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STATISTICALLY CONVERGENT DIFFERENCE DOUBLE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTION

Abstract. In this article we introduce some statistically convergent difference double sequence spaces defined by Orlicz function. Completeness of the spaces will be proved. We study some of their other properties like solidness, symmetricity etc. and prove some inclusion results.

Key words: Orlicz function, statistical convergence, difference double sequence space, completeness, regular convergence, solid space, symmetric space etc.

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1. Introduction

Throughout, a double sequence is denoted by $A = < a_{nk} >$, a double infinite array of elements $a_{nk} \in X$ for all $n, k \in N$, where X is the set of real or complex numbers.

The initial works on double sequences is found in Bromwich [3]. Later on it is studied by Hardy [5], Moricz [10] and many others.

Throughout the article ${}_2w$, ${}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$ denote the spaces of *all, bounded, convergent in Pringsheim's sense, null in Pringsheim's sense, regularly convergent and regularly null* double sequences of complex numbers.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [7] as follows.

$$Z(\Delta) = \{(x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in N$.

Following Hardy [5] the difference of double sequences is defined by Tripathy and Sarma [13] as follows.

$$\Delta a_{nk} = a_{nk} - a_{n+1,k} - a_{n,k+1} + a_{n+1,k+1} \quad \text{for all } n, k \in N.$$

The notion of statistical convergence of double sequences is introduced by Tripathy [11]. The idea depends on the density of subsets of $N \times N$. A subset E of $N \times N$ is said to have density $\rho(E)$ if

$$\rho(E) = \lim_{p,q \to \infty} \frac{1}{pq} \sum_{n \le p} \sum_{k \le q} \chi_E \text{ exists.}$$

A double sequence $< a_{nk} >$ is said to be statistically convergent in Pringsheim's sense to a number L if for given $\varepsilon > 0$, $\rho(\{(n,k) : |a_{nk} - L| \ge \varepsilon\}) = 0$.

A double sequence $< a_{nk} >$ is said to regularly statistically convergent to a number L if $< a_{nk} >$ converges statistically in Pringsheim's sense to L and the following statistical limits exist.

$$\text{stat} - \lim_{n \to \infty} a_{nk} = x_k, \text{ exist for each } k \in N.$$

and

$$\text{stat} - \lim_{n \to \infty} a_{nk} = y_k, \text{ exist for each } k \in N.$$

2. Definitions and preliminaries

An *Orlicz function* M is a mapping $M : [0, \infty) \to [0, \infty)$ such that it is *continuous*, *non-decreasing* and *convex* with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to construct the sequence space,

$$\ell^M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$||(x_k)|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\}.$$

Remark 1. Let $0 < \lambda < 1$, then $M(\lambda x) \le \lambda M(x)$, for all $x \ge 0$.

Definition 1. *A double sequence space E is said to be solid if $< \alpha_{nk}\, a_{nk} > \in E$ whenever $< a_{nk} > \in E$ for all double sequences $< \alpha_{nk} >$ of scalars with $|\alpha_{nk}| \le 1$ for all $n, k \in N$.*

Definition 2. *Let $K = \{(n_i, k_i) : i \in N; n_1 < n_2 < n_3 < \ldots$ and $k_1 < k_2 < k_3 < \ldots\} \subseteq N \times N$ and E be a double sequence space. A K-step space of E is a sequence space $\lambda_k^E = \{< a_{n_i k_i} > \in {}_2w : < a_{nk} > \in E\}$. A canonical* *pre-image of a sequence* $< a_{n_i k_i} > \in \lambda_k^E$ *is a sequence* $< b_{nk} > \in E$ *defined as follows:*

$$b_{nk} = \begin{cases} a_{nk}, & if \ \ (n,k) \in K, \\ 0, & otherwise. \end{cases}$$

A canonical pre-image of a step space λ_k^E *is a set of canonical pre-images of all elements in* λ_k^E*.*

Definition 3. *A double sequence space* E *is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark 2. From the above notions, it follows that 'If a sequence space E solid then E is monotone'.

