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ON THE DYNAMICS OF THE RECURSIVE SEQUENCE $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}}$*

Abstract. In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum\limits_{k=1}^{t} x_{n-2k}^{p} \prod\limits_{k=1}^{t} x_{n-2k}^{q}}, \quad n = 0, 1, \ldots,$$

where β is a positive parameter and α, γ are non-negative parameters and non-negative initial conditions.

Key words: difference equations, recursive sequences, oscillation, global asymptotic behavior, period two solutions, semicycles.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

Consider the higher-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum\limits_{k=1}^{t} x_{n-2k}^{p} \prod\limits_{k=1}^{t} x_{n-2k}^{q}}, \quad n = 0, 1, ... \tag{1}$$

where the parameters, β is positive and α, γ are non-negative real numbers and the initial conditions $x_{-2t}, ..., x_{-2}, x_{-1}$ and x_0 are non-negative real numbers such that

$$0 < \beta + \gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}, \quad n = 0, 1, \ldots .$$

and if $\alpha = 0$ the equation $x_{n+1} = 0$ is trivial, if $\gamma = 0$ the equation $x_{n+1} = \frac{\alpha}{\beta} x_{n-1}$ is linear. We assume that all parameters in equations are positive.

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We investigate the global asymptotic behavior and the periodic character of the solutions of the difference (1), by generalizing the results due to El-Owaidy et al. [1] corresponding to the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \quad n = 0, 1, \dots .$$

where the parameters α, β and γ are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary non-negative real numbers. Similar recursive sequences were studied previously; for example, see Refs. [1-22].

We need the following definitions and theorem [23]:

Definition 1. *Let I be an interval of the real numbers and let $f : I^{2t+1} \longrightarrow I$ be a continuously differantiable function. Consider the difference equation*

$$x_{n+1} = f\left(x_n, x_{n-1}, x_{n-2}, ..., x_{n-2t}\right), \quad n = 0, 1, \dots . \tag{2}$$

with x_{-i} for $i = 0, 1, ..., 2t \in I$. Let $\overline{x}$ be the equilibrium point of (2). The linearized equation of (2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{2t+1} y_{n-2t}, \quad n = 0, 1, \dots \tag{3}$$

where

$$\begin{aligned} c_1 &= \frac{\partial f}{\partial x_n}\left(\overline{x}, \overline{x}, ..., \overline{x}\right), \\ c_2 &= \frac{\partial f}{\partial x_{n-1}}\left(\overline{x}, \overline{x}, ..., \overline{x}\right), \\ &\vdots \\ c_{2t+1} &= \frac{\partial f}{\partial x_{n-2t}}\left(\overline{x}, \overline{x}, ..., \overline{x}\right). \end{aligned}$$

The characteristic equation of (3) is

$$\lambda^{2t+1} - c_1 \lambda^{2t} - \dots - c_{2t-1}\lambda^2 - c_{2t}\lambda - c_{2t+1} = 0 \tag{4}$$

Definition 2. *Let $\overline{x}$ be an equilibrium point of (2).*

(*a*) *The equilibrium $\overline{x}$ is called* locally stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0, \dots, x_{-2t} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-2t} - \overline{x}| < \delta$, then $|x_n - \overline{x}| < \varepsilon$, for all $n \geq -2t$.*

(*b*) *The equilibrium $\overline{x}$ is called* locally asymptotically stable *if it is locally stable and if there exists $\gamma > 0$ such that if $x_0, \dots, x_{-2t} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-2t} - \overline{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \overline{x}$.*

(*c*) *The equilibrium* $\overline{x}$ *is called* global attractor *if for every* $x_0, \dots, x_{-2t} \in I$ *we have* $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(*d*) *The equilibrium* $\overline{x}$ *is called* globally asymptotically stable *if it is locally stable and is a global attractor.*

Definition 3. *A positive semicycle of* $\{x_n\}_{n=-2t}^{\infty}$ *of (2) consists of a 'string' of terms* $\{x_l, x_{l+1}, \dots, x_m\}$*, all greater than or equal to* $\overline{x}$*, with* $l \geq -2t$ *and* $m < \infty$ *and such that either* $l = -2t$ *or* $l > -2t$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} < \overline{x}$.

A negative semicycle of $\{x_n\}_{n=-2t}^{\infty}$ *of (2) consists of a 'string' of terms* $\{x_l, x_{l+1}, \dots, x_m\}$ *all less than* $\overline{x}$*, with* $l \geq -2t$ *and* $m < \infty$ *and such that either* $l = -2t$ *or* $l > -2t$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geq \overline{x}$.

Definition 4. *A solution* $\{x_n\}_{n=-2t}^{\infty}$ *of (2) is called nonoscillatory if there exists* $N \geq -2t$ *such that either*

$$x_n > \overline{x} \text{ or } x_n < \overline{x} \quad for\ \forall n \geq N$$

and it is called oscillatory if it is not nonoscillatory.

