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SINGULAR INTEGRAL EQUATIONS WITH MULTIPLICATIVE CAUCHY-TYPE KERNELS

Abstract. In this paper we consider singular integral equations of the first kind with multiplicative Cauchy–type kernels defined on n–dimensional domains. We give their general solutions in the class of Hölder continuous functions and propose the statements of uniqueness problem.

Key words: singular integral equation, Cauchy kernel, multiplicative kernel.

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1. Introduction

In the one–dimensional case the theory of singular integral equations is more fully developed [2, 11, 13, 20, 23, 26], and its results are formulated more simply than the corresponding results in the multi–dimensional case. Furthermore, it is well known that one–dimensional equations often arise in certain problems in fluid and solid mechanics [7, 10, 18, 23] and are closely related to the theory of boundary value problems [3, 13, 22, 23, 26]. In the monograph [18] (and in [5, 29]) the stationary linear problem of ideal fluid flow around a finite–span wing was reduced to the solution of the singular integral equation with multiplicative Cauchy kernel of the form

$$\frac{1}{\pi^2}\iint\limits_D \frac{\varphi\left(\sigma_1,\sigma_2\right)}{\left(\sigma_1-x\right)\left(\sigma_2-y\right)}d\sigma_1 d\sigma_2 = f\left(x,y\right), \quad (x,y)\in D,$$

where D is a rectangle. The theory of this equation was well developed in the works [4, 6, 12, 18, 19, 26]. Moreover, three–dimensional equation, where D is a cube, was discussed in [27]. General solution in the class of Hölder continuous functions with the statement of uniqueness problem was obtained there. It is worth noting that various kinds of multidimensional singular integral equations were studied in detail in the books [9, 21].

Our main objective in this paper is to determine general formulas for the solutions and give the statement of uniqueness problems for the following equations:

$$\frac{1}{\pi^n}\int_{-1}^{1}\cdots\int_{-1}^{1}\frac{\varphi(t_1,\dots,t_n)}{(t_1-x_1)\cdot\ldots\cdot(t_n-x_n)}\,dt_1\dots dt_n = f(x_1,\dots,x_n), \quad (x_1,\dots,x_n)\in(-1,\,1)^n, \tag{1}$$

$$\frac{1}{\pi^n}\int_{0}^{+\infty}\cdots\int_{0}^{+\infty}\frac{\varphi(\sigma_1,\dots,\sigma_n)}{(\sigma_1-x_1)\dots(\sigma_n-x_n)}d\sigma_1\dots d\sigma_n = f(x_1,\dots,x_n), \quad (x_1,\dots,x_n)\in(0,\,+\infty)^n, \tag{2}$$

$$\frac{1}{(\pi i)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{\varphi(\sigma_1,\dots,\sigma_n)}{(\sigma_1-x_1)\dots(\sigma_n-x_n)}d\sigma_1\dots d\sigma_n = f(x_1,\dots,x_n), \quad (x_1,\dots,x_n)\in\mathbb{R}^n, \tag{3}$$

assuming that all the appearing functions are Hölder continuous. Equation (1) was discussed in [18] but without general formula for solution and without statement of uniqueness conditions.

Solving an integral equation defined on an infinite domain can be a challenging problem which many papers and books were dedicated to (cf. [1]). Such equations have received significant attention during the last years. In the case $n = 1$, equation (2) was considered in [11, 13, 24]. The case $n = 2$ was fully described in [24], and for $n = 3$ the inversion formulas with uniqueness conditions were presented in [28]. In the case $n = 1$ (this problem is known as Hilbert transform) (3) was considered in [8, 13, 24]. As previously, the results for $n = 2, 3$ were presented in [24, 28]. Finally, exact solution of equations with double Cauchy–type kernels in the case of half–plane was described in [14] and numerical solutions of equations in the case of infinite domains of integration were investigated in [15, 16, 17, 25]. The theory of the first kind singular integral equations defined on the infinite areas differs from the similar theory for the finite domains of integration. For instance, in the case $n = 1$ to solve (2) one has to transform it to the equation defined on the finite interval and then solve it as the airfoil equation (cf. [13, 23, 26]). The conditions for solvability in the class of Hölder continuous functions lead to a system of equations defined on infinite area. This system is always solvable. The equation defined on the finite integration domain can be unsolvable in the class of Hölder continuous functions (bounded on the whole domain). Similarly, in the case of (3) one transforms it to the equation defined on a unit circle, which theory is simple [23], but the solution formulas for the equation defined on the real axis are more complicated [24].