Definition 4. *A double sequence space* E *is said to be symmetric if* $< a_{nk} > \in E$ *implies* $< a_{\pi(n)\pi(k)} > \in E$*, where* π *is a permutation of* N*.*

Definition 5. *A double sequence space* E *is said to be convergence free if* $< b_{nk} > \in E$ *whenever* $< a_{nk} > \in E$ *where* $a_{nk} = 0$ *implies* $b_{nk} = 0$*.*

Let M be an Orlicz function. We have the following double sequence spaces.

$$2\ell_\infty(M,q) = \left\{ < a_{nk} > \in {}_2w(q) : \sup_{n,k} M\left(q\left(\frac{a_{nk}}{\rho}\right)\right) < \infty, \text{ for some } \rho > 0 \right\}$$

$${}_2\overline{c}(M,q) = \left\{ < a_{nk} > \in {}_2w(q) : \text{stat} - \lim_{n,k} M\left(q\left(\frac{a_{nk} - L}{\rho}\right)\right) = 0, \text{ for some } \rho > 0 \right\}.$$

Also $< a_{nk} > \in {}_2\overline{c}^R(M,q)$ i.e. regularly convergent if $< a_{nk} > \in_2 \overline{c}(M,q)$ and the following limits hold:

There exists $L_k \in X$, such that $\text{stat} - \lim\limits_{n} M\left(q\left(\frac{a_{nk}-L_k}{\rho}\right)\right) = 0$, for some $\rho > 0$ and all $k \in N$.

There exists $J_n \in X$, such that $\text{stat} - \lim\limits_{k} M\left(q\left(\frac{a_{nk}-J_n}{\rho}\right)\right) = 0$, for some $\rho > 0$ and all $n \in N$.

The definition of ${}_2\overline{c}_0(M,q)$ and ${}_2\overline{c}_0^R(M,q)$ follows from the above definition on taking $L = L_k = J_n = \theta$, for all $n,k \in N$.

We introduce the following difference double sequence spaces.

$${}_2\ell_\infty(M,\Delta) = \left\{ < a_{nk} > \in {}_2w : \sup_{n,k} M\left(\frac{|\Delta a_{nk}|}{r}\right) < \infty, \text{ for some } r > 0 \right\}$$

$${}_2\overline{c}(M,\Delta) = \left\{ < a_{nk} > \in {}_2w : \text{stat} - \lim M\left(\frac{|\Delta a_{nk} - L|}{r}\right) = 0, \text{ for some } r > 0 \right\}.$$

Also $< a_{nk} > \in {}_2\bar{c}^R(M,\Delta)$ i.e. regularly convergent if $< a_{nk} > \in {}_2\bar{c}(M,\Delta)$ and the following limits hold:

There exists $L_k \in X$, such that stat $- \lim\limits_n M\left(\frac{|\Delta a_{nk}-L_k|}{r}\right) = 0$, for some $r>0$ and all $k \in N$.

There exists $J_n \in X$, such that stat $- \lim\limits_k M\left(\frac{|\Delta a_{nk}-J_n|}{r}\right) = 0$, for some $r>0$ and all $n \in N$.

The definition of ${}_2\bar{c}_0(M,\Delta)$ and ${}_2\bar{c}_0^R(M,\Delta)$ follows from the above definition on taking $L = L_k = J_n = 0$, for all $n,k \in N$.

3. Main results

Theorem 1. *The classes $Z(M,\Delta)$, where $Z = {}_2\bar{c}$, ${}_2\bar{c}_0$, ${}_2\bar{c}^R$, ${}_2\bar{c}_0^R$ and ${}_2\ell_\infty$ are linear spaces.*

Theorem 2. *The spaces $Z(M,\Delta)$, where $Z = {}_2\bar{c}^R$, ${}_2\bar{c}_0^R$ and ${}_2\ell_\infty$ are Banach spaces normed by*

$$f(< a_{nk} >) = \sup_n |a_{n1}| + \sup_k |a_{1k}| + \inf\left\{r>0 : \sup_{n,k} M\left(\frac{|\Delta a_{nk}|}{r}\right) \le 1\right\}.$$

Proof. We prove the theorem for the space ${}_2\bar{c}^R(M,\Delta)$ and the proof for the other cases can be established following similar technique. Let $A^i =< a^i_{nk} >$ be a Cauchy sequence in ${}_2\bar{c}^R(M,\Delta)$. We have to show the following:

(i) $a^i_{nk} \to a_{nk}$ as $i \to \infty$, for each $(n,k) \in N \times N$,

(ii) $a_i \to a$ as $i \to \infty$ where stat $- \lim a^i_{nk} = a_i$ for each $i \in N$.

(iii) $a_{nk} \to a$ (statistically relative to M).

