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**QUALITATIVE PROPERTIES FOR A HIGHER ORDER
RATIONAL DIFFERENCE EQUATION**

ABSTRACT. The main objective of this paper is to study the behavior of solutions of the difference equation

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-q}x_{n-l}}{cx_{n-q} + dx_{n-p}}, \quad n = 0, 1, \dots,$$

where the initial conditions $x_{-r}, x_{-r+1}, \dots, x_0$ are arbitrary positive real numbers, $r = \max\{q, l, p\}$ is nonnegative integer and a, b, c, d are positive constants. Also, we give the solution of some special cases of this equation.

KEY WORDS: stability, boundedness, solutions of the difference equations.

AMS Mathematics Subject Classification: 39A10, 39A11.

1. Introduction

In this paper we deal with the behavior of solutions of the difference equation

$$(1) \quad x_{n+1} = ax_{n-l} + \frac{bx_{n-q}x_{n-l}}{cx_{n-q} + dx_{n-p}}, \quad n = 0, 1, \dots,$$

where the initial conditions $x_{-r}, x_{-r+1}, \dots, x_0$ are arbitrary positive real numbers, $r = \max\{q, l, p\}$ is nonnegative integer and a, b, c, d are positive constants. Moreover, we obtain the form of the solution of some special cases of equation (1) and some numerical simulations to the equation are given to illustrate our results. Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$f : I^{k+1} \rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$(2) \quad x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots,$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1 (Equilibrium Point). *A point $\bar{x} \in I$ is called an equilibrium point of equation (2) if*

$$\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of (2), or equivalently, \bar{x} is a fixed point of f .

Definition 2 (Periodicity). *A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.*

Definition 3 (Fibonacci sequence). *The sequence $\{F_m\}_{m=0}^{\infty} = \{1, 2, 3, 5, 8, 13, \dots\}$, that is, $F_m = F_{m-1} + F_{m-2}$, $m \geq 0$, $F_{-2} = 0$, $F_{-1} = 1$ is called Fibonacci sequence.*

Definition 4 (Stability). *(i) The equilibrium point \bar{x} of (2) is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with*

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point \bar{x} of (2) is called locally asymptotically stable if \bar{x} is locally stable solution of (2), and there exist $\gamma > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have $\lim_{n \rightarrow \infty} x_n = \bar{x}$.

(iii) The equilibrium point \bar{x} of (2) is called a global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ we have $\lim_{n \rightarrow \infty} x_n = \bar{x}$.

(iv) The equilibrium point \bar{x} of (2) is called globally asymptotically stable if \bar{x} is locally stable and \bar{x} is also global attractor.

(v) The equilibrium point \bar{x} of (2) is called unstable if \bar{x} is not locally stable.

The linearized equation of equation (2) about the equilibrium \bar{x} is the linear difference equation

$$(3) \quad y_{n+1} = \sum_{i=0}^k \frac{\partial f(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}.$$

The following tws will be useful for the proof of our results in this paper.

Theorem 1 ([18]). Assume $p_1, p_2, \dots, p_k \in R$ and $k \in \{1, 2, \dots\}$. Then

$$\sum_{i=1}^k |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$(4) \quad x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, \dots$$

Consider the following equation

$$(5) \quad x_{n+1} = g(x_{n-q}, x_{n-l}, x_{n-p}).$$

The next theorem will be useful for the proof of our results in this paper.

Theorem 2 (see [20]). Let $[a, b]$ be an interval of real numbers and assume that

$$g : [a, b]^3 \rightarrow [a, b],$$

is a continuous function satisfying the following properties:

(a) $g(x, y, z)$ is nondecreasing in x and y in $[a, b]$ for each $z \in [a, b]$, and is nonincreasing in $z \in [a, b]$ for each x and y in $[a, b]$.

(b) If $(m, M) \in [a, b] \times [a, b]$ is a solution of the system

$$M = g(M, M, m) \text{ and } m = g(m, m, M),$$

then

$$m = M.$$

Then equation (5) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of equation (5) converges to \bar{x} .

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: Elabbasy et al. [7] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^k x_{n-i}}.$$

Also Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of special case of the following recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}.$$

In [13] Elsayed dealt with the behavior of solutions of the difference equation of order two and gave the form of the solution of some special cases of this equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}.$$

Elsayed [15] studied the behavior of solutions of the difference equation of order four and gave the form of the solution of some special cases of this equation

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}x_{n-3}}{cx_{n-1} + dx_{n-3}}.$$

Other related results on rational difference equations can be found in refs. [1-15, 19-27].