2. Function classes

Let us introduce function classes that will be used when solving (1), (2) and (3).

Definition 1. *We write* $\varphi(x_1,\dots,x_n) \in h(-1,1)$, *if it satisfies the Hölder inequality*

$$|\varphi(x_1,\dots,x_n) - \varphi(x'_1,\dots,x'_n)| \leqslant \sum_{j=1}^{n} K_j |x_j - x'_j|^{\mu}, \tag{4}$$

where $0 < \mu \leqslant 1$, *and* $K_j > 0$ *are constants independent of the choice of points* $(x_1,\dots,x_n), (x'_1,\dots,x'_n) \in [-1,1]^n$, *i.e.* $\varphi(x_1,\dots,x_n) \in H([-1,1]^n)$.

Definition 2. *We write* $\varphi(x_1,\dots,x_n) \in h_0$, *if it satisfies an inequality of the form* (4) *in each closed subset contained in the domain* $(-1,1)^n$, *and the representation*

$$\varphi(x_1,\dots,x_n) = \frac{\varphi^*(x_1,\dots,x_n)}{(1 \pm x_1)^{\alpha_1} \cdot \ldots \cdot (1 \pm x_n)^{\alpha_n}}, \quad 0 \leqslant \alpha_j < 1,\ j = 1,\dots,n,$$

where $\varphi^*(x_1,\dots,x_n) \in H([-1,1]^n)$, *is valid near the boundary points* $x_j = \pm 1$, $j = 1,\dots,n$.

Definition 3. *We write* $\varphi(x_1,\dots,x_n) \in h(0,+\infty)$, $(x_1,\dots,x_n) \in (0,+\infty)^n$, *if the function*

$$\varphi^*(t_1,\dots,t_n) = \varphi\left(\frac{1+t_1}{1-t_1},\dots,\frac{1+t_n}{1-t_n}\right), \quad -1 \leqslant t_j < 1, \quad j = 1,\dots,n,$$

satisfies the inequality

$$|\varphi^*(t'_1,\dots,t'_n) - \varphi^*(t''_1,\dots,t''_n)| \leqslant \sum_{j=1}^{n} K_j |t'_j - t''_j|^{\mu_j}, \tag{5}$$

$K_j > 0$, $0 < \mu_j \leqslant 1$, $j = 1,\dots,n$, $(t'_1,\dots,t'_n) \in [-1,1)^n$, $(t''_1,\dots,t''_n) \in [-1,1)^n$, *and*

$$\lim_{t_j \to 1-0} \varphi^*(t_1,\dots,t_n) = \lim_{x_j \to +\infty} \varphi(x_1,\dots,x_n) = 0, \quad \forall (t_1,\dots,t_{j-1},t_{j+1},\dots,t_n) \in [-1,1)^{n-1}. \tag{6}$$

Definition 4. *We write* $\varphi(x_1,\dots,x_n)\in h(+\infty)$, $(x_1,\dots,x_n)\in(0,+\infty)^n$, *if the function* $\varphi^*(t_1,\dots,t_n)$ *satisfies the inequality (5) in each closed domain contained in* $(-1,1)^n$, *and in a neighbourhood of* $t_j=-1$, $j=1,\dots,n$, *the condition*

$$\varphi^*(t_1,\dots,t_n)=\frac{\varphi^{**}(t_1,\dots,t_n)}{(1+t_j)^{\alpha_j}},\quad 0\leqslant\alpha_j<1,\quad j=1,\dots,n,$$

is fulfilled. Here $\varphi^{**}(t_1,\dots,t_n)$ *satisfies Hölder condition on* $[-1,1]^n$ *and the condition (6) holds.*

Definition 5. *We write* $\varphi(x_1,\dots,x_n)\in h(\infty)$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, *if the function*

$$\varphi^*(t_1,\dots,t_n)=\varphi\left(i\frac{1+t_1}{1-t_1},\dots,i\frac{1+t_n}{1-t_n}\right),$$

$(t_1,\dots,t_n)\in L_1\times\ldots\times L_n$, $L_j=\{t_j:|t_j|\leqslant 1\}$, $j=1,\dots,n$, *satisfies the inequality of the form (5).*