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $t > 0$ such that $M\left(\frac{tx_0}{2}\right) \ge 1$ and $m_0 \in N$ be such that

$$f(< a^i_{nk} - a^j_{nk} >) < \frac{\varepsilon}{tx_0} \quad \text{for all } i,j \ge m_0. \tag{1}$$

By the definition of f we have

$$|a^i_{n1} - a^j_{n1}| < \frac{\varepsilon}{tx_0}, \quad |a^i_{1k} - a^j_{1k}| < \frac{\varepsilon}{tx_0}, \quad M\left(\frac{|\Delta a^i_{nk} - \Delta a^j_{nk}|}{r}\right) \le 1$$

$$\Rightarrow M\left(\frac{|\Delta a^i_{nk} - \Delta a^j_{nk}|}{f(a^i_{nk} - a^j_{nk})}\right) \le 1 \le M\left(\frac{tx_0}{3}\right) \quad \text{for all } i,j \ge m_0$$

$$\Rightarrow |\Delta a^i_{nk} - \Delta a^j_{nk}| < \frac{tx_0}{3}\frac{\varepsilon}{tx_0} = \frac{\varepsilon}{3} \quad \text{for all } i,j \ge m_0.$$

Hence $< a^j_{n1} >$, $< a^j_{1k} >$ and $< \Delta a^j_{nk} >$ are Cauchy sequences of complex numbers and so there exists complex numbers a_{n1}, a_{1k} and y_{nk} such that

$$\lim_{j\to\infty} a^j_{n1} = a_{n1}, \quad \lim_{j\to\infty} a^j_{1k} = a_{1k}, \quad \lim_{j\to\infty} \Delta a^j_{nk} = y_{nk}.$$

From this it is clear that $\lim\limits_{j\to\infty} a^j_{nk}$ exists. Using continuity of f, from (1) we have

$$a^i_{nk} \to a_{nk} \quad \text{as} \quad i \to \infty. \tag{2}$$

(ii) We have $\text{stat}-\lim a^i_{nk} = a$ for each $i \in N$. Thus there exists a subset $E_i \subset N \times N$ such that $\rho(E_i) = 1$ and

$$M\left(\frac{|a^i_{nk} - a_i|}{r}\right) \le M\left(\frac{\varepsilon}{3r}\right) \text{ for all } (n,k) \in E_i, \tag{3}$$

$$\text{for each } i \in N \text{ and for some } t > 0.$$

$$\Rightarrow |a^i_{nk} - a_i| < \frac{\varepsilon}{3}.$$

for all $(n,k) \in E_i$, for each $i \in N$ and by continuity of M.

Let $i, j \ge m_0$ and $(n,k) \in E_i \cap E_j$. Then we have

$$\begin{aligned} |a_i - a_j| &\le |a^i_{nk} - a_i| + |a^i_{nk} - a^j_{nk}| + |a^j_{nk} - a_j| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ by (2) and (3).} \end{aligned}$$

Hence $< a_i >$ is a Cauchy sequence in X, which is complete. Thus $< a_i >$ converges in X and let $\lim\limits_{i\to\infty} a_i = a$.

(iii) Let $\varepsilon_1 > 0$ be given. Let $i \ge m_0$ and $t > 0$ be so chosen that $M\left(\frac{\varepsilon}{t}\right) < \varepsilon_1$. From (ii) we have a subset $E \subset N \times N$ with $\rho(E) = 1$ such that

$$|a^i_{nk} - a_i| < \frac{\varepsilon}{3}.$$

By (i) we have $|a_{nk} - a^i_{nk}| < \frac{\varepsilon}{3}$ for all $i \ge m_0$. By (ii) we have $|a^i - a| < \frac{\varepsilon}{3}$ for all $i \ge m_0$. Hence for all $i \ge m_0$ and for all $(n,k) \in E$ with $\rho(E) = 1$, we have

$$\begin{aligned} |a_{nk} - a| &\le |a_{nk} - a^i_{nk}| + |a^i_{nk} - a_i| + |a_i - a| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \\ &\Rightarrow M\left(\frac{|a_{nk} - a|}{t}\right) \le M\left(\frac{\varepsilon}{t}\right) = \varepsilon_1 \text{ for some } t > 0 \\ &\qquad \text{and all } (n,k) \in E \text{ with } \rho(E) = 1 \\ &\Rightarrow \text{stat} - \lim a_{nk} = a. \end{aligned}$$

Hence $< a_{nk} > \in {}_2\bar{c}^R(M,\Delta)$. Thus ${}_2\bar{c}^R(M,\Delta)$ is a Banach space. ■

Proposition 1. *(i) $Z(M,\Delta) \subset_2 \ell_\infty(M,\Delta)$ for $Z =_2 \bar{c}^R,_2 \bar{c}^R_0$. The inclusions are strict.*

(ii) $Z(M) \subset Y(M,\Delta)$ for $Z = {}_2\bar{c}^R$, ${}_2\bar{c}$ and $Y = {}_2\bar{c}^R_0$, ${}_2\bar{c}_0$ respectively. The inclusions are strict.

Proposition 2. *The spaces* $Z(M,\Delta)$ *for* $Z = {}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *are nowhere dense subset of* ${}_2\ell_\infty(M,\Delta)$.

