F A S C I C U L I M A T H E M A T I C I
Nr 50 2013

A. A. Nasef and A. I. El-Maghrabi

WEAK AND STRONG FORMS OF γ-IRRESOLUTENESS

Abstract. In this paper we consider new weak and strong forms of γ-irresoluteness and γ-closure via the concept of $g\gamma$-closed sets which we call ap-γ-irresolute, ap-γ-closed and contra-γ-irresolute maps. Moreover, we use ap-γ-irresolute and ap-γ-closed maps to obtain a characterization of $\gamma - T_{\frac{1}{2}}$-spaces.

Key words: topological spaces, generalized γ-closed sets, γ-open sets, γ-closed maps, γ-irresolute maps.

(2000) *AMS Mathematics Subject Classification* : 54B05, 54C08.

1. Introduction and preliminaries

A.A. El-Atik [6] introduced the notion of γ-open sets and γ-continuity in topological spaces. Andrijevic [1] defined and investigated b-open sets which are equivalent with γ-open sets. El-Atik [6] introduced a new map called γ-irresolute which is contained in the class of γ-continuous maps. In this paper, we introduce weak and strong forms of γ-irresoluteness called ap-γ-irresoluteness and ap-γ-closedness by using gγ-closed sets and obtain some basic properties of such maps. This definition enables us to obtain conditions under which maps and inverse maps preserve gγ-closed sets. Also, in this paper we present a new generalization of contra γ-continuity due to the present Author and EL-Maghrabi [14, 7] called contra-γ-irresoluteness. We define this last class of maps by the requirement that the inverse of each γ-open set in the codommain is γ-closed in the domain. This notion is a stronger form of ap-γ-irresoluteness. Finally, we characterize the class of $\gamma - T_{\frac{1}{2}}$ spaces in terms of ap-γ-irresolute and ap-γ-closed maps.

Throughout the present paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of (X, τ). The subset A of a topological space (X, τ) is called γ-open [6] or b-open [1] or sp-open [5] (resp. α-open [15], semi-open [10]) if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ (resp. $A \subseteq Int(Cl(Int(A)))$, $A \subseteq Cl(Int(A))$), where $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively. The complement of a γ-open (resp. α-open, semi-open) set is called γ-closed (resp. α-closed, semi-closed). The intersection of all γ-closed (resp. α-closed, semi-closed) sets containig A is called the γ-closure (res. α-closure, semi-closure) of A and is denoted by $\gamma Cl(A)$ resp. $\alpha Cl(A)$, $sCl((A))$. The interior of A is the union of all γ-open sets in X and is denoted by γ-$Int(A)$. The family of all γ-open (resp. γ-closed, α-open, semi-open) sets in X (resp. $\gamma C(X,\tau)$, $\alpha O(X,\tau)$, $SO(X,\tau)$) is denoted by $\gamma O(X,\tau)$. A subset A of (X,τ) is said to be :

(i) generalized closed (briefly, g-closed) [11] set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),

(ii) generalized α-closed (briefly, $g\alpha$-closed) [12] set if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X,τ),

(iii) generalized semi-closed (briefly, gs-closed) [2] set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ),

(iv) semi-generalized closed (briefly, sg-closed) [3] set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X,τ),

(v) generalized γ-closed (briefly, $g\gamma$-closed) [8] (equivalently, gb-closed) [9] set if $\gamma cl(A) \subseteq U$ whenever $A \subseteq U$ and U is γ-open in (X,τ).

It shoud be noted that this notion is a particular case of the notion of generalized (m_1, m_2)-closed sets introduced by Noiri [16]. A subset B is said to be generalized γ-open (breifly, $g\gamma$-open) in (X,τ) [8] if its complemen $B^c = X - B$ is $g\gamma$-closed in (X,τ).

A map $f : (X,\tau) \to (Y,\sigma)$ is called:

(i) γ-irresolute [6] if for each $V \in \gamma O(Y,\sigma)$, $f^{-1}\ (V) \in \gamma O(X,\tau)$.

(ii) pre-γ-closed [6] (resp. pre-γ-open [6]), if for every γ-closed (resp. γ-open) set A of (X,τ), $f(A)$ is γ-closed (resp. γ-open) in (Y,σ).