3. Solutions of equations

Theorem 1. *Let a function* $f(x_1,\dots,x_n)\in h(-1,1)$ *be defined in the domain* $[-1,1]^n$. *Then the general solution of (1) in the class* h_0 *can be represented by the formula*

$$\varphi(x_1,\dots,x_n)=\frac{1}{\sqrt{1-x_1^2}\cdot\ldots\cdot\sqrt{1-x_n^2}}R(f;x_1,\dots,x_n)+\sum_{j=1}^{n}\frac{C_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}{\sqrt{1-x_j^2}},$$

where

$$R(f;x_1,\dots,x_n)=\frac{(-1)^n}{\pi^n}\int_{-1}^{1}\!\!\dots\!\int_{-1}^{1}\sqrt{1-t_1^2}\cdot\ldots\cdot\sqrt{1-t_n^2}\,f(t_1,\dots,t_n)\times\frac{dt_1\dots dt_n}{(t_1-x_1)\cdot\ldots\cdot(t_n-x_n)},\quad(x_1,\dots,x_n)\in(-1,1)^n,$$

and $C_j(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$, $j=1,\dots,n$ *are arbitrary functions of the class* h_0.

In addition, if the solution $\varphi(x_1,\dots,x_n)$ satisfies the conditions

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(x_1,\dots,x_{i-1},t_i,x_{i+1},\dots,x_n)\,dt_i = g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n), \tag{7}$$

where $g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $i=1,\dots,n$, are given functions of the class h_0 such that

$$\frac{1}{\pi^k}\int_{-1}^{1}\dots\int_{-1}^{1}\varphi\left(x_1,\dots,t_{i_1},\dots,t_{i_k},\dots,x_n\right)dt_{i_1}\dots dt_{i_k} = \frac{1}{\pi^{k-1}}\int_{-1}^{1}\dots\int_{-1}^{1}g_{i_j}\left(x_1,\dots,x_n\right)dt_{i_1}\dots dt_{i_{j-1}}dt_{i_{j+1}}\dots dt_{i_k},$$
$$k=2,\dots,n-1,\;\; j=1,\dots,k,\;\; i_j=j,\dots,n-k+j, \tag{8}$$

and

$$\frac{1}{\pi^n}\int_{-1}^{1}\dots\int_{-1}^{1}\varphi\left(t_1,\dots,t_n\right)dt_1\dots dt_n = \frac{1}{\pi^{n-1}}\int_{-1}^{1}\dots\int_{-1}^{1}g_1(t_2,\dots,t_n)dt_2\dots dt_n = \dots = \frac{1}{\pi^{n-1}}\int_{-1}^{1}\dots\int_{-1}^{1}g_n(t_1,\dots,t_{n-1})dt_1\dots dt_{n-1} = A, \tag{9}$$

then the unique solution of the problem (1), (7)-(9) is given by the formula

$$\varphi(x_1,\dots,x_n) = \frac{R(f;x_1,\dots,x_n)}{\sqrt{\left(1-x_1^2\right)\dots\left(1-x_n^2\right)}} + \sum_{i=1}^{n}\frac{g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)}{\sqrt{1-x_i^2}} + \sum_{k=2}^{n-1}(-1)^{k-1}\sum_{i_1=1}^{n-k+1}\sum_{i_2=i_1+1}^{n-k+2}\dots\sum_{i_k=i_{k-1}+1}^{n}\frac{1}{\sqrt{1-x_{i_1}^2}\dots\sqrt{1-x_{i_k}^2}} \times\frac{1}{\pi^{k-1}}\int_{-1}^{1}\dots\int_{-1}^{1}g_{i_1}(x_1,\dots,t_{i_2},\dots,t_{i_k},\dots,x_n)dt_{i_2}\dots dt_{i_k} \tag{10}$$

$$+ \ (-1)^{n-1} \frac{A}{\sqrt{1-x_{i_1}^2} \cdot \ldots \cdot \sqrt{1-x_{i_n}^2}}.$$

The proof of the theorem (and of the others in this section) is based on mathematical induction. Initial cases were proved in the works [23, 24, 26, 28].