2. Local stability of the equilibrium point

In this section we investigate the local stability character of the solutions of equation (1). Equation (1) has a unique equilibrium point

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{c\bar{x} + d\bar{x}},$$

or

$$\bar{x}^2 (1 - a) (c + d) = b\bar{x}^2,$$

if $(1 - a)(c + d) \neq b$, then the unique equilibrium point is $\bar{x} = 0$.

Let $f : (0, \infty)^3 \rightarrow (0, \infty)$ be a function defined by

$$(6) \quad f(u, v, w) = av + \frac{buv}{cu + dw}.$$

Therefore it follows that

$$f_u(u, v, w) = \frac{bdvw}{(cu + dw)^2},$$

$$f_v(u, v, w) = a + \frac{bu}{cu + dw},$$

and

$$f_w(u, v, w) = \frac{-bduv}{(cu + dw)^2},$$

we see that

$$f_u(\bar{x}, \bar{x}, \bar{x}) = \frac{bd}{(c + d)^2} = -\alpha_2,$$

$$f_v(\bar{x}, \bar{x}, \bar{x}) = a + \frac{b}{c + d} = -\alpha_1,$$

$$f_w(\bar{x}, \bar{x}, \bar{x}) = \frac{-bd}{(c+d)^2} = -\alpha_0.$$

The linearized equation of equation (1) about \bar{x} is

$$(7) \quad y_{n+1} + \alpha_2 y_{n-q} + \alpha_1 y_{n-l} + \alpha_0 y_{n-p} = 0.$$

Theorem 3. *Assume that*

$$b(c+3d) < (c+d)^2(1-a).$$

Then the equilibrium point of equation (1) is locally asymptotically stable.

Proof. It follows by Theorem 1 that equation (7) is asymptotically stable if

$$|\alpha_2| + |\alpha_1| + |\alpha_0| < 1,$$

then

$$\left| \frac{bd}{(c+d)^2} \right| + \left| a + \frac{b}{c+d} \right| + \left| \frac{bd}{(c+d)^2} \right| < 1,$$

or

$$a + \frac{b}{c+d} + \frac{2bd}{(c+d)^2} < 1,$$

and so

$$\frac{b(c+3d)}{(c+d)^2} < (1-a).$$

The proof is completed. ■

3. Global attractor of the equilibrium point of equation (1)

In this section, we investigate the global attractivity character of solutions of equation (1).

Theorem 4. *The equilibrium point \bar{x} of equation (1) is global attractor if $c(1-a) \neq b$.*

Proof. Let p, q are a real numbers and assume that $g : [p, q]^3 \rightarrow [p, q]$ be a function defined by

$$g(u, v, w) = av + \frac{buw}{cu + dw},$$

then we can see that the function $g(u, v, w)$ is increasing in u, v and decreasing in w .

Suppose that (m, M) is a solution of the system

$$M = g(M, M, m) \quad \text{and} \quad m = g(m, m, M).$$

Then from equation (1), we have

$$M = aM + \frac{bM^2}{cM + dm}, \quad m = am + \frac{bm^2}{cm + dM}.$$

Therefore

$$M(1 - a) = \frac{bM^2}{cM + dm}, \quad m(1 - a) = \frac{bm^2}{cm + dM},$$

or

$$\begin{aligned} c(1 - a)M^2 + d(1 - a)Mm &= bM^2, \\ c(1 - a)m^2 + d(1 - a)Mm &= bm^2. \end{aligned}$$

Subtracting we obtain

$$c(1 - a)(M^2 - m^2) = b(M^2 - m^2), \quad c(1 - a) \neq b.$$

Thus

$$M = m.$$

It follows by Theorem 2 that \bar{x} is a global attractor of equation (1) and then the proof is complete. \blacksquare

4. Boundedness of solutions of equation (1)

In this section we study the boundedness of solutions of equation (1).

Theorem 5. *Every solution of equation(1) is bounded if*

$$\left(a + \frac{b}{c}\right) < 1.$$

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ be a solution of equation (1). It follows from equation (1) that

$$\begin{aligned} x_{n+1} &= ax_{n-l} + \frac{bx_{n-q}x_{n-l}}{cx_{n-q} + dx_{n-p}} \\ &\leq ax_{n-l} + \frac{bx_{n-q}x_{n-l}}{cx_{n-q}} = \left(a + \frac{b}{c}\right)x_{n-l}. \end{aligned}$$

Then

$$x_{n+1} \leq x_{n-l} \quad \text{for all } n \geq 0.$$

Then the subsequences

$$\{x_{(l+1)n-l}\}_{n=0}^{\infty}, \{x_{(l+1)n-l+1}\}_{n=0}^{\infty}, \dots, \{x_{(l+1)n-1}\}_{n=0}^{\infty}, \{x_{(l+1)n}\}_{n=0}^{\infty}$$

are decreasing and so are bounded from above by $H = \max\{x_{-r}, \dots, x_{-1}, x_0\}$. ■

5. Special case of equation (1)

Our goal in this section is to find a specific form of the solutions of some special cases of equation (1) when $a = b = c = d = 1$ and give numerical example then draw it by using MATLAB 7.13.

