F A S C I C U L I M A T H E M A T I C I

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FUZZY C-FLATS

Abstract. In this paper, we introduce the notion of fuzzy C-matroids, a new class of fuzzy matroids. We study properties of this class and define several new types of fuzzy maps between fuzzy matroids. In addition, we define fuzzy C-inner and fuzzy C-closure operators in this class and characterize fuzzy C-matroids and fuzzy maps between matroids in terms of these notions.

Key words: fuzzy matroid, fuzzy flat, fuzzy closure, fuzzy strong map, fuzzy hesitant map.

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1. Introduction

Matroid theory has several interesting applications in system analysis, operations research and economics. Since most of the time the aspects of matroid problems are uncertain, it is nice to deal with these aspects via the methods of fuzzy logic. The notion of fuzzy matroids was first introduced by Geotschel and Voxman in their landmark paper [2] using the notion of fuzzy independent set. The notion of fuzzy independent set was also explored in [7, 8]. Some constructions, fuzzy spanning sets, fuzzy rank and fuzzy closure axioms were also studied in [3, 4, 5, 11]. Several other fuzzifications of matroids were also discussed in [6, 9]. Since the notion of flats in traditional matroids is one of the most significant notions that plays a very important rule in characterizing strong maps (see for example [10]), in [1], the notions of fuzzy flats and fuzzy C flats were introduced and several examples were provided. Thus in [1], fuzzy matroids are defined via fuzzy flats axioms and it was shown that the levels of the fuzzy matroid introduced are indeed crisp matroids. Moreover, fuzzy strong maps and fuzzy hesitant maps are introduced and explored. We remark that this approach in [1] is different from those mentioned above.

Let E be any be any non-empty set. By $\wp(1)$ we denote the set of all fuzzy sets on E. That is $\wp(1) = [0,1]^E$, which is a completely distributive lattice. Thus let 0^E and 1^E denote its greatest and smallest elements, respectively. That is $0^E(e) = 0$ and $1^E(e) = 1$ for every $e \in E$. Let $\mathfrak{A}$ be a set of fuzzy sets and $\mu_1, \mu_2 \in \mathfrak{A}$. Then μ_1 is a subset of μ_2 , written $\mu_1 \leq \mu_2$, if $\mu_1(e) \leq \mu_2(e)$ for all $e \in E$. If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, then μ_1 is a proper subset of μ_2, written $\mu_1 < \mu_2$. Moreover, $\mu_1 \prec \mu_2$ if $\mu_1 < \mu_2$ and there does not exist $\mu_3 \in \mathfrak{A}$ such that $\mu_1 < \mu_3 < \mu_2$. Finally, $\mu_1 \vee \mu_2 = \sup\{\mu_1, \mu_2\}$ and $\mu_1 \wedge \mu_2 = \inf\{\mu_1, \mu_2\}$.

Next we recall some basic definitions and results from [1].

Definition 1. *Let E be a finite set and let $\mathfrak{F}$ be a finite family of fuzzy sets satisfying the following three conditions:*

(i) $1^E \in \mathfrak{F}$.

(ii) If $\mu_1, \mu_2 \in \mathfrak{F}$, then $\mu_1 \wedge \mu_2 \in \mathfrak{F}$.

(iii) If $\mu \in \mathfrak{F}$ and $\mu_1, \mu_2, \ldots, \mu_n$ are all minimal members of $\mathfrak{F}$ (with respect to standard fuzzy inclusion) that properly contain μ (in this case we write $\mu \prec \mu_i$ for all $i = 1, 2, \ldots, n$), then the fuzzy union of $\mu_1, \mu_2, ..., \mu_n$ is equal to 1^E (i.e. $\vee_{i=1}^{n}\mu_i = 1^E$).