Theorem 2. *Let $f(x_1,\ldots,x_n) \in h(-1,1)$. If the solution of equation (1) is sought in the function class $h(-1,1)$ and the following necessary and sufficient conditions*

$$\frac{1}{\pi} \int_{-1}^{1} \frac{f(x_1,\ldots,\tau_i,\ldots,x_n)}{\sqrt{1-\tau_i^2}}\, d\tau_i = 0, \quad i = 1,\ldots,n, \tag{11}$$

hold, then the unique solution of the problem (1), (11) is given by

$$\varphi(x_1,\ldots,x_n) = \sqrt{1-x_1^2} \cdot \ldots \cdot \sqrt{1-x_n^2}$$
$$\times \frac{(-1)^n}{\pi^n} \int_{-1}^{1} \ldots \int_{-1}^{1} \frac{1}{\sqrt{1-t_1^2} \cdot \ldots \cdot \sqrt{1-t_n^2}} \frac{f(t_1,\ldots,t_n)\, dt_1 \ldots dt_n}{(t_1-x_1) \cdot \ldots \cdot (t_n-x_n)}.$$

Theorem 3. *Let $f(x_1,\ldots,x_n) \in h(0,+\infty)$, $(x_1,\ldots,x_n) \in (0,+\infty)^n$. Then the general solution of (2) in the function class $h(+\infty)$ is given by the formula*

$$\begin{aligned} \varphi(x_1,\ldots,x_n) = {} & (-1)^n R(f;x_1,\ldots,x_n) \\ & + \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} C_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), \end{aligned} \tag{12}$$

where

$$R(f;x_1,\ldots,x_n) = \frac{1}{\sqrt{x_1 \ldots x_n}} \frac{1}{\pi^n} \int_0^{+\infty} \ldots \int_0^{+\infty} \sqrt{\sigma_1 \ldots \sigma_n}$$
$$\times \frac{x_1+1}{\sigma_1+1} \ldots \frac{x_n+1}{\sigma_n+1} \frac{f(\sigma_1,\ldots,\sigma_n)}{(\sigma_1-x_1)\ldots(\sigma_n-x_n)} d\sigma_1 \ldots d\sigma_n,$$

and $C_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$, $i = 1,\ldots,n$, are arbitrary functions from the class $h(+\infty)$.

If the solution $\varphi(x_1,\ldots,x_n)$ is subjected to the conditions

$$\begin{aligned} & \frac{1}{\pi} \int_0^{+\infty} \frac{\varphi(x_1,\ldots,x_{i-1},\sigma_i,x_{i+1},\ldots,x_n)}{\sigma_i+1} d\sigma_i \\ & \qquad = g_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n), \quad i = 1, \ldots, n, \end{aligned} \tag{13}$$

where $g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$, $i=1,\dots,n$, *are given functions from the class* $h(+\infty)$ *satisfying*

$$(14)\qquad \frac{1}{\pi^k}\int_0^{+\infty}\dots\int_0^{+\infty}\frac{g_i(x_1,\dots,x_{j_1-1},\sigma_{j_1},x_{j_1+1},\dots,x_{j_k-1},\sigma_{j_k},x_{j_k+1},\dots,x_n)}{(\sigma_{j_1}+1)\dots(\sigma_{j_k}+1)}\,d\sigma_{j_1}\dots d\sigma_{j_k}$$

$$=\frac{1}{\pi^k}\int_0^{+\infty}\dots\int_0^{+\infty}\frac{g_r(x_1,\dots,x_{j_1-1},\sigma_{j_1},x_{j_1+1},\dots,x_{j_k-1},\sigma_{j_k},x_{j_k+1},\dots,x_n)}{(\sigma_{j_1}+1)\dots(\sigma_{j_k}+1)}\,d\sigma_{j_1}\dots d\sigma_{j_k},$$

$$k=1,\dots,n-2,\quad i,r\in\{1,\dots,n\},\quad i\neq r,$$

$$(15)\qquad \frac{1}{\pi^{n-1}}\int_0^{+\infty}\dots\int_0^{+\infty}\frac{g_i(\sigma_1,\dots,\sigma_{i-1},\sigma_{i+1},\dots,\sigma_n)}{(\sigma_1+1)\dots(\sigma_{i-1}+1)(\sigma_{i+1}+1)\dots(\sigma_n+1)}\,d\sigma_1\dots d\sigma_{i-1}d\sigma_{i+1}\dots d\sigma_n = A,\quad i=1,\dots,n,$$