Theorem 1. (*i*) *If all roots of (4) have absolute values less than one, then the equilibrium point* $\overline{x}$ *of (2) is locally asymptotically stable.*

(*ii*) *If at least one of the roots of (4) has absolute value greater than one, then the equilibrium point* $\overline{x}$ *of (2) is unstable.*

(*iii*) *The equilibrium point* $\overline{x}$ *of (2) is called saddle point if (4) has roots both inside and outside the unit disk.*

2. Dynamics of equation (1)

In this section, we investigate the dynamics of (1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\beta/\gamma)^{1/qt+p}\, y_n$ reduces (1) to the difference equation

$$(5) \qquad y_{n+1} = \frac{r y_{n-1}}{1 + y_{n-2}^{q+p} y_{n-4}^{q} \cdots y_{n-2t}^{q} + y_{n-2}^{q} y_{n-4}^{q+p} \cdots y_{n-2t}^{q} + \dots + y_{n-2}^{q} y_{n-4}^{q} \cdots y_{n-2t}^{q+p}}$$

where $r = \alpha/\beta > 0$ and $n = 0, 1, \dots$.

Note that $\overline{y}_1 = 0$ is always an equilibrium point of (5). When $r > 1$, (5) also possesses the unique positive equilibrium $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$.

Theorem 2. *The following statements are true.*

(*i*) *If* $r < 1$*, then the equilibrium point* $\overline{y}_1 = 0$ *of (5) is locally asymptotically stable.*

(ii) If $r > 1$, then the equilibrium point $\overline{y}_1 = 0$ of (5) is a saddle point.

(iii) When $r > 1$, then the positive equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ of (5) is unstable.

Proof. The linearized equation of (5) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} = rz_{n-1}, \quad n = 0, 1, \ldots$$

so, the characteristic equation of (5) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{2t+1} - r\lambda^{2t-1} = 0$$

hence the proof of (i) and (ii) follows Theorem 1.

For (iii) we assume that $r > 1$; then the linearized equation of (5) about the equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ has the form

$$\begin{aligned} z_{n+1} &= z_{n-1} - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-2} - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-4} \\ &\quad - \ldots - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-2t} = 0 \end{aligned}$$

where $n = 0, 1, \ldots$. So the characteristic equation of (5) about the equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ is

$$\lambda^{2t+1} - \lambda^{2t-1} + \frac{(qt+p)}{t}\frac{(r-1)}{r}\lambda^{2t-2} + \ldots + \frac{(qt+p)}{t}\frac{(r-1)}{r} = 0. \tag{6}$$

It is clear that (6) has a root in the interval $(-\infty, -1)$ and so $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ is an unstable equilibrium point. This completes the proof. ■

Theorem 3. *Assume that $r > 1$. Let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5) such that*

$$y_{-2t}, \ldots y_0 \geq \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}, \quad y_{-2t+1}, \ldots, y_{-1} < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \tag{7}$$

or

$$y_{-2t}, \ldots y_0 < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}, \quad y_{-2t+1}, \ldots, y_{-1} \geq \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \tag{8}$$

Then $\{y_n\}_{n=-2t}^{\infty}$ oscillates about $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ with semicycle of length 1.

Proof. Assume that (7) holds. (The case where (8) holds is similar and will be omitted.) Then,

$$\begin{aligned} y_1 &= \frac{ry_{-1}}{1 + y_{-2}^{q+p}y_{-4}^{q}\cdots y_{-2t}^{q} + y_{-2}^{q}y_{-4}^{q+p}\cdots y_{-2t}^{q} + y_{-2}^{q}y_{-4}^{q}\cdots y_{-2t}^{q+p}} \\ &< \frac{r\overline{y}_2}{1 + t\overline{y}_2^{qt+p}} = \frac{r\overline{y}_2}{1 + r - 1} = \overline{y}_2, \\ &< \overline{y}_2 \end{aligned}$$

and

$$\begin{aligned} y_2 &= \frac{ry_0}{1+y_{-1}^{q+p}y_{-3}^{q}\dots y_{-2t+1}^{q}+y_{-1}^{q}y_{-3}^{q+p}\dots y_{-2t+1}^{q}+y_{-1}^{q}y_{-3}^{q}\dots y_{-2t+1}^{q+p}} \\ &\geq \frac{r\overline{y}_2}{1+t\overline{y}_2^{qt+p}} = \frac{r\overline{y}_2}{1+r-1} = \overline{y}_2, \\ &\geq \overline{y}_2 \end{aligned}$$

then the proof follows by induction. ■

Theorem 4. *Assume that $r<1$; then the equilibrium point $\overline{y}_1=0$ of (5) is globally asymptotically stable.*

Proof. We know by Theorem 2 that the equilibrium point $\overline{y}_1=0$ of (5) is locally asymptotically stable. So let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5). It suffices to show that

$$\lim_{n\to\infty} y_n = 0$$

Since

$$\begin{aligned} y_{n+1} &= \frac{ry_{n-1}}{1+y_{n-2}^{q+p}y_{n-4}^{q}\cdots y_{n-2t}^{q}+y_{n-2}^{q}y_{n-4}^{q+p}\cdots y_{n-2t}^{q}+y_{n-2}^{q}y_{n-4}^{q}\cdots y_{n-2t}^{q+p}} \\ y_{2n-1} &< r^n y_{-1} \quad \text{and} \quad y_{2n} < r^n y_0 \end{aligned}$$

we have

$$\lim_{n\to\infty} y_n = 0.$$

This completes the proof. ■

Theorem 5. *Assume that $r=1$; then (5) possesses the prime period 2 solutions*

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots \tag{9}$$

with $\Phi>0$. Furthermore, every solution of (5) converges to a period 2 solution (9) with $\Phi\geq 0$.