Then the system $FM = (E, \mathfrak{F})$ is called fuzzy matroid *and the elements of $\mathfrak{F}$ are* fuzzy flats of FM: *The set $\mathcal{O} = \{1^E - \lambda : \lambda \in \mathfrak{F}\}$ is called the collection of fuzzy open sets of FM.*

Definition 2. *For $r \in (0, 1]$, let $C^r(\mu) = \{e \in E | \mu(e) \geq r\}$ be the r-level of a fuzzy set $\mu \in \mathfrak{F}$, and let $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ be the r-level of the family $\mathfrak{F}$ of fuzzy flats. Then for $r \in (0, 1]$, $(E, \mathfrak{F}^r)$ is the*r-level *of the fuzzy set system $(E, \mathfrak{F})$.*

Theorem 1. *For every $r \in (0, 1]$, the r-levels $\mathfrak{F}^r = \{C^r(\mu) : \mu \in \mathfrak{F}\}$ of a family of fuzzy flats $\mathfrak{F}$ of a fuzzy matroid $FM = (E, \mathfrak{F})$ is a family of crisp flats.*

Definition 3. *Let E be any set with n-elements and $\mathfrak{F} = \{\chi_A : A \leq E$, $|A| = n$ or $|A| < m\}$ where m is a positive integer such that $m \leq n$. Then $(E, \mathfrak{F})$ is a fuzzy matroid called the fuzzy uniform matroid on n-elements and rank m, denoted by $F_{m,n}$. $F_{m,m}$ is called the free fuzzy uniform matroid on n-elements.*

We remark that the rank notion in the preceding definition coincides with that in [4].

Definition 4. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then the* fuzzy closure of μ *is* $\bar{\mu} = \bigwedge_{\lambda \in \mathfrak{F};\, \mu \leq \lambda} \lambda$.

Theorem 2. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and X be a non-empty subset of E. Then $(X, \mathfrak{F}_X)$ is a fuzzy matroid, where $\mathfrak{F}_X = \{\chi_X \wedge \mu : \mu \in \mathfrak{F}\}$.*

Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid, X be a non-empty subset of E and μ be a fuzzy set in X. We may realize μ as a fuzzy set in E by the convention that $\mu(e) = 0$ for all $e \in E - X$. It can be easily shown that $\mathfrak{F}_X = \{\mu|_X : \mu \in \mathfrak{F}\}$, where $\mu|_X$ is the restriction of μ to X.

Let E_1 and E_2 be two sets, μ_1 is a fuzzy set in E_1, μ_2 is a fuzzy set in E_2 and $f : E_1 \to E_2$ be a map. Then we define the fuzzy sets $f(\mu_1)$ (the image of μ_1) and $f^{-1}(\mu_2)$ (the preimage of μ_2) by

$$f(\mu_1)(y) = \begin{cases} \sup\{\mu_1(x) : x \in f^{-1}(\{y\})\}, & y \in Range(f) \\ 0\,, & O.W. \end{cases}$$

and

$$f^{-1}(\mu_2)(x) = \mu_2(f(x)) \quad \text{for all } \ x \in E_1.$$

Definition 5. *A fuzzy strong map from a fuzzy matroid $FM_1 = (E_1, \mathfrak{F}_1)$ into a fuzzy matroid $FM_2 = (E_2, \mathfrak{F}_2)$ is a map $f : E_1 \to E_2$ such that the preimage of every fuzzy flat in FM_2 is a fuzzy flat in FM_1.*

Theorem 3. *Let $FM_1 = (E_1, \mathfrak{F}_1)$ and $FM_2 = (E_2, \mathfrak{F}_2)$ be fuzzy matroids and $f : E_1 \to E_2$ be a map. Then the following are equivalent:*
(i) f is fuzzy strong.
(ii) For every fuzzy set μ_1 in FM_1, $f(\overline{\mu_1}) \leq \overline{f(\mu_1)}$.
(iii) For every fuzzy set μ_2 in FM_2, $\overline{f^{-1}(\mu_2)} \leq f^{-1}(\overline{\mu_2})$.

Definition 6. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy C flat if $\bigvee_{1^E - \lambda \in \mathfrak{F}, \lambda \leq \overline{\mu}} \lambda \leq \mu$.*

Clearly, every fuzzy flat is a fuzzy C flat, but the converse need not be true.

Example 1. Let $E = \{a, b, c, d\}$ and $\mathfrak{F} = \{1^E, 0, \chi_{\{b,d,c\}}, \chi_{\{a,c,d\}}, \chi_{\{c,d\}}\}$. By Theorem 2, $FM = (E, \mathfrak{F})$ is a fuzzy matroid. Since $\overline{\chi_{\{d\}}} = \chi_{\{c,d\}}$ and $\bigvee_{1^E - \lambda \in \mathfrak{F}, \lambda \leq \chi_{\{c,d\}}} \lambda = 0 \leq \chi_{\{c,d\}}$, $\chi_{\{d\}}$ is a fuzzy C flat that is not a fuzzy flat.