then the unique solution is given by

$$\varphi(x_1,\dots,x_n)=(-1)^n R(f;x_1,\dots,x_n) + \sum_{i=1}^{n}\frac{1}{\sqrt{x_i}}g_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$$
$$+\sum_{i=2}^{n-1}\sum_{j_1=1}^{n-i+1}\sum_{j_2=j_1+1}^{n-i+2}\sum_{j_3=j_2+1}^{n-i+3}\dots\sum_{j_i=j_{i-1}+1}^{n}\frac{(-1)^{i+1}}{\sqrt{x_{j_1}\dots x_{j_i}}}\frac{1}{\pi^{i-1}}$$
$$\times\int_0^{+\infty}\dots\int_0^{+\infty}\frac{g_{j_1}(x_1,\dots,x_{j_1-1},x_{j_1+1},\dots,x_{j_2-1},\sigma_{j_2},}{(\sigma_{j_2}+1)\dots}\,\frac{x_{j_2+1},\dots,x_{j_i-1},\sigma_{j_i},x_{j_i+1},\dots,x_n)}{(\sigma_{j_i}+1)}\cdot d\sigma_{j_2}\dots d\sigma_{j_i}+(-1)^{n+1}\frac{A}{\sqrt{x_1\dots x_n}}.$$

Remark 1. Note that using the identity $\frac{x+1}{(\sigma+1)(\sigma-x)}=\frac{1}{\sigma-x}-\frac{1}{\sigma+1}$, formula (12) can be rewritten as

$$\varphi(x_1,\dots,x_n)=\frac{(-1)^n}{\sqrt{x_1\dots x_n}}\frac{1}{\pi^n}\int_0^{+\infty}\dots\int_0^{+\infty}\sqrt{\sigma_1\dots\sigma_n}f(\sigma_1,\dots,\sigma_n)$$

$$\times \frac{d\sigma_1 \dots d\sigma_n}{(\sigma_1 - x_1) \dots (\sigma_n - x_n)} + \sum_{i=1}^{n} \frac{1}{\sqrt{x_i}} C_i^* \left(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \right),$$

where $C_i^* \left(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \right)$, $i = 1, \dots, n$, are arbitrary functions from the class $h\,(+\infty)$.

Theorem 4. *Let* $f\,(x_1, \dots, x_n) \in h\,(0, +\infty)\,, (x_1, \dots, x_n) \in (0, +\infty)^n$. *Then the solution of the system of (2) and the following equations:*

$$(16) \qquad \frac{1}{\pi} \int_0^{+\infty} \frac{\varphi\,(x_1, \dots, \sigma_k, \dots, x_n)}{\sigma_k + 1} d\sigma_k$$
$$= \sqrt{x_1 \dots x_{k-1} x_{k+1} \dots x_n} \frac{(-1)^{n-1}}{\pi^n}$$
$$\times \int_0^{+\infty} \dots \int_0^{+\infty} \frac{f\,(\sigma_1, \dots, \sigma_n)}{\sqrt{\sigma_1 \dots \sigma_n}\,(\sigma_1 - x_1) \dots (\sigma_{k-1} - x_{k-1})}$$
$$\frac{d\sigma_1 \dots d\sigma_n}{(\sigma_k + 1)\,(\sigma_{k+1} - x_{k+1}) \dots (\sigma_n - x_n)}$$

$k = 1, \dots, n$, *in the class of functions* $h\,(0, +\infty)$ *is given by*

$$\varphi\,(x_1, \dots, x_n) = \sqrt{x_1 \dots x_n} \frac{(-1)^n}{\pi^n} \int_0^{+\infty} \dots \int_0^{+\infty} \frac{f\,(\sigma_1, \dots, \sigma_n)\, d\sigma_1 \dots d\sigma_n}{\sqrt{\sigma_1 \dots \sigma_n}\,(\sigma_1 - x_1) \dots (\sigma_n - x_n)}.$$

Theorem 5. *Let* $f\,(x_1, \dots, x_n) \in h\,(\infty)$, $(x_1, \dots, x_n) \in \mathbb{R}^n$ *and let* $\lim\limits_{|x_j| \to +\infty} f\,(x_1, \dots, x_n) = 0$, $-\infty < x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n < +\infty$, $j = 1, \dots, n$. *Then the general solution of (3) in the function class* $h\,(\infty)$ *is given by the formula*

$$(17) \qquad \varphi\,(x_1, \dots, x_n) = R\,(f; x_1, \dots, x_n) + \sum_{k=1}^{n} C_k \left(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \right),$$

where

$$R\,(f; x_1, \dots, x_n) = \frac{1}{(\pi i)^n} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \frac{x_1 + i}{\sigma_1 + i} \dots$$
$$\frac{x_n + i}{\sigma_n + i} \frac{f\,(\sigma_1, \dots, \sigma_n)}{(\sigma_1 - x_1) \dots (\sigma_n - x_n)} d\sigma_1 \dots d\sigma_n$$

and the functions $C_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)$, $k=1,\ldots,n$, *are arbitrary functions from the class* $h(\infty)$.