5.1. On the Difference Equation $x_{n+1} = x_{n-1} + \frac{x_{n-2}x_{n-1}}{x_{n-2}+x_{n-4}}$

In this subsection we study the following special case of equation (1):

$$(8) \quad x_{n+1} = x_{n-1} + \frac{x_{n-2}x_{n-1}}{x_{n-2} + x_{n-4}},$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers.

Theorem 6. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of equation (8). Then for $n = 0, 1, 2, \dots$*

$$\begin{aligned} x_{6n-1} &= h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right), \\ x_{6n} &= k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right), \\ x_{6n+1} &= h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right), \\ x_{6n+2} &= k \prod_{i=0}^n \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right), \\ x_{6n+3} &= h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right), \\ x_{6n+4} &= k \prod_{i=0}^n \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right), \end{aligned}$$

where $x_{-4} = r, x_{-3} = s, x_{-2} = t, x_{-1} = h, x_0 = k, \prod_{i=0}^{-1} A_i = 1$, and $\{F_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{6n-7} &= h \prod_{i=0}^{n-2} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right), \\
x_{6n-6} &= k \prod_{i=0}^{n-2} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right), \\
x_{6n-5} &= h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right), \\
x_{6n-4} &= k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right), \\
x_{6n-3} &= h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right), \\
x_{6n-2} &= k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right),
\end{aligned}$$

Now, it follows from equation (8) that

$$\begin{aligned}
x_{6n-1} &= x_{6n-3} + \frac{x_{6n-4}x_{6n-3}}{x_{6n-4} + x_{6n-6}} \\
&= x_{6n-3} \left(1 + \frac{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right)}{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right)} \right. \\
&\quad \left. + k \prod_{i=0}^{n-2} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right) \right) \\
&= x_{6n-3} \left(1 + \frac{\left(\frac{F_{4n-1}h + F_{4n-2}s}{F_{4n-2}h + F_{4n-3}s} \right)}{\left(\frac{F_{4n-1}h + F_{4n-2}s}{F_{4n-2}h + F_{4n-3}s} \right) + 1} \right) \\
&= x_{6n-3} \left(1 + \frac{F_{4n-1}h + F_{4n-2}s}{F_{4n-1}h + F_{4n-2}s + F_{4n-2}h + F_{4n-3}s} \right) \\
&= x_{6n-3} \left(1 + \frac{F_{4n-1}h + F_{4n-2}s}{F_{4n}h + F_{4n-1}s} \right) \\
&= x_{6n-3} \left(\frac{F_{4n}h + F_{4n-1}s + F_{4n-1}h + F_{4n-2}s}{F_{4n}h + F_{4n-1}s} \right) \\
&= x_{6n-3} \left(\frac{F_{4n+1}h + F_{4n}s}{F_{4n}h + F_{4n-1}s} \right) \\
&= h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right).
\end{aligned}$$

From equation (8), we have

$$\begin{aligned}
 x_{6n} &= x_{6n-2} + \frac{x_{6n-3}x_{6n-2}}{x_{6n-3} + x_{6n-5}} \\
 &= x_{6n-2} \left(1 + \frac{h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right)}{h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right)} \right. \\
 &\quad \left. + h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right) \right) \\
 &= x_{6n-2} \left(1 + \frac{\left(\frac{F_{4n-1}k + F_{4n-2}t}{F_{4n-2}k + F_{4n-3}t} \right)}{\left(\frac{F_{4n-1}k + F_{4n-2}t}{F_{4n-2}k + F_{4n-3}t} \right) + 1} \right) \\
 &= x_{6n-2} \left(1 + \frac{F_{4n-1}k + F_{4n-2}t}{F_{4n-1}k + F_{4n-2}t + F_{4n-2}k + F_{4n-3}t} \right) \\
 &= x_{6n-2} \left(1 + \frac{F_{4n-1}k + F_{4n-2}t}{F_{4n}k + F_{4n-1}t} \right) \\
 &= x_{6n-2} \left(\frac{F_{4n}k + F_{4n-1}t + F_{4n-1}k + F_{4n-2}t}{F_{4n}k + F_{4n-1}t} \right) \\
 &= x_{6n-2} \left(\frac{F_{4n+1}k + F_{4n}t}{F_{4n}k + F_{4n-1}t} \right) \\
 &= k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right).
 \end{aligned}$$

