the unique common fixed point of f, g, F and G. ■

Corollary 1. *Let f, g be self mappings of a symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs (f, F) and (g, G) are owc. If*

$$\begin{aligned}(4)\quad \psi(\delta(Fx, Gy)) \ < \ &\max\{\psi(d(fx, gy)), \psi(D(fx, Fx)),\\ &\psi(D(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))\}\end{aligned}$$

for all $x, y \in X$ for which $fx \neq gy$, where $\psi(t)$ is an weakly altering distance, then f, g, F and G have a unique common fixed point.

Proof. The proof follows from Theorem 6 and Example 1. ■

Corollary 2. *Let f, g be self mappings of a symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs (f, F) and (g, G) are owc. If*

$$\begin{aligned}(5)\quad \psi(\delta(Fx, Gy)) \ < \ &h\max\{\psi(d(fx, gy)), \psi(D(fx, Fx)),\\ &\psi(D(gy, Gy)), \frac{1}{2}[\psi(\delta(fx, Gy)) + \psi(\delta(gy, Fx))]\},\end{aligned}$$

for all $x, y \in X$ for which $fx \neq gy$, $h \in (0, 1)$ and ψ is an weakly altering distance, then f, g, F and G have a unique common fixed point.

Proof. The proof follows from Theorem 6 and Example 2. ■

Corollary 3. *Let f, g be self mappings of a symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs (f, F) and (g, G) are owc. If*

$$(6)\ \psi(\delta(Fx, Gy)) < a\psi(d(fx, gy)) + b\max\{\psi(D(fx, Fx)), \psi(D(gy, Gy))\} + c\max\{\psi(\delta(fx, gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))\}$$

for all $x, y \in X$ for which $fx \neq gy$, $a, b, c \geq 0$ and $a + b + c = 1$ and ψ is an weakly altering distance, then f, g, F and G have a unique common fixed point.

Proof. The proof follows from Theorem 6 and Example 3. ■

Remark 3. 1) In the proof of Theorem 6 is used a similar technique as in the proof of Theorem 2.2 [6].

2) From Theorem 6 and Example 4 - 15 we obtain new results.

For $\psi(t) = t$ we obtain:

Theorem 7 (Theorem 3.5 [4])**.** *Let f, g be self mappings of a symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs (f, F) and (g, G) are owc. If*

$$(7)\qquad \phi(\delta(Fx, Gy), d(fx, gy), D(fx, Fx), D(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)) < 0$$

for all $x, y \in X$ for which $fx \neq gy$ and $\phi \in \mathcal{F}_W$, then f, g, F and G have a unique common fixed point.

Remark 4. From Theorem 7 and Example 1, 2, 3 we obtain Theorem 1, 2, 3.

If f, g, F and G are self mappings of a symmetric space (X, d), by Theorem 6 we obtain

Theorem 8. *Let f, g, F, G be self mappings of a symmetric space (X, d) such that (f, F) and (g, G) are owc. If*

$$(8)\qquad \phi(\psi(d(Fx, Gy)), \psi(d(fx, gy)), \psi(d(fx, Fx)), \psi(d(gy, Gy)), \psi(d(fx, Gy)), \psi(d(gy, Fx))) < 0$$

for all $x, y \in X$ for which $fx \neq gy$, where $\psi(t)$ is an weakly altering distance and $\phi \in \mathcal{F}_W$, then f, g, F and G have a unique common fixed point.

If $\psi(t) = t$ by Theorem 8 we obtain:

Theorem 9. *Let f, g, F, G be self mappings of a symmetric space (X, d) such that (f, F) and (g, G) are owc. If*

$$\phi(d(Fx, Gy), d(fx, gy), d(fx, Fx), d(gy, Gy), d(fx, Gy), d(gy, Fx)) < 0 \tag{9}$$

for all $x, y \in X$ for which $fx \neq gy$ and $\phi \in \mathcal{F}_W$, then f, g, F and G have a unique common fixed point.

Remark 5. *i*) From Theorem 9 and Example 1, 2, 3 we obtain Theorem 1, Corollary 1 and Theorem 2 from [15].

ii) By Theorem 9 and Example 4 we obtain a correct form of Theorem 3 from [15].