If the solution $\varphi(x_1,\ldots,x_n)$ *of (3) is subjected to the conditions*

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(x_1,\ldots,x_{k-1},\sigma_k,x_{k+1},\ldots,x_n)\,d\sigma_k}{\sigma_k+i} = g_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n), \quad -\infty<x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n<+\infty,\quad k=1,\ldots,n, \tag{18}$$

where $g_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)$, $k=1,\ldots,n$, *are given functions from the class* $h(\infty)$ *such that*

$$\frac{1}{(\pi i)^k}\int_{-\infty}^{+\infty}\!\!\ldots\int_{-\infty}^{+\infty}\frac{g_p(x_1,\ldots,x_{j_1-1},\sigma_{j_1},x_{j_1+1},\ldots,x_{j_k-1},\sigma_{j_k},x_{j_k+1},\ldots,x_n)}{(\sigma_{j_1}+i)\ldots(\sigma_{j_k}+i)}\cdot d\sigma_{j_1}\ldots d\sigma_{j_k} = \frac{1}{(\pi i)^k}\int_{-\infty}^{+\infty}\!\!\ldots\int_{-\infty}^{+\infty}\frac{g_r(x_1,\ldots,x_{j_1-1},\sigma_{j_1},x_{j_1+1},\ldots,x_{j_k-1},\sigma_{j_k},x_{j_k+1},\ldots,x_n)}{(\sigma_{j_1}+i)\ldots(\sigma_{j_k}+i)}\cdot d\sigma_{j_1}\ldots d\sigma_{j_k}, \quad k=1,\ldots,n-2,\ p,r\in\{1,\ldots,n\},\ p\neq r, \tag{19}$$

$$\frac{1}{(\pi i)^{n-1}}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{\infty}\frac{g_k(\sigma_1,\ldots,\sigma_{k-1},\sigma_{k+1},\ldots,\sigma_n)}{(\sigma_1+i)\ldots(\sigma_{k-1}+i)(\sigma_{k+1}+i)\ldots(\sigma_n+i)}\cdot d\sigma_1\ldots d\sigma_{k-1}d\sigma_{k+1}\ldots d\sigma_n = A,\quad k=1,\ldots,n, \tag{20}$$

then the unique solution of (3) has the form

$$\begin{aligned}\varphi(x_1,\ldots,x_n) &= R(f;x_1,\ldots,x_n)\\ &-\sum_{k=1}^{n} g_k(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)\\ &-\sum_{k=2}^{n-1}\sum_{j_1=1}^{n-k+1}\sum_{j_2=j_1+1}^{n-k+2}\sum_{j_3=j_2+1}^{n-k+3}\ldots\\ &\qquad\sum_{j_k=j_{k-1}+1}^{n}\int_{-\infty}^{+\infty}\!\!\ldots\int_{-\infty}^{+\infty} g_{j_1}\big(x_1,\ldots,x_{j_1-1},x_{j_j+1},\ldots,x_{j_2-1},\sigma_{j_2},\\ &\qquad x_{j_2+1},\ldots,x_{j_k-1},\sigma_{j_k},x_{j_k+1},\ldots,x_n\big)\frac{d\sigma_{j_2}\ldots d\sigma_{j_k}}{(\sigma_{j_2}+i)\ldots(\sigma_{j_k}+i)} - A.\end{aligned}$$

Remark 2. Note that using the identity $\frac{x+i}{(\sigma+i)(\sigma-x)} = \frac{1}{\sigma-x} - \frac{1}{\sigma+i}$, one can rewrite formula (17) in the form

$$\varphi\left(x_1,\ldots,x_n\right) = \frac{1}{(\pi i)^n}\int\limits_{-\infty}^{+\infty}\ldots\int\limits_{-\infty}^{+\infty}\frac{f\left(\sigma_1,\ldots,\sigma_n\right)}{\left(\sigma_1-x_1\right)\ldots\left(\sigma_n-x_n\right)}d\sigma_1\ldots d\sigma_n + \sum_{k=1}^{n} C_k^*\left(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n\right),$$

where $C_k^*\left(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n\right)$, $k=1,\ldots n$, are arbitrary functions from the class $h\left(\infty\right)$.