Also, we get from equation (8)

$$\begin{aligned}
 x_{6n+1} &= x_{6n-1} + \frac{x_{6n-2}x_{6n-1}}{x_{6n-2} + x_{6n-4}} \\
 &= x_{6n-1} \left(1 + \frac{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right)}{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right)} \right. \\
 &\quad \left. + k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right) \right) \\
 &= x_{6n-1} \left(1 + \frac{\left(\frac{F_{4n+1}t + F_{4n}r}{F_{4n}r + F_{4n-1}t} \right)}{\left(\frac{F_{4n+1}t + F_{4n}r}{F_{4n}r + F_{4n-1}t} \right) + 1} \right) \\
 &= x_{6n-1} \left(1 + \frac{F_{4n+1}t + F_{4n}r}{F_{4n+1}t + F_{4n}r + F_{4n}r + F_{4n-1}t} \right) \\
 &= x_{6n-1} \left(1 + \frac{F_{4n+1}t + F_{4n}r}{F_{4n+2}t + F_{4n+1}r} \right) \\
 &= x_{6n-1} \left(\frac{F_{4n+2}t + F_{4n+1}r + F_{4n+1}t + F_{4n}r}{F_{4n+2}t + F_{4n+1}r} \right)
 \end{aligned}$$

$$\begin{aligned}
&= x_{6n-1} \left(\frac{F_{4n+3}t + F_{4n+2}r}{F_{4n+2}t + F_{4n+1}r} \right) \\
&= h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right).
\end{aligned}$$

Using equation (8), we obtain

$$\begin{aligned}
x_{6n+2} &= x_{6n} + \frac{x_{6n-1}x_{6n}}{x_{6n-1} + x_{6n-3}} \\
&= x_{6n} \left(1 + \frac{h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right)}{h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right)} \right. \\
&\quad \left. + h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right) \right) \\
&= x_{6n} \left(1 + \frac{\left(\frac{F_{4n+1}h + F_{4n}s}{F_{4n}h + F_{4n-1}s} \right)}{\left(\frac{F_{4n+1}h + F_{4n}s}{F_{4n}h + F_{4n-1}s} \right) + 1} \right) \\
&= x_{6n} \left(1 + \frac{F_{4n+1}h + F_{4n}s}{F_{4n+1}h + F_{4n}s + F_{4n}h + F_{4n-1}s} \right) \\
&= x_{6n} \left(1 + \frac{F_{4n+1}h + F_{4n}s}{F_{4n+2}h + F_{4n+1}s} \right) \\
&= x_{6n} \left(\frac{F_{4n+2}h + F_{4n+1}s + F_{4n+1}h + F_{4n}s}{F_{4n+2}h + F_{4n+1}s} \right) \\
&= x_{6n} \left(\frac{F_{4n+3}h + F_{4n+2}s}{F_{4n+2}h + F_{4n+1}s} \right) \\
&= k \prod_{i=0}^n \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}t + F_{4j+4}r}{F_{4j+4}t + F_{4j+3}r} \right) \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right).
\end{aligned}$$

Also, we see from equation (8)

$$\begin{aligned}
x_{6n+3} &= x_{6n+1} + \frac{x_{6n}x_{6n+1}}{x_{6n} + x_{6n-2}} \\
&= x_{6n+1} \left(1 + \frac{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right)}{k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \left(\frac{F_{4i+5}k + F_{4i+4}t}{F_{4i+4}k + F_{4i+3}t} \right)} \right. \\
&\quad \left. + k \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}h + F_{4i+2}s}{F_{4i+2}h + F_{4i+1}s} \right) \left(\frac{F_{4i+5}t + F_{4i+4}r}{F_{4i+4}t + F_{4i+3}r} \right) \prod_{j=0}^{n-2} \left(\frac{F_{4j+5}k + F_{4j+4}t}{F_{4j+4}k + F_{4j+3}t} \right) \right) \\
&= x_{6n+1} \left(1 + \frac{\left(\frac{F_{4n+1}k + F_{4n}t}{F_{4n}k + F_{4n-1}t} \right)}{\left(\frac{F_{4n+1}k + F_{4n}t}{F_{4n}k + F_{4n-1}t} \right) + 1} \right)
\end{aligned}$$