Lemma 1. *The intersection of fuzzy C flats is a fuzzy C flat.*

Definition 7. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. The fuzzy C closure of μ is $\overline{\mu}^F = \bigwedge\{\acute{\mu} : \acute{\mu}$ is a fuzzy C flat and $\mu \leq \acute{\mu}\}$.*

Theorem 4. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*
(i) $\overline{0}^F = 0$.
(ii) $\overline{\mu}^F$ is a fuzzy C flat.
(iii) $\mu \leq \overline{\mu}^F$.
(iv) If $\mu \leq \lambda$, then $\overline{\mu}^F \leq \overline{\lambda}^F$.
(v) $\overline{\overline{\mu}^F}^F = \overline{\mu}^F$.

Theorem 5. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ be a fuzzy set. Then μ is a fuzzy C flat if and only if $\overline{\mu}^F = \mu$.*

Theorem 6. *Let $FM = (E, \mathfrak{F})$ be a fuzzy matroid and μ, λ be fuzzy sets. Then*

(i) $\overline{\mu \vee \lambda}^F \geq \overline{\mu}^F \vee \overline{\lambda}^F$.

(ii) $\overline{\mu \wedge \lambda}^F \leq \overline{\mu}^F \wedge \overline{\lambda}^F$.

Definition 8. *A map $f : FM_1 \to FM_2$ is*

(i) fuzzy C strong *if the inverse image of every fuzzy flat of FM_2 is a fuzzy C flat of FM_1.*

(ii) fuzzy hesitant *if the inverse image of every fuzzy C flat of FM_2 is a fuzzy C flat of FM_1.*

Clearly, a fuzzy strong (fuzzy hesitant) map is fuzzy C strong, but the converse need not be true since a fuzzy C flat need not be a fuzzy flat as we have seen in Example 1.

A map $f : FM_1 \to FM_2$ is said to be *fuzzy* if the image of every fuzzy flat of FM_1 is a fuzzy flat of FM_2. The following is a trivial result.

Lemma 2. *Let $f : FM_1 \to FM_2$ be a fuzzy map that is also fuzzy strong. Then $f^{-1}(\bar{\mu}) = \overline{f^{-1}(\mu)}$ for every fuzzy set μ of FM_2.*

Theorem 7. *A fuzzy map $f : FM_1 \to FM_2$ that is also fuzzy strong is fuzzy hesitant.*

Theorem 8. *The following are equivalent for a map $f : FM_1 \to FM_2$:*

(i) *f is hesitant.*

(ii) *$f(\overline{\mu}^F) \leq \overline{f(\mu)}^F$ for every fuzzy set μ of FM_1.*

(iii) *$\overline{f^{-1}(\lambda)}^F \leq f^{-1}(\overline{\lambda}^F)$ for every fuzzy set λ of FM_2.*

2. Fuzzy C-matroid

In this section, we define and study several properties of *fuzzy C-matroids*. In addition, we characterize *fuzzy C-open* sets in terms of the *fuzzy C-inner* and *fuzzy weak closure* of fuzzy sets. The following result is immediate:

Lemma 3. *A fuzzy map $f : FM_1 \to FM_2$ is fuzzy strong if and only if $f(\overline{\lambda}) \leq \overline{f(\lambda)}$ for every subset λ of the ground set of FM_1.*

Definition 9. *Let $FM = (E,\mathcal{O})$ be a fuzzy matroid. A fuzzy set $\lambda \in E$ is called a* fuzzy C-open set *in FM if there exists a fuzzy open set μ such that $\mu \leq \lambda \leq \bar{\mu}$.*

The collection of all fuzzy C-open sets in FM is denoted by $FO(FM)$ and the pair $(E, FO(FM))$ is called the *fuzzy C-matroid* associated with FM. A subset $\lambda \in E$ is *fuzzy C-closed* if its complement is fuzzy C-open.

Example 2. Let $E = \{a, b, c\}$ and $\mathcal{O} = \{0, \chi_{\{a\}}\}$. Consider the matroid $FM = (E, \mathcal{O})$. Then 0 and $\chi_{\{a\}}$ are fuzzy C-open sets. In fact, every fuzzy open set is fuzzy C-open and thus, the class of fuzzy C-matroids is not empty.