5. Applications

Theorem 10. *Let f, g be self mappings of the symmetric space (X, d) and F, G be maps of X into $B(X)$ such that the pairs (f, F) and (g, G) are owc. If*

$$\phi\Big(\int_0^{\delta(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt, \int_0^{D(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt\Big) < 0 \tag{10}$$

for all $x, y \in X$ for which $fx \neq gy$, where $\phi \in \mathcal{F}_W$ and $h(t)$ is as in Theorem 4, then f, g, F and G have a unique common fixed point.

Proof. As in Lemma 1 we have

$$\begin{aligned}
\psi(\delta(Fx, Gy)) &= \int_0^{\delta(Fx,Gy)} h(t)dt, \psi(d(fx, gy)) = \int_0^{d(fx,gy)} h(t)dt, \\
\psi(D(fx, Fx)) &= \int_0^{D(fx,Fx)} h(t)dt, \psi(D(gy, Gy)) = \int_0^{D(gy,Gy)} h(t)dt, \\
\psi(\delta(fx, Gy)) &= \int_0^{\delta(fx,Gy)} h(t)dt, \psi(\delta(gy, Fx)) = \int_0^{\delta(gy,Fx)} h(t)dt.
\end{aligned}$$

Then, by (10) we have

$$\phi(\psi(\delta(Fx, Gy)), \psi(d(fx, gy)), \psi(D(fx, Fx)), \psi(D(gy, Gy)), \psi(\delta(fx, Gy)), \psi(\delta(gy, Fx))) < 0$$

for all $x, y \in X$ with $fx \neq gy$ and $\phi \in \mathcal{F}_W$. Because by Lemma 1, then $\psi(t) = \int_0^t h(t)dt$ is an weakly altering distance, the conditions of Theorem 6 are satisfied and Theorem 10 follows from Theorem 6. ■

For example, by Theorem 10 and Example 2 we obtain

Corollary 4. *Let f, g be self mappings of the symmetric space (X,d) and F, G be maps of X into $B(X)$ such that the pairs (f,F) and (g,G) are owc and*

$$\int_0^{\delta(Fx,Gy)} h(t)dt < k \max\left\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{D(fx,Fx)} h(t)dt,\right.$$
$$\left.\int_0^{D(gy,Gy)} h(t)dt, \int_0^{\delta(fx,Gy)} h(t)dt, \int_0^{\delta(gy,Fx)} h(t)dt\right\} < 0$$

for all $x,y \in X$ with $fx \neq gy$, where $k \in (0,1)$ and $h(t)$ is as in Theorem 4. Then f,g,F and G have a unique common fixed point.

If f,g,F and G are single valued mappings, by Theorem 10 we obtain

Theorem 11. *Let f, g, F and G be self mappings of a symmetric space (X,d) such that (f,F) and (g,G) are owc and*

$$\phi\left(\int_0^{d(Fx,Gy)} h(t)dt, \int_0^{d(fx,gy)} h(t)dt, \int_0^{d(fx,Fx)} h(t)dt,\right.$$
$$\left.\int_0^{d(gy,Gy)} h(t)dt, \int_0^{d(fx,Gy)} h(t)dt, \int_0^{d(gy,Fx)} h(t)dt\right) < 0 \tag{11}$$

for all $x,y \in X$ with $fx \neq gy$, $\phi \in \mathcal{F}_W$ and $h(t)$ is as in Theorem 4. Then f, g, F and G have a unique common fixed point.

For example, by Theorem 11 and Example 3 we have

Corollary 5. *Let f, g, F and G be self mappings of a symmetric space (X,d) such that (f,F) and (g,G) are owc and*

$$\int_0^{d(Fx,Gy)} h(t)dt < a\int_0^{d(fx,gy)} h(t)dt \tag{12}$$
$$+ b\max\left\{\int_0^{d(fx,Fx)} h(t)dt, \int_0^{d(gy,Gy)} h(t)dt\right\}$$
$$+ c\max\left\{\int_0^{d(fx,gy)} h(t)dt, \int_0^{d(fx,Gy)} h(t)dt, \int_0^{d(gy,Fx)} h(t)dt\right\} < 0$$

for all $x,y \in X$ with $fx \neq gy$, where $a,b,c \geq 0$, $a+b+c=1$ and $h(t)$ is as in Theorem 4. Then f, g, F and G have a unique common fixed point.

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