(iii) contra-γ-closed [7] if, $f(U)$ is γ-open in Y, for each closed set U of X.

2. Ap-γ-irresolute, ap-γ-closed and contra-γ-irresolute maps

Definition 1. *A map $f : (X,\tau) \to (Y,\sigma)$ is said to be approximately γ-irresolute (briefly, ap-γ-irresolute) if $\gamma Cl(A) \subseteq f^{-1}(G)$ whenever G is a γ-open subset of (Y,σ), A is a $g\gamma$-closed subset of (X,τ) and $A \subseteq f^{-1}(G)$.*

Definition 2. *A map $f : (X,\tau) \to (Y,\sigma)$ is said to be approximately γ-closed (briefly, ap-γ -closed) if, $f(A) \subseteq \gamma Int(H)$ whenever H is a $g\gamma$-open subset of (Y,σ), A is a γ-closed subset of (X,τ) and $f(A) \subseteq H$.*

Theorem 1. (i) *$f : (X,\tau) \to (Y,\sigma)$ is ap-γ-irresolute if $f^{-1}(G)$ is γ-closed in (X,τ), for every $G \in \gamma O(Y,\sigma)$.*

(ii) *$f : (X,\tau) \to (Y,\sigma)$ is ap-γ-closed if, $f(A) \in \gamma O(Y,\sigma)$, for every γ-closed subset A of (X,τ).*

Proof. (i) Let $A \subseteq f^{-1}(G)$, where $G \in \gamma O(Y, \sigma)$ and A is a $g\gamma$-closed subset of (X, τ). Therefore $\gamma Cl(A) \subseteq \gamma Cl(f^{-1}(G)) = f^{-1}(G)$. Thus f is ap-γ-irresolute.

(ii) Let $f(A) \subseteq H$, where A is a γ-closed subset of (X, τ) and H is a $g\gamma$-open subset of (Y, σ). Therefore $\gamma Int(f(A) \subseteq \gamma Int(H)$. Then $f(A) \subseteq \gamma Int(H)$. Thus f is ap-γ-closed. ∎

Clearly, γ-irresolute maps are ap-γ-irresolute. Also, pre-γ-closed maps are ap-γ-closed. The converse implications do not hold as it is shown in the following example.

Example 1. Let $X = \{a, b\}$ be the Sierpinski space with the topology $\tau = \{X, \phi, \{a\}\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = b$ and $f(b) = a$. Since the image of every γ-closed set is γ-open, then f is ap-γ-closed (similarly, since the inverse image of every γ-open set is γ-closed , then f ap-γ-irresolute). However $\{b\}$ is γ-closed in (X, τ) (resp. $\{a\}$ is γ-open), but $f(\{b\})$ is not γ-closed (resp. $f^{-1}(\{a\})$ is not γ-open) in (X, τ). Therefore f is not pre-γ-closed (resp. f is not γ-irresolute).

Remark 1. Let be (X, τ) a space as defined in Example 1. Then the identity map on (X, τ) is both ap-γ-irresolute and ap-γ-closed. It is clear that the converses of (i) and (ii) in Theorem 1 do not hold.

In the following result, the converses of (i) and (ii) in Theorem 1 are true under certain conditions.

Theorem 2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a space (X, τ) to a space (Y, σ).*

(i) *Let all subsets of (X, τ) be clopen, then f is ap-γ-irresolute if and only if $f^{-1}(G)$ is γ-closed in (X, τ), for every $G \in \gamma O(Y, \sigma)$,*

(ii) *Let all subsets of (Y, σ) be clopen, then f is ap-γ-closed if and only if $f(A) \in \gamma O(Y, \sigma)$, for every γ-closed subset A of (X, τ).*

Proof. (i) The sufficiency is stated in Theorem 1.

Necessity. Assume that f is ap-γ-irresolute. Let A be an arbitrary subset of (X, τ) such that $A \subseteq H$, where $H \in \gamma O(X, \tau)$. Then by hypothesis $\gamma Cl(A) \subseteq \gamma Cl(H) = H$. Therefore all subsets of (X, τ) are $g\gamma$-closed (hence and all are $g\gamma$-open). So, for any $G \in \gamma O(Y, \sigma)$, $f^{-1}(G)$ is γ-closed in (X, τ). Since f is ap-γ-irresolute, $\gamma Cl(f^{-1}(G) \subseteq f^{-1}(G)$. Therefore $\gamma Cl(f^{-1}(G)) = f^{-1}(G)$, i.e., $f^{-1}(G)$ is γ-closed in (X, τ).