Feeble-closure $\underline{\lambda}$ of λ can be defined in a manner analogous to the fuzzy closure $\bar{\lambda}$ of λ. The *fuzzy C-inner* λ_o of λ and *fuzzy C-spanning* sets can be defined an analogous manner to the fuzzy inner and fuzzy spanning set notions.

A map $f : FM_1 \to FM_2$ is *fuzzy C-strong* if the inverse image of any fuzzy open set in FM_2 is a fuzzy C-open set in FM_1, f is *fuzzy hesitant* if the inverse image of any fuzzy C-open set in FM_2 is a fuzzy C-open set in FM_1 and f is *fuzzy pre-C-open* if the image of any fuzzy C-open set in FM_1 is a fuzzy C-open set in FM_2. Two fuzzy matroids FM_1 and FM_2 are *fuzzy C-isomorphic* if there exists a fuzzy map $h : FM_1 \to FM_2$ which is bijective, fuzzy hesitant and fuzzy pre-C-open. Such an h is called *fuzzy C-isomorphism.*

Our main goal is to study properties of fuzzy C-matroids and the preceding maps between fuzzy matroids. We characterize fuzzy C-matroids and the maps between fuzzy matroids purely in terms of fuzzy C-inner and fuzzy C-closure operators. Finally, we define an equivalence relation on a certain collection of fuzzy matroids which partitions that collection into two classes of fuzzy matroids with the same collection of fuzzy C-open sets.

We next show that every fuzzy matroid is a fuzzy C-matroid, but the converse needs not be true.

Theorem 9. *Let* $FM = (E,\mathcal{O})$ *be a fuzzy matroid. Then*
(a) $\mathcal{O} \leq FO(FM)$;
(b) *for* $\lambda \in FO(FM)$ *and* $\lambda \leq \mu \leq \bar{\lambda}$, *then* $\mu \in FO(FM)$.

Proof. (a) Is trivial. (b) As λ is fuzzy C-open, there exists a fuzzy open set η such that $\eta \leq \lambda \leq \bar{\eta}$. Thus by Theorem 4, $\eta \leq \lambda \leq \mu \leq \bar{\lambda} \leq \bar{\bar{\eta}} = \bar{\eta}$ and hence $\mu \in FO(FM)$. ■

Corollary 1. *Every fuzzy matroid is a fuzzy C-matroid.*

In the following example, we show that the converse of the preceding corollary needs not be true.

Example 3. Let $E = \{a, b, c, d\}$ and $\mathcal{O} = \{0, 1, \chi_{\{a,b\}}, \chi_{\{c,d\}}\}$. Consider the matroid $FM = (E, \mathcal{O})$. Then $FO(FM) = \{1, 0, \chi_{\{a\}}, \chi_{\{b\}}, \chi_{\{d\}}, \chi_{\{a,b\}}, \chi_{\{c,d\}}\}$, but $\{1 - \mu : \mu \in FO(FM)\}$ does not satisfy (iii) of Definition 1 and hence the fuzzy C-matroid $(E, FO(FM))$ is not a matroid.

Next, we show that a strong map which is also open, is a pre-fuzzy C-open.

Theorem 10. *Let* $FM_1 = (E_1,\mathcal{O}_1)$ *and* $FM_2 = (E_2,\mathcal{O}_2)$ *be fuzzy matroids. Let* $f : FM_1 \to FM_2$ *be a fuzzy strong and fuzzy open map. If* $\lambda \in FO(FM_1)$, *then* $f(\lambda) \in FO(FM_2)$.

Proof. As $\lambda \in FO(FM_1)$, there exists a fuzzy open set $\mu \in \ \mu_1$ such that $\mu \leq \lambda \leq \bar{\mu}$. Thus $f(\mu) \leq f(\lambda) \leq f(\bar{\mu})$. As f is a strong map, by Lemma 3, $f(\bar{\mu}) \leq \overline{f(\mu)}$ and as f is open, $f(\mu) \in \ \mathcal{O}_2$. Therefore, $f(\lambda) \in FO(FM_2)$. ■

Next, we show all non-trivial fuzzy C-open sets in a fuzzy matroid, must contain non-trivial fuzzy open sets.