Remark 3. If we look for the solution of (3) fulfilling the condition

$$\lim_{|x_1|\to+\infty,\ldots,|x_n|\to+\infty}\varphi\left(x_1,\ldots,x_n\right)=0,$$

then the functions $C_k\left(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n\right)$, $k=1,\ldots,n$, appearing in (17) satisfy the identity

$$\lim_{|x_1|\to+\infty,\ldots,|x_n|\to+\infty}\sum_{k=1}^{n} C_k\left(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n\right) = \frac{1}{(\pi i)^n}\int\limits_{-\infty}^{+\infty}\ldots\int\limits_{-\infty}^{+\infty}\frac{f\left(\sigma_1,\ldots,\sigma_n\right)}{\left(\sigma_1+i\right)\ldots\left(\sigma_n+i\right)}d\sigma_1\ldots d\sigma_n.$$

In the case $n=3$ this statement was presented in [28]. The proof of the general case is similar.

4. Examples

Example 1. Let $n=3$ and

$$f\left(x_1,x_2,x_3\right) = \frac{1}{\left(x_1-a_1\right)\left(x_2-a_2\right)\left(x_3-a_3\right)},$$

where $a_i>1$ for $i=1,2,3$. Then the operator $R(f;x_1,x_2,x_3)$ has the form

$$R(f;x_1,x_2,x_3) = -\left(1+\frac{\sqrt{a_1^2-1}}{x-a_1}\right)\left(1+\frac{\sqrt{a_2^2-1}}{x-a_2}\right)\left(1+\frac{\sqrt{a_3^2-1}}{x-a_3}\right).$$

Thus, the general solution of (2) in the h_0 class takes the form

$$\varphi(x_1,x_2,x_3) = \frac{1}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}\sqrt{1-x_3^2}}R(f;x_1,x_2,x_3) + \frac{C_1(x_2,x_3)}{\sqrt{1-x_1^2}} + \frac{C_2(x_1,x_3)}{\sqrt{1-x_2^2}} + \frac{C_3(x_1,x_2)}{\sqrt{1-x_3^2}},\quad -1<x_1,x_2,x_3<1,$$

where $C_j(\cdot)$, $j=1,2,3$ are arbitrary functions of the class h_0. The functions $C_j(\cdot)$ on the right-hand side in the above expression are uniquely determined if (2) is supplemented by the conditions (7)-(9), where

$$g_1(x_2,x_3)=\frac{1-x_2x_3}{\sqrt{1-x_2^2}\sqrt{1-x_3^2}},\quad g_2(x_1,x_3)=\frac{1-x_1x_3}{\sqrt{1-x_1^2}\sqrt{1-x_3^2}},$$
$$g_3(x_1,x_2)=\frac{1-x_1x_2}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}}.$$

By (10) the solution of the problem (2), (7)-(9) is given by the formula

$$\varphi(x_1,x_2,x_3)=\frac{R(f;x_1,x_2,x_3)-x_2x_3-x_1x_3-x_1x_2+1}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}\sqrt{1-x_3^2}}.$$

Example 2. Let $n=3$,

$$f\left(x_1,x_2,x_3\right)=\frac{1}{\left(1+x_1\right)\left(1+x_2\right)\left(1+x_3\right)},\quad x_1,\,x_2,\,x_3\in\left(0,\,+\infty\right),$$

$$g_1\left(x_2,x_3\right)=\frac{\sqrt{3}}{2}\frac{1}{x_2^{\frac{1}{2}}x_3^{\frac{2}{3}}},\quad g_2\left(x_1,x_3\right)=\frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{x_1^{\frac{1}{4}}x_3^{\frac{2}{3}}},\quad g_3\left(x_1,x_2\right)=\frac{1}{\sqrt{2}}\frac{1}{x_1^{\frac{1}{4}}x_2^{\frac{1}{2}}}.$$

Then $A=1$ and the solution of the problem (2), (13), (14), (15) in the function class $h\left(+\infty\right)$ is of the form

$$\varphi\left(x_1,x_2,x_3\right)=\frac{1}{8\sqrt{x_1x_2x_3}}\frac{\left(x_1-1\right)\left(x_2-1\right)\left(x_3-1\right)}{\left(x_1+1\right)\left(x_2+1\right)\left(x_3+1\right)}+\frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{x_1^{\frac{1}{4}}x_2^{\frac{1}{2}}x_3^{\frac{2}{3}}}.$$

Moreover, the solution of (2), (16) in the function class $h\left(0,+\infty\right)$ is given by

$$\varphi\left(x_1,x_2,x_3\right)=\frac{\sqrt{x_1x_2x_3}}{\left(x_1+1\right)\left(x_2+1\right)\left(x_3+1\right)}.$$