(ii) The sufficiency is clear by Theorem 1.

Necessity. Assume that f is ap-γ-closed. As in (i), we obtain that all subsets of (Y, σ) are $g\gamma$-open. Therefore for any γ-closed subset A of (X, τ), $f(A)$ is $g\gamma$-open in Y. Since f is ap-γ-closed $f(A) \subseteq \gamma Int(f(A))$. Hence $f(A) = \gamma Int(f(A))$, i.e., $f(A)$ is γ-open. ∎

As an immediate consequence of Theorem 2, we have the following.

Corollary 1. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) to a topological space (Y, σ).*

(i) *Let all subsets of (X, τ) be clopen, then f is ap-γ-irresolute if and only if, f is γ-irresolute,*

(ii) *Let all subsets of (Y, σ) be clopen, then f is ap-γ-closed if and only if, f is pre γ-closed.*

Definition 3. *A map $f : (X, \tau) \to (Y, \sigma)$ is called:*

(i) *contra-γ-irresolute if $f^{-1}(G)$ is γ-closed in (X, τ) for each $G \in \gamma O(Y,\sigma)$,*

(ii) *contra-pre-γ-closed if $f(A) \in \gamma O(Y, \sigma)$, for each γ-closed set A of (X, τ).*

Remark 2. In fact, contra-γ-irresoluteness and γ-irresoluteness are independent notions. Example 1 shows that contra-γ-irresoluteness does not imply γ-irresoluteness while the converse is shown in the following example.

Example 2. A γ-irresolute map need not be contra-γ-irresolute. The identity map on the topological space (X, τ), where $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $X = \{a, b, c\}$ is an example of a γ-irresolute map which is not contra-γ-irresolute.

Recall that a map $f : (X, \tau) \to (Y, \sigma)$ is contra-γ-continuous [7, 14] if, $f^{-1}(G)$ is γ-closed in (X, τ), for each open set G of (Y, σ).

Every contra-γ-irresolute map is contra-γ-continuous, but not conversely as the following example shows.

Example 3. *Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $Y = \{p, q\}$, $\sigma = \{Y, \phi, \{p\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = p$ and $f(b) = f(c) = q$. Then f is contra-γ-continuous, but f is not contra-γ-irresolute.*

The following result can be easily verfied. Therefore we omitted its proof.

Theorem 3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a map. Then the following conditions are equivalent:*

(i) *f is contra-γ-irresolute,*

(ii) *The inverse image of each γ-closed set of Y is γ-open in X.*

Remark 3. By Theorem 1, we have that every contra-γ-irresolute map is ap-γ-irresolute and every contra-γ-closed map is ap-γ-closed, the converse implications do not hold (see Remak 1).

A map $f : (X, \tau) \to (Y, \sigma)$ is called perfectly contra-γ-irresolute if the inverse of every γ-open set of Y is γ-clopen in X.

Lemma 1. *Every perfectly contra-γ-irresolute map is contra-γ-irresolute and γ-irresolute. But the converse may not be true.*

Example 4. Remark 2 is an example of a contra-γ-irresolute map which is not perfectly contra-γ-irresolute and Example 3 is an example of a γ-irresolute map which is not perfectly contra-γ-irresolute.

Remark 4. For the defintions of ap-irresolute (resp. ap-α-irresolute), contra-irresolute (resp. contra-α-irresolute), perfectly contra -irresolute (resp. perfectly contra-α-irresolute) and irresolute (resp. α-irresolute) see [3, 4, 13].