Theorem 11. *Let* $FM = (E, \mathcal{O})$ *be a matroid and* λ *be a fuzzy C-open set such that* $\lambda \neq 0$ *and* $\lambda \neq 1$. *Then there exists* $\mu \in O - \{0, 1\}$ *such that* $\mu \leq \lambda$.

Proof. As λ is fuzzy C-open, there exists a fuzzy open set μ such that $\mu \leq \lambda \leq \bar{\mu}$. As $\lambda \neq 1$ and $\mu \leq \lambda$, $\mu \neq 1$. Also as $\lambda \neq 0$, $\bar{\mu} \neq 0$ and hence $\mu \neq 0$. ■

In the next result, we characterize fuzzy C-open sets in terms of the fuzzy C-inner and fuzzy C-closure notions.

Theorem 12. *Let* $FM = (E, \mathcal{O})$ *be a matroid and* $\lambda \in E$. *Then*
(a) λ *is fuzzy C-open if and only if* $\lambda_o = \lambda$.
(b) λ *is fuzzy C-closed if and only if* $\underline{\lambda} = \lambda$.

Proof. Only the proof of part (a) is given. The proof of (b) is similar.

For every $x \in \lambda_o$, there exists a fuzzy C-open set μ such that $x \in \mu \leq \lambda$. Thus $x \in \lambda$ and hence $\lambda_o \leq \lambda$. On the other hand, for every $x \in \lambda$, as λ is fuzzy C-open, $x \in \lambda_o$. Thus $\lambda \leq \lambda_o$ and hence $\lambda_o = \lambda$. ■

We end this section with two main results related to fuzzy C-open sets, that shall be used in the next two sections.

Theorem 13. *Let* $FM = (E, \mathcal{O})$ *be a matroid,* $\mu \in \mathcal{O}$ *and* λ *be a fuzzy C-open set. Then* $\mu \vee \lambda$ *is fuzzy C-open.*

Proof. As λ is fuzzy C-open, there exists a fuzzy open set η such that $\eta \leq \lambda \leq \bar{\eta}$. Thus $\eta \vee \mu \leq \lambda \vee \mu \leq \bar{\eta} \vee \mu \leq \overline{\eta \vee \mu}$ and as $\eta \vee \mu \in \mathcal{O}$, $\eta \vee \mu$ is fuzzy C-open. ■

Corollary 2. *Let* $FM(E, \mathcal{O})$ *be a matroid,* η *be a fuzzy flat of* M *and* λ *be a fuzzy C-closed set. Then* $\eta \wedge \lambda$ *is fuzzy C-closed.*

Theorem 14. *Let $FM = (E, \mathcal{O})$ be a matroid and $\lambda \in E$. Then*
(a) $1 - (\bar{\lambda} - \lambda)$ *is a fuzzy C-spanning set.*
(b) $(\bar{\lambda})^o \leq (\underline{\lambda})_o$

Proof. *(a)* Suppose there exists $e \in 1 - \overline{1 - (\bar{\lambda} - \lambda)}$. Then there exists a fuzzy C open set μ containing e such that $\mu \wedge \big(1 - (\bar{\lambda} - \lambda)\big) = 0$. Thus $\mu \wedge (1 - \bar{\lambda}) = 0$ and $\mu \wedge \lambda = 0$ and hence $\mu \leq \bar{\lambda} \wedge (1 - \lambda)$. Then $\bar{\mu} \leq \bar{\lambda} \wedge \overline{(1 - \lambda)}$ and thus $\bar{\mu} \leq \lambda - \lambda^o \leq \lambda$ and as $\mu \leq \bar{\mu}$, $\mu \leq \lambda$. Therefore, $\mu \leq \lambda \wedge (1 - \lambda) = 0$, which is impossible.