$$\begin{aligned}
 &= x_{6n+1} \left(1 + \frac{F_{4n+1}k + F_{4n}t}{F_{4n+1}k + F_{4n}t + F_{4n}k + F_{4n-1}t} \right) \\
 &= x_{6n+1} \left(1 + \frac{F_{4n+1}k + F_{4n}t}{F_{4n+2}k + F_{4n+1}t} \right) \\
 &= x_{6n+1} \left(\frac{F_{4n+2}k + F_{4n+1}t + F_{4n+1}k + F_{4n}t}{F_{4n+2}k + F_{4n+1}t} \right) \\
 &= x_{6n+1} \left(\frac{F_{4n+3}k + F_{4n+2}t}{F_{4n+2}k + F_{4n+1}t} \right) \\
 &= h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right).
 \end{aligned}$$

From equation (8), we get

$$\begin{aligned}
 x_{6n+4} &= x_{6n+2} + \frac{x_{6n+1}x_{6n+2}}{x_{6n+1} + x_{6n-1}} \\
 &= x_{6n+2} \left(1 + \frac{h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right)}{h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+3}k + F_{4j+2}t}{F_{4j+2}k + F_{4j+1}t} \right) \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right)} \right. \\
 &\quad \left. + h \prod_{i=0}^{n-1} \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \left(\frac{F_{4i+5}h + F_{4i+4}s}{F_{4i+4}h + F_{4i+3}s} \right) \right) \\
 &= x_{6n+2} \left(1 + \frac{\left(\frac{F_{4n+3}t + F_{4n+2}r}{F_{4n+2}t + F_{4n+1}r} \right)}{\left(\frac{F_{4n+3}t + F_{4n+2}r}{F_{4n+2}t + F_{4n+1}r} \right) + 1} \right) \\
 &= x_{6n+2} \left(1 + \frac{F_{4n+3}t + F_{4n+2}r}{F_{4n+3}t + F_{4n+2}r + F_{4n+2}t + F_{4n+1}r} \right) \\
 &= x_{6n+2} \left(1 + \frac{F_{4n+3}t + F_{4n+2}r}{F_{4n+4}t + F_{4n+3}r} \right) \\
 &= x_{6n+2} \left(\frac{F_{4n+4}t + F_{4n+3}r + F_{4n+3}t + F_{4n+2}r}{F_{4n+4}t + F_{4n+3}r} \right) \\
 &= x_{6n+2} \left(\frac{F_{4n+5}t + F_{4n+4}r}{F_{4n+4}t + F_{4n+3}r} \right) \\
 &= h \prod_{i=0}^n \left(\frac{F_{4i+3}t + F_{4i+2}r}{F_{4i+2}t + F_{4i+1}r} \right) \left(\frac{F_{4i+3}k + F_{4i+2}t}{F_{4i+2}k + F_{4i+1}t} \right) \prod_{j=0}^{n-1} \left(\frac{F_{4j+5}h + F_{4j+4}s}{F_{4j+4}h + F_{4j+3}s} \right).
 \end{aligned}$$

Hence, the proof is completed. ■

Example 1. For confirming the results of this subsection, we consider numerical example for $x_{-4} = 15$, $x_{-3} = 11$, $x_{-2} = 9$, $x_{-1} = 7$, $x_0 = 5$. [see Figure 1].

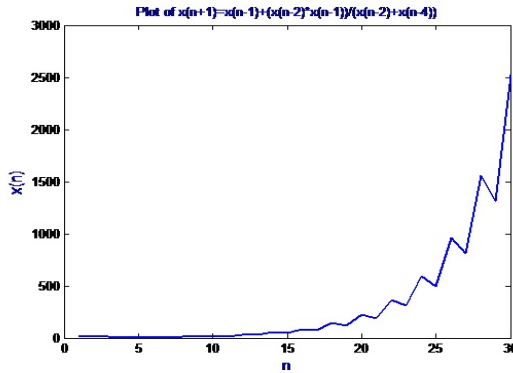


Figure 1.

6. On the solutions of difference equation

$$x_{n+1} = x_{n-1} + \frac{x_{n-2}x_{n-1}}{x_{n-2} - x_{n-4}}$$

In this section we give a specific form of the solutions of the difference equation

$$(9) \quad x_{n+1} = x_{n-1} + \frac{x_{n-2}x_{n-1}}{x_{n-2} - x_{n-4}},$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers with $x_{-2} \neq x_{-4} \neq x_0, x_{-1} \neq x_{-3}$.