Example 3. Let

$$f\left(x_1,\ldots,x_n\right)=\frac{1}{\left(1+x_1\right)^2\ldots\left(1+x_n\right)^2},\quad \left(x_1,\ldots,x_n\right)\in\left(0,\,+\infty\right)^n,$$

$$g_i\left(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\right)=\frac{1}{\sqrt{x_1\ldots x_{i-1}x_{i+1}\ldots x_n}},\quad i=1,\ldots,n.$$

Then

$$A=\frac{1}{\pi^{n-1}}\int\limits_0^{+\infty}\!\!\ldots\!\int\limits_0^{+\infty}\frac{1}{\sqrt{\sigma_1\ldots\sigma_{i-1}\sigma_{i+1}\ldots\sigma_n}}$$
$$\cdot\frac{d\sigma_1\ldots d\sigma_{i-1}d\sigma_{i+1}\ldots d\sigma_n}{\left(\sigma_1+1\right)\ldots\left(\sigma_{i-1}+1\right)\ldots\left(\sigma_{i+1}+1\right)\ldots\left(\sigma_n+1\right)}=1$$

and the solution of (2), (13), (14), (15) in the class of functions $h\,(+\infty)$ is

$$\varphi\left(x_{1},\ldots,x_{n}\right)=\frac{1}{8^{n}}\frac{1}{\sqrt{x_{1}\ldots x_{n}}}\frac{x_{1}^{2}+6x_{1}-3}{\left(x_{1}+1\right)^{2}}\cdots\frac{x_{n}^{2}+6x_{n}-3}{\left(x_{n}+1\right)^{2}}+\frac{1}{\sqrt{x_{1}\ldots x_{n}}}.$$

The solution of the system (2), (16) in the class $h\,(0,+\infty)$ is given by

$$\varphi\left(x_{1},\ldots,x_{n}\right)=\sqrt{x_{1}\ldots x_{n}}\frac{1}{2^{n}}\frac{x_{1}+3}{\left(x_{1}+1\right)^{2}}\cdots\frac{x_{n}+3}{\left(x_{n}+1\right)^{2}}.$$

Example 4. Let

$$f\left(x_{1},\ldots,x_{n}\right)=\frac{1}{\left(x_{1}+i\right)^{2}\ldots\left(x_{n}+i\right)^{2}},\quad\left(x_{1},\ldots,x_{n}\right)\in\mathbb{R}^{n},$$

$$g_{k}\left(x_{1},\ldots,x_{k-1},x_{k+1},\ldots,x_{n}\right)=\frac{1}{\left(x_{1}+i\right)\ldots\left(x_{k-1}+i\right)\left(x_{k+1}+i\right)\ldots\left(x_{n}+i\right)},$$

$k=1,\ \ldots,\ n$. Then $A=0$ and the solution of the problem (2), (18), (19), (20) is given by

$$\varphi\left(x_{1},\ldots,x_{n}\right)=\frac{1}{\left(x_{1}+i\right)^{2}\ldots\left(x_{n}+i\right)^{2}}-\sum_{k=1}^{n}\frac{1}{\left(x_{1}+i\right)\ldots\left(x_{k-1}+i\right)\left(x_{k+1}+i\right)\ldots\left(x_{n}+i\right)}.$$

5. Future work

As future work, the authors are going to present the solutions of the equation

$$\begin{aligned}&\frac{1}{\pi^{n}}\int\limits_{L}\ldots\int\limits_{L}\frac{\varphi\left(\sigma_{1},\ldots,\sigma_{n}\right)}{\left(\sigma_{1}-x_{1}\right)\ldots\left(\sigma_{n}-x_{n}\right)}d\sigma_{1}\ldots d\sigma_{n}\\&\qquad+\frac{\lambda}{\pi^{n}}\int\limits_{L}\ldots\int\limits_{L}k\left(x_{1},\ldots,x_{n},\sigma_{1},\ldots,\sigma_{n}\right)\varphi\left(\sigma_{1},\ldots,\sigma_{n}\right)d\sigma_{1}\ldots d\sigma_{n}\\&\quad=f\left(x_{1},\ldots,x_{n}\right),\quad\left(x_{1},\ldots,x_{n}\right)\in L^{n},\quad n\geqslant 2,\end{aligned}$$

where L is $(-1,1)$, $(0,+\infty)$ or $(-\infty,+\infty)$.

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