(b) We show $x \notin (\underline{\lambda})_o$ implies $x \notin (\bar{\lambda})^o$. Assume $x \notin (\underline{\lambda})_o$ and let μ be any fuzzy open set containing x. Then by Theorem 9, μ is fuzzy C-open and as $x \notin (\underline{\lambda})_o$, $\mu \not\leq \underline{\lambda}$. Thus there exists $y \in \mu$ such that $y \notin \underline{\lambda}$ and as μ is fuzzy C-open containing y, $\mu \wedge \lambda = 0$. Hence $y \notin \bar{\lambda}$ and thus $\mu \not\leq \bar{\lambda}$. Therefore, $x \notin (\bar{\lambda})^o$. ■

3. Fuzzy hesitant maps

In this section, properties of fuzzy hesitant maps are studied. In addition, connections between fuzzy hesitant maps and fuzzy strong maps, fuzzy open maps, fuzzy C-strong maps, fuzzy pre-C-strong maps and fuzzy inner and fuzzy closure notions are also studied. The following trivial result is an elementary fuzzy matroid result.

Lemma 4. *If $f : FM_1 \to FM_2$ is a fuzzy strong and fuzzy open map, then $f^{-1}(\bar{A}) = \overline{f^{-1}(\lambda)}$ for any subset λ of the ground set of FM_2.*

Theorem 15. *Any fuzzy map which is fuzzy open and fuzzy strong is fuzzy hesitant.*

Proof. Let $f : FM_1 \to FM_2$ be a fuzzy open and fuzzy strong map. If $\lambda \in FO(FM_2)$, then there exists a fuzzy open set μ in FM_2 such that $\mu \leq \lambda \leq \bar{\mu}$. By Lemma 4, $f^{-1}(\bar{\mu}) = \overline{f^{-1}(\mu)}$. Also, $f^{-1}(\mu) \leq f^{-1}(\lambda) \leq f^{-1}(\bar{\mu}) = \overline{f^{-1}(\mu)}$, and since f is fuzzy strong, $f^{-1}(\ \mu)$ is fuzzy open. Thus $f^{-1}(\lambda) \in FO(FM_1)$ and hence f is fuzzy hesitant. ■

In the following example, we show that a fuzzy map which is fuzzy strong and fuzzy hesitant needs not be fuzzy open.

Example 4. Let $E = \{a, b, c\}$, $\mathrm{O}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, E\}$ and $\mathrm{O}_2 = \mathrm{O}_1 \cup \{\{a, c\}\}$. Then $FO(E,\mathrm{O}_1) = FO(E, \mathcal{O}_2)$. Thus the identity map $i : (E,\mathrm{O}_2) \to (E,\mathrm{O}_1)$ is fuzzy strong and fuzzy hesitant but not fuzzy open.

Next, we show fuzzy strong maps and fuzzy hesitant maps are fuzzy C-strong.

Theorem 16. *Let $S(FM_1, FM_2)$, $FS(FM_1, FM_2)$ and $H(FM_1, FM_2)$ denote respectively, the class of fuzzy strong, fuzzy C-strong and fuzzy hesitant maps. Then $S(FM_1, FM_2) \subseteq FS(FM_1, FM_2)$ and $H(FM_1, FM_2) \subseteq FS(FM_1, FM_2)$.*

Proof. $S(FM_1, FM_2) \subseteq FS(FM_1, FM_2)$ because if the inverses of fuzzy open sets are fuzzy open, it follows by Theorem 9 that the inverses are fuzzy C-open. $H(FM_1, FM_2) \subseteq FS(FM_1, FM_2)$ because if the inverses of fuzzy C-open sets are fuzzy C-open, it follows by Theorem 9 that the inverses of fuzzy open sets are fuzzy C-open. ■

Next, we characterize fuzzy hesitant maps in terms of fuzzy C-closed sets and fuzzy C-closure of fuzzy sets.

Theorem 17. *A fuzzy map $f : FM_1 \to FM_2$ is fuzzy hesitant if and only if the inverses of fuzzy C-closed sets in FM_2 are fuzzy C-closed in FM_1.*

Proof. Follows directly from definitions. ■

Theorem 18. *A fuzzy map $f : FM_1 \to FM_2$ is fuzzy hesitant if and only if $f(\underline{\lambda}) \leq \underline{f(\lambda)}$ for every subset $\lambda \in E(FM_1)$.*