Theorem 7. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of equation (9). Then for $n = 0, 1, 2, \dots$*

$$\begin{aligned} x_{6n-1} &= h \prod_{i=0}^{n-1} \left(\frac{F_{2i+3}t - F_{2i+1}r}{F_{2i+1}t - F_{2i-1}r} \right) \left(\frac{F_{2i+3}k - F_{2i+1}t}{F_{2i+1}k - F_{2i-1}t} \right) \left(\frac{F_{2i+3}h - F_{2i+1}s}{F_{2i+1}h - F_{2i-1}s} \right), \\ x_{6n} &= k \prod_{i=0}^{n-1} \left(\frac{F_{2i+3}h - F_{2i+1}s}{F_{2i+1}h - F_{2i-1}s} \right) \left(\frac{F_{2i+4}t - F_{2i+2}r}{F_{2i+2}t - F_{2i}r} \right) \left(\frac{F_{2i+4}k - F_{2i+2}t}{F_{2i+2}k - F_{2i}t} \right), \\ x_{6n+1} &= h \prod_{i=0}^n \left(\frac{F_{2i+3}t - F_{2i+1}r}{F_{2i+1}t - F_{2i-1}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+3}k - F_{2j+1}t}{F_{2j+1}k - F_{2j-1}t} \right) \left(\frac{F_{2j+4}h - F_{2j+2}s}{F_{2j+2}h - F_{2j}s} \right), \\ x_{6n+2} &= k \prod_{i=0}^n \left(\frac{F_{2i+3}h - F_{2i+1}s}{F_{2i+1}h - F_{2i-1}s} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+4}t - F_{2j+2}r}{F_{2j+2}t - F_{2j}r} \right) \left(\frac{F_{2j+4}k - F_{2j+2}t}{F_{2j+2}k - F_{2j}t} \right), \\ x_{6n+3} &= h \prod_{i=0}^n \left(\frac{F_{2i+3}t - F_{2i+1}r}{F_{2i+1}t - F_{2i-1}r} \right) \left(\frac{F_{2i+3}k - F_{2i+1}t}{F_{2i+1}k - F_{2i-1}t} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+4}h - F_{2j+2}s}{F_{2j+2}h - F_{2j}s} \right), \\ x_{6n+4} &= k \prod_{i=0}^n \left(\frac{F_{2i+3}h - F_{2i+1}s}{F_{2i+1}h - F_{2i-1}s} \right) \left(\frac{F_{2i+4}t - F_{2i+2}r}{F_{2i+2}t - F_{2i}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+2}k - F_{2j}t}{F_{2j+2}k - F_{2j}t} \right), \end{aligned}$$

where $x_{-4} = r$, $x_{-3} = s$, $x_{-2} = t$, $x_{-1} = h$, $x_0 = k$, $\prod_{i=0}^{-1} A_i = 1$, and $\{F_m\}_{m=-1}^\infty = \{1, 0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

Proof. As the proof of Theorem 6. ■

Example 2. Assume that $x_{-4} = 0.5$, $x_{-3} = 8$, $x_{-2} = 5$, $x_{-1} = 10$, $x_0 = 15$. [see Figure 2], and for $x_{-4} = 15$, $x_{-3} = 17$, $x_{-2} = 10$, $x_{-1} = 5$, $x_0 = 1$. [see Figure 3].

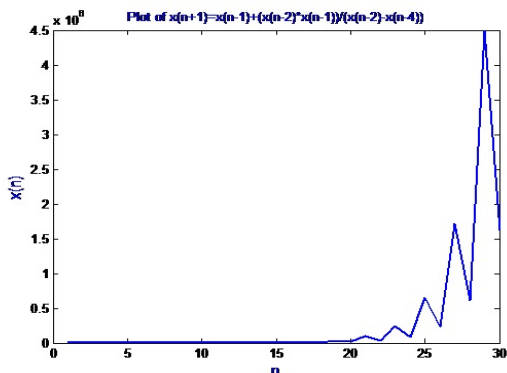


Figure 2.

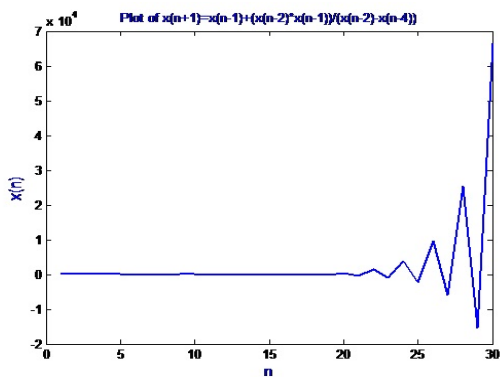


Figure 3.