Proof. Let

$$\dots,\Phi,\Psi,\Phi,\Psi,\dots$$

be a period two solution of (5). Then

$$\Phi = \frac{r\Phi}{1+\Psi^{q+p}\Psi^{q}\dots\Psi^{q}+\Psi^{q}\Psi^{q+p}\dots\Psi^{q}+\Psi^{q}\Psi^{q}\dots\Psi^{q+p}}$$

and

$$\Psi = \frac{r\Psi}{1+\Phi^{q+p}\Phi^{q}\dots\Phi^{q}+\Phi^{q}\Phi^{q+p}\dots\Phi^{q}+\Phi^{q}\Phi^{q}\dots\Phi^{q+p}}.$$

So

$$t\Phi\Psi = \frac{(\Phi-\Psi)(r-1)}{\Psi^{qt+p-1}-\Phi^{qt+p-1}} \geq 0,$$

which implies that $r-1\leq 0$.

If $r<1$, then this implies that $\Phi<0$ or $\Psi<0$, which is impossible, so $r=1$. If $r>1$, then this implies that $\Phi=\Psi=\left(\frac{r-1}{t}\right)^{1/qt+p}\neq 0$, which contradicts that

$\Phi \neq \Psi$, so $r = 1$. To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5); then

$$y_{n+1} - y_{n-1}$$
$$= \frac{-y_{n-1}y_{n-2}^{q+p}y_{n-4}^{q}\cdots y_{n-2t}^{q} - y_{n-1}y_{n-2}^{q}y_{n-4}^{q+p}\cdots, y_{n-2t}^{q} - y_{n-1}y_{n-2}^{q}y_{n-4}^{q}\cdots y_{n-2t}^{q+p}}{1 + y_{n-2}^{q+p}y_{n-4}^{q}\cdots y_{n-2t}^{q} + y_{n-2}^{q}y_{n-4}^{q+p}\cdots, y_{n-2t}^{q} + y_{n-2}^{q}y_{n-4}^{q}\cdots y_{n-2t}^{q+p}}$$
$$y_{n+1} - y_{n-1} \leq 0.$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$) and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus

$$\Phi = \frac{\Phi}{1 + \Psi^{q+p}\Psi^{q}...\Psi^{q} + \Psi^{q}\Psi^{q+p}\ldots\Psi^{q} + \Psi^{q}\Psi^{q}\ldots\Psi^{q+p}}$$

and

$$\Psi = \frac{\Psi}{1 + \Phi^{q+p}\Phi^{q}\ldots\Phi^{q} + \Phi^{q}\Phi^{q+p}\ldots\Phi^{q} + \Phi^{q}\Phi^{q}\ldots\Phi^{q+p}},$$

which implies that

$$t\Phi\Psi^{qt+p} = 0 \ \ and \ \ t\Psi\Phi^{qt+p} = 0.$$

This completes the proof. ■

Theorem 6. *Assume that $r > 1$; then (5) possesses an unbounded solution.*

Proof. From Theorem 3, we can assume without loss of generality that the solution $\{y_n\}_{n=-2t}^{\infty}$ of (5) is such that

$$y_{2n-1} < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \quad \text{and} \quad y_{2n} > \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p},$$

for $n \geq 0$. Then

$$y_{2n+2}$$
$$= \frac{ry_{2n}}{1 + y_{2n-1}^{q+p}y_{2n-3}^{q}\cdots y_{2n-2t+1}^{q} + y_{2n-1}^{q}y_{2n-3}^{q+p}\cdots y_{2n-2t+1}^{q} + y_{2n-1}^{q}y_{2n-3}^{q}\cdots y_{2n-2t+1}^{q+p}}$$
$$y_{2n+2} > \frac{ry_{2n}}{1 + (r-1)} = y_{2n}$$

and

$$y_{2n+3} = \frac{ry_{2n+1}}{1 + y_{2n}^{q+p}y_{2n-2}^{q}\cdots y_{2n-2t+2}^{q} + y_{2n}^{q}y_{2n-2}^{q+p}\cdots y_{2n-2t+2}^{q} + y_{2n}^{q}y_{2n-2}^{q}\cdots y_{2n-2t+2}^{q+p}}$$
$$y_{2n+3} < \frac{ry_{2n+1}}{1 + (r-1)} = y_{2n+1}$$

from which it follows that

$$\lim n \to \infty y_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete. ■

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