Proof. If $\lambda \in E(FM_1)$, then consider $\underline{f(\lambda)}$ which is fuzzy C-closed in M_2. Thus by Theorem 17, $f^{-1}(\underline{f(\lambda)})$ is fuzzy C-closed in M_1. Furthermore, $\lambda \leq f^{-1}(f(\lambda)) \leq f^{-1}(\underline{f(\lambda)})$. Therefore, by definition of fuzzy C-closure, $\underline{\lambda} \leq f^{-1}(\underline{f(\lambda)})$, and consequently,

$$f(\underline{\lambda}) \leq f(f^{-1}(\underline{f(\lambda)})) = f(\underline{\lambda}) \wedge f(1^{E_1})) \leq \underline{f(\lambda)}.$$

Conversely, if μ is fuzzy C-closed in M_2, consider $f^{-1}(\mu)$. Note that

$$f(\underline{f^{-1}(\mu)}) \leq \underline{f(f^{-1}(\mu))} = \underline{\mu \wedge f(1^{E_1}))} \leq \underline{\mu} = \mu.$$

Hence, $\underline{f^{-1}(\mu)} \leq f^{-1}(\mu)$, so that $f^{-1}(\mu) = \underline{f^{-1}(\mu)}$, and by Theorem 12, $f^{-1}(\mu)$ is fuzzy C-closed. Thus f is fuzzy hesitant by Theorem 17. ■

Theorem 19. *A fuzzy map $f : FM_1 \to FM_2$ is fuzzy hesitant if and only if $\underline{f^{-1}(\mu)} \leq f^{-1}(\underline{\mu})$ for every subset $\mu \in E(FM_2)$.*

Proof. The proof is similar to the proof of Theorem 18. ■

Theorem 20. *If $f : FM_1 \to FM_2$ and $g : FM_2 \to FM_3$ are fuzzy hesitant, then $g \circ f$ is fuzzy hesitant.*

Proof. Obvious. ■

Theorem 21. *If $f : FM_1 \to FM_2$ is a fuzzy strong and fuzzy open map, then f is fuzzy hesitant and fuzzy pre-C-open.*

Proof. The result follows from Theorem 10 and Theorem 15. ∎

4. C-isomorphisms

In this section, we study the properties of fuzzy C-isomorphisms. We show that every fuzzy isomorphism is a fuzzy C-isomorphism and then we characterize fuzzy C-isomorphisms purely in terms of fuzzy C-inner and fuzzy C-closure of sets. The proof of the following result follows directly from Theorem 21.

Theorem 22. *Every fuzzy isomorphism is a fuzzy C-isomorphism.*

The converse of the preceding theorem needs not be true. For example, the fuzzy map $i : (E, O_2) \to (E, O_1)$ in Example 4 is a fuzzy C-isomorphism, but i is not a fuzzy isomorphism.

Theorem 23. *If $h : FM_1 \to FM_2$ is a fuzzy C-isomorphism, then $\underline{h^{-1}(\mu)} = h^{-1}(\underline{\mu})$ for all $\mu \in E(FM_2)$.*

Proof. By Theorem 19, $\underline{h^{-1}(\mu)} \leq h^{-1}(\underline{\mu})$ since h is fuzzy hesitant. h^{-1} is a fuzzy hesitant map, so by Theorem 18, $h^{-1}(\underline{\mu}) \leq \underline{h^{-1}(\mu)}$ which completes the proof. ∎

Corollary 3. *If $h : FM_1 \to FM_2$ is a fuzzy C-isomorphism, then $\underline{h(\lambda)} = h(\underline{\lambda})$ for all $\lambda \in E(FM_1)$.*

Corollary 4. *The property of having a fuzzy C-spanning set is preserved under fuzzy C-isomorphism.*

Corollary 5. *If $h : FM_1 \to FM_2$ is a fuzzy C-isomorphism, then $h(\lambda_o) = (h(\lambda))_o$ for all $\lambda \in E(FM_1)$.*

Proof. $\lambda_o = 1^{E_1} - \underline{(1^{E_1} - \lambda)}$. Thus,

$$h(\lambda_o) = 1^{E_2} - h(\underline{(1^{E_1} - \lambda)}) = 1^{E_2} - \underline{h(1^{E_1} - \lambda)}.$$

Therefore, $h(\lambda_o) = 1^{E_2} - \underline{1^{E_2} - h(\lambda))} = (h(\lambda))_o$. ∎

Corollary 6. *If $h : FM_1 \to FM_2$ is a fuzzy C-isomorphism, then $h^{-1}(\mu_o) = (h^{-1}(\mu))_o$ for all $\mu \in E(FM_2)$.*

Theorem 24. $(\underline{\lambda})_o = 0$ *if and only if* $(\bar{\lambda})^o = 0$.