7. On the solutions of difference equation

$$x_{n+1} = x_{n-1} - \frac{x_{n-2}x_{n-1}}{x_{n-2} + x_{n-4}}$$

In this section, we investigate the solutions of the difference equation

$$(10) \quad x_{n+1} = x_{n-1} - \frac{x_{n-2}x_{n-1}}{x_{n-2} + x_{n-4}},$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers.

Theorem 8. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of equation (10). Then for $n = 0, 1, 2, \dots$

$$\begin{aligned}
 x_{6n-1} &= h \prod_{i=0}^{n-1} \left(\frac{F_{2i}t + F_{2i+1}r}{F_{2i+1}t + F_{2i+2}r} \right) \left(\frac{F_{2i}k + F_{2i+1}t}{F_{2i+1}k + F_{2i+2}t} \right) \left(\frac{F_{2i+1}h + F_{2i+2}s}{F_{2i+2}h + F_{2i+3}s} \right), \\
 x_{6n} &= k \prod_{i=0}^{n-1} \left(\frac{F_{2i}h + F_{2i+1}s}{F_{2i+1}h + F_{2i+2}s} \right) \left(\frac{F_{2i+1}t + F_{2i+2}r}{F_{2i+2}t + F_{2i+3}r} \right) \left(\frac{F_{2i+1}k + F_{2i+2}t}{F_{2i+2}k + F_{2i+3}t} \right), \\
 x_{6n+1} &= h \prod_{i=0}^n \left(\frac{F_{2i}t + F_{2i+1}r}{F_{2i+1}t + F_{2i+2}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j}k + F_{2j+1}t}{F_{2j+1}k + F_{2j+2}t} \right) \left(\frac{F_{2j+1}h + F_{2j+2}s}{F_{2j+2}h + F_{2j+3}s} \right), \\
 x_{6n+2} &= k \prod_{i=0}^n \left(\frac{F_{2i}h + F_{2i+1}s}{F_{2i+1}h + F_{2i+2}s} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+1}t + F_{2j+2}r}{F_{2j+2}t + F_{2j+3}r} \right) \left(\frac{F_{2j+1}k + F_{2j+2}t}{F_{2j+2}k + F_{2j+3}t} \right), \\
 x_{6n+3} &= h \prod_{i=0}^n \left(\frac{F_{2i}t + F_{2i+1}r}{F_{2i+1}t + F_{2i+2}r} \right) \left(\frac{F_{2i}k + F_{2i+1}t}{F_{2i+1}k + F_{2i+2}t} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+1}h + F_{2j+2}s}{F_{2j+2}h + F_{2j+3}s} \right), \\
 x_{6n+4} &= k \prod_{i=0}^n \left(\frac{F_{2i}h + F_{2i+1}s}{F_{2i+1}h + F_{2i+2}s} \right) \left(\frac{F_{2i+1}t + F_{2i+2}r}{F_{2i+2}t + F_{2i+3}r} \right) \prod_{j=0}^{n-1} \left(\frac{F_{2j+1}k + F_{2j+2}t}{F_{2j+2}k + F_{2j+3}t} \right),
 \end{aligned}$$

where $x_{-4} = r$, $x_{-3} = s$, $x_{-2} = t$, $x_{-1} = h$, $x_0 = k$, and $\{F_m\}_{m=0}^{\infty} = \{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

Proof. As the proof of Theorem 6. ■

Example 3. Figure 4 shows the solution of equation (10) when $x_{-4} = 3$, $x_{-3} = 5$, $x_{-2} = 7$, $x_{-1} = 8$, $x_0 = 2$.

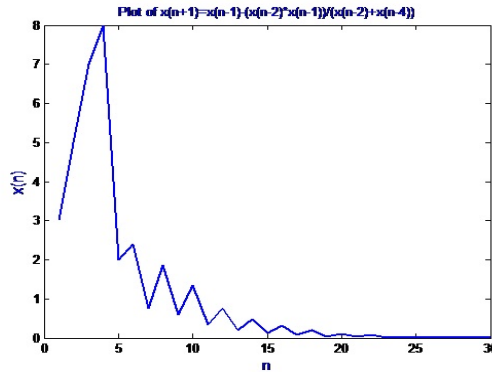


Figure 4.