Proof. The proof is clear from the Theorem 2 and Proposition 1(i). ■

Proposition 3. *Let* M, M_1, M_2 *be Orlicz functions. Then*
(i) $Z(M_2,\Delta) \subseteq Z(M_1,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *if* $M_1(x) \le M_2(x)$ *for all* $x \in [0,\infty)$.
(ii) $Z(M_1,\Delta) \cap Z(M_2,\Delta) \subseteq Z(M_1 + M_2,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$.
(iii) $Z(M_1,\Delta) \subseteq Z(M \circ M_1,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$.

Proof. The proof of (i) and (ii) are obvious.
(iii) Consider $Z = {}_2\overline{c}$. Let $< a_{nk} > \in {}_2\overline{c}(M_1,\Delta)$. Then for some $r > 0$,

$$\text{stat} - \lim M_1\left(\frac{|\Delta a_{nk} - L|}{r}\right) = 0.$$

Let $b_{nk} = M_1\left(\frac{|\Delta a_{nk} - L|}{r}\right)$. Since $b_{nk} \to 0(\text{stat})$, there exists $J \subseteq N \times N$ with $\rho(J) = 1$ such that $b_{nk} < 1$ for all $(n,k) \in J$.
Now by the Remark 1, we have $M(b_{nk}) \le b_{bk} M(1)$.

$$\Rightarrow \text{stat} - \lim M\left(M_1\left(\frac{|\Delta a_{nk} - L|}{r}\right)\right) = 0$$

$$\Rightarrow \text{stat} - \lim (M \circ M_1)\left(\frac{|\Delta a_{nk} - L|}{r}\right) = 0.$$

Hence $< a_{nk} > \in_2 \overline{c}(M \circ M_1,\Delta)$. Similarly the result can be proved for the other cases also. ■

Proposition 4. *The spaces* $Z(M,\Delta)$ *for* $Z = {}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *and* ${}_2\ell_\infty$ *are* K*-spaces.*

Property 1. *The spaces* $Z(M,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *and* ${}_2\ell_\infty$ *are not symmetric.*

Proof. The result follows from the following example. ■

Example 1. *Consider the sequence space* ${}_2\overline{c}_0$. *Let* $M(x) = x^3$. *Let the sequence* $< a_{nk} >$ *be defined by*

$$a_{nk} = \begin{cases} 1, & \text{if } n \text{ is odd for all } k \in N, \\ -1, & \text{otherwise.} \end{cases}$$

Then $\Delta a_{nk} = 0$ *for all* $n, k \in N$.

Let $< b_{nk} >$ *be a rearrangement of the sequence* $< a_{nk} >$ *defined by*

$$b_{nk} = \begin{cases} -1, & \text{if } n+k \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\Delta b_{nk} = \begin{cases} -4, & \text{if } n+k \text{ is even,} \\ 4, & \text{otherwise.} \end{cases}$$

The sequence $< a_{nk} > \in {}_2\overline{c}_0(M,\Delta)$ *but* $< b_{nk} > \notin {}_2\overline{c}_0(M,\Delta)$.

Hence the space ${}_2\overline{c}_0(M,\Delta)$ *is not symmetric. Similarly the other spaces are also not symmetric.*

Property 2. *The spaces* $Z(M,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *and* ${}_2\ell_\infty$ *are not monotone and hence are not solid.*

Proof. To prove the results, consider the following example. ■

Example 2. *We prove the result for* $Z = {}_2\overline{c}$ *and the other cases can be proved using similar technique. Let* $M(x) = x$. *Let the sequence* $< a_{nk} >$ *be defined by*

$$a_{nk} = 1 \quad \text{for all } n,k \in N.$$

Then $< a_{nk} > \in_2 \overline{c}(M,\Delta)$. *Let* $J = \{(n,k) \in N \times N : n \geq k\}$. *Let the sequence* $< b_{nk} >$ *be defined by*

$$b_{nk} = \begin{cases} a_{nk}, & \text{for all } (n,k) \in J, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $< b_{nk} >$ *belongs to the canonical pre-image of* J*-step space of* ${}_2\overline{c}(M,\Delta)$, *but* $< b_{nk} > \notin_2 \overline{c}(M,\Delta)$. *Hence* ${}_2\overline{c}(M,\Delta)$ *is not monotone. Similarly it can be shown that the other spaces are not monotone.*

Thus by Remark 2 the spaces are not solid. The proof of the following result is obvious.

Property 3. *The spaces* $Z(M,\Delta)$ *for* $Z = {}_2\overline{c}$, ${}_2\overline{c}_0$, ${}_2\overline{c}^R$, ${}_2\overline{c}_0^R$ *and* ${}_2\ell_\infty$ *are not convergence free.*

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