Proof. If $(\underline{\lambda})_o = 0$, then by Theorem 14, $(\bar{\lambda})^o \leq (\underline{\lambda})_o$. Thus $(\bar{\lambda})^o = 0$. Conversely, if $(\bar{\lambda})^o = 0$, then as $\underline{\lambda} \leq \bar{\lambda}$, 0 is the only fuzzy C-open set contained in $\underline{\lambda}$. Thus by Theorem 11, 0 is the only fuzzy open set contained in $\underline{\lambda}$. Hence 0 is the only fuzzy C-open set contained in $\underline{\lambda}$. Therefore, $(\underline{\lambda})_o = 0$. ■

Theorem 25. *If $h : FM_1 \to FM_2$ is a fuzzy C-isomorphism and $\lambda \in E(FM_1)$ such that $(\bar{\lambda})^o = 0$, then $(\overline{h(\lambda)})^o = 0$.*

Proof. By Theorem 24, since $(\bar{\lambda})^o = 0$, $(\underline{\lambda})_o = 0$. Now consider $h(\lambda)$. Note that by Corollary 3, $\underline{h(\lambda)} = h(\underline{\lambda})$. Thus $(\underline{h(\lambda)})_o = (h(\underline{\lambda}))_o = h(\underline{\lambda}_o)$ by Corollary 5. Hence $(\underline{h(\lambda)})_o = h(0) = 0$, and by Theorem 24, $(\overline{h(\lambda)})^o = 0$. ■

Theorem 26. *Feeble-isomorphism is an equivalence relation between matroids.*

Proof. Reflexivity and symmetry are immediate and transitivity follows from Theorem 20. ■

5. C-matroid classes

If E is a finite fuzzy set, let $FM(E)$ denote the collection of all fuzzy matroids which have E as their ground set. If (E,O_1) and $(E, \mathcal{O}_2)$ are two elements of $FM(E)$, then $(E, \mathcal{O}_1)$ is *fuzzy C-correspondent* to $(E, \mathcal{O}_2)$ if $FO(E,\mathrm{O}_1) = FO(E, \mathcal{O}_2)$.

Theorem 27. *Fuzzy C-correspondent is an equivalence relation on $FM(E)$.*

Proof. Clearly $(E, \mathcal{O})$ is fuzzy C-correspondent to itself for any collection of fuzzy open sets O. Symmetry and transitivity follow from symmetry and transitivity of fuzzy set equality, respectively. ■

Thus, the collection $M(E)$ is partitioned into equivalence classes. Denote the equivalence class of fuzzy matroids with the same collection of fuzzy C-open sets as $FM(E,\mathrm{O})$ by $[E, FO(E, \mathcal{O})]$. Then clearly, $[E, FO(E, \mathcal{O})]$ contains a maximal fuzzy matroid in the sense that the fuzzy matroid induced on E by the fuzzy C-closure operator is finer than the fuzzy matroid on any other collection of fuzzy open sets in $[E, FO(E, \mathcal{O})]$, and of course the fuzzy matroid so induced gives a fuzzy matroid in $[E, FO(E, \mathcal{O})]$. We end this section with two powerful results that have trivial proofs follow immediately from definitions.

Theorem 28. *If $f : (E_1, \mathcal{O}_1) \to (E_2, \mathcal{O}_2)$ is fuzzy hesitant, and if $(E_1, \mathcal{O}_3)$ is an element of $[E_1, FO(E_1, \mathcal{O}_1)]$ and $(E_2, \mathcal{O}_4)$ is an element of $[E_2, FO(E_2, \mathcal{O}_2)]$, then $f : (E_2, \mathcal{O}_3) \to (E_2, \mathcal{O}_4)$ is fuzzy hesitant.*

Theorem 29. *If $f : (E_1, \mathcal{O}_1) \to (E_2, \mathcal{O}_2)$ is fuzzy C-strong, and if $(E_1, \mathcal{O}_3) \in [E_1, FO(E_1, \mathcal{O}_1)]$, then $f : (E_1, \mathcal{O}_3) \to (E_2, \mathcal{O}_2)$ is fuzzy C-fuzzy strong.*

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