8. On the solutions of difference equation

$$x_{n+1} = x_{n-1} - \frac{x_{n-2}x_{n-1}}{x_{n-2}-x_{n-4}}$$

Here the specific form of the solutions of the difference equation

$$(11) \quad x_{n+1} = x_{n-1} - \frac{x_{n-2}x_{n-1}}{x_{n-2} - x_{n-4}},$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary real numbers with $x_0 \neq x_{-4} \neq x_{-2}, x_{-1} \neq x_{-3}$ and $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0 \neq 0$, will be derived.

Theorem 9. *Let $\{x_n\}_{n=-4}^\infty$ be a solution of equation (11). Then every solution of equation (11) is periodic with period 36. Moreover $\{x_n\}_{n=-4}^\infty$ takes the form*

$$\left\{ r, s, t, h, k, \frac{rh}{r-t}, \frac{sk}{s-h}, \frac{rht}{(t-r)(k-t)}, \frac{sk(t-r)}{t(s-h)}, \frac{rt(h-s)}{(t-r)(k-t)}, \right. \\ \left. \frac{s(t-r)(k-t)}{t(s-h)}, \frac{t^2(s-h)}{(t-r)(k-t)}, \frac{sh(t-r)(k-t)}{t(s-h)}, \frac{tk(h-s)}{(t-r)(k-t)}, \right. \\ \left. \frac{rh(k-t)}{t(h-s)}, \frac{tks}{(r-t)(k-t)}, \frac{rh}{s-h}, \frac{sk}{t-k}, -r, -s, -t, -h, -k, \frac{rh}{t-r}, \right. \\ \left. \frac{sk}{h-s}, \frac{rht}{(r-t)(k-t)}, \frac{sk(t-r)}{t(h-s)}, \frac{rt(s-h)}{(t-r)(k-t)}, \frac{s(t-r)(k-t)}{t(h-s)}, \right. \\ \left. \frac{t^2(s-h)}{(t-r)(k-t)}, \frac{h(t-r)(k-t)}{t(h-s)}, \frac{tk(s-h)}{(t-r)(k-t)}, \frac{rh(t-k)}{t(h-s)}, \right. \\ \left. \frac{tks}{(r-t)(k-t)}, \frac{rh}{h-s}, \frac{ks}{k-t}, r, s, t, h, k, \dots \right\},$$

where $x_{-4} = r, x_{-3} = s, x_{-2} = t, x_{-1} = h, x_0 = k$.

Proof. As the proof of Theorem 6 and will be omitted. ■

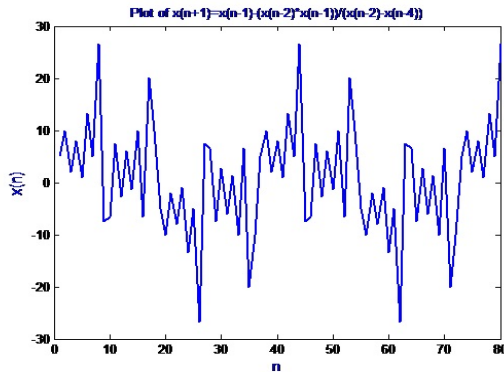


Figure 5.

Example 4. The following table shows the solution of equation (11) when $x_{-4} = 5$, $x_{-3} = 10$, $x_{-2} = 2$, $x_{-1} = 8$, $x_0 = 1$. [See Figure 5].

n	x_n	n	x_n	n	x_n	n	x_n	n	x_n
-4	5	14	-5	32	5	50	-5	68	5
-3	10	15	-10	33	10	51	-10	69	10
-2	2	16	-2	34	2	52	-2	70	2
-1	8	17	-8	35	8	53	-8	71	8
0	1	18	-1	36	1	54	-1	72	1
1	13.3333	19	-13.3333	37	13.3333	55	-13.3333	73	13.3333
2	5	20	-5	38	5	56	-5	74	5
3	26.6667	21	-26.6667	39	26.6667	57	-26.6667	75	26.6667
4	-7.5	22	7.5	40	-7.5	58	7.5	76	-7.5
5	-6.6667	23	6.6667	41	-6.6667	59	6.6667	77	-6.6667
6	7.5	24	-7.5	42	7.5	60	-7.5	78	7.5
7	-2.6667	25	2.6667	43	-2.6667	61	2.6667	79	-2.6667
8	6	26	-6	44	6	62	-6	80	6
9	-1.3333	27	1.3333	45	-1.3333	63	1.3333	81	-1.3333
10	10	28	-10	46	10	64	-10	82	10
11	-6.6667	29	6.6667	47	-6.6667	65	6.6667	83	-6.6667
12	20	30	-20	48	20	66	-20	84	20
13	10	31	-10	49	10	67	-10	85	10

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Received on 04.01.2012 and, in revised form, on 11.04.2012.