Example 5. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$, $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$, $\tau_2 = \{\phi, X, \{c\}, \{a, b\}\}$ and $\tau_3 = \{\phi, X\}$. Then,

$$SO(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\} = \gamma O(X, \tau),$$
$$\alpha O(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{a, b\}\},$$
$$SO(X, \tau_1) = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\} = \alpha O(X, \tau_1) = \gamma O(X, \tau_1),$$
$$SO(X, \tau_2) = \{\phi, X, \{c\}, \{a, b\}\} = \alpha O(X, \tau_2),$$
$$\gamma O(X, \tau_2) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$ Then,

(a) Let $f : (X, \tau) \to (X, \tau_2)$ be defined as $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then:

(i) f is contra-irresolute (hence, ap-irresolute), but f is not contra-α-irresolute (hence f is not perfectly contra-α-irresolute);

(ii) f is irresolute but f is not γ-irresolute;

(iii) f is irresolute but f is not α-irresolute.

(b) Let $f : (X, \tau_3) \to (X, \tau)$ be the identity map. Then:

(i) f is γ-irresolute but f is not irresolute;

(ii) f is γ-irresolute but f is not α-irresolute.

(c) Let $f : (X, \tau_2) \to (X, \tau_1)$ be the identity map. Then:

(i) f is contra-γ-irresolute but f is not contra-irresolute;

(ii) f is contra-γ-irresolute but f is not contra-α-irresolute;

(d) Let $f : (X, \tau_2) \to (X, \tau_2)$ be the identity map. Then:

(i) f is perfectly contra-γ-irresolute but f is not perfectly contra-irresolute;

(ii) f is perfectly contra-γ-irresolute but f is not perfectly contra-α-irresolute.

Example 6. EL-Atik [6] For any countable set X, the identity maps from an indiscrete space into any other one is γ-irresolute but it is not irresolute.

Example 7. EL-Atik [6] The identity function from a particular point topological space on any countable set with any particular point into an indiscrete one is irresolute but not γ-irresolute.

Clearly, the following diagram holds and none of its implications are reversible:

contra-γ-irresolute

$\nearrow$ $\searrow$

perfectly contra-γ-irresolute ap-γ-irresolute

$\searrow$ $\nearrow$

γ-irresolute

$\rceil\Updownarrow$

irresolute

$\nearrow$ $\searrow$

perfectly contra-irresolute ap-irresolute

$\searrow$ $\nearrow$

contra-irresolute

$\rceil\Updownarrow$

contra-α-irresolute

$\nearrow$ $\searrow$

perfectly contra-α-irresolute ap-α-irresolute

$\searrow$ $\nearrow$

α-irresolute

The following theorem is a decomposition of perfectly contra-γ-irresoluteness.

Theorem 4. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*
(i) f is perfectly contra-γ-irresolute,
(ii) f is contra-γ-irresolute and γ-irresolute.

Theorem 5. *If a map $f : (X, \tau) \to (Y, \sigma)$ is γ-irresolute and ap-γ-closed, then $f^{-1}(A)$ is $g\gamma$-closed (resp. $g\gamma$-open) whenever A is a $g\gamma$-closed (resp. $g\gamma$-open) subset of (Y, σ).*

Proof. Let A be a $g\gamma$-closed subset of (Y, σ). Suppose that $f^{-1}(A) \subseteq G$ where $G \in \gamma O(X, \tau)$. Taking complements, we obtain $G^c \subseteq f^{-1}(A^c)$ or $f(G^c) \subseteq A^c$. Since f is ap-γ-closed, then $f(G^c) \subseteq \gamma Int(A^c) = (\gamma Cl(A))^c$. It follows that $G^c \subseteq (f^{-1}(\gamma Cl(A)))^c$ and hence $f^{-1}(\gamma Cl(A)) \subseteq G$. Since f is γ-irresolute, $f^{-1}(\gamma Cl(A))$ is γ-closed. Thus we have

$$\gamma Cl(f^{-1}(A)) \subseteq \gamma Cl(f^{-1}(\gamma Cl(A)) = f^{-1}(\gamma Cl(A)) \subseteq G.$$

This implies that $f^{-1}(A)$ is $g\gamma$-closed in (X, τ). ■

A similar argument shows that inverse images of $g\gamma$-open sets are $g\gamma$-open.

Theorem 6. *If a map $f : (X, \tau) \to (Y, \sigma)$ is ap-γ-irresolute and pre-γ-closed, then for every $g\gamma$-closed subset V of (X, τ) $f(V)$ is a $g\gamma$-closed set of (Y, σ).*

Proof. Let V be a $g\gamma$-closed subset of (X, τ). Let $f(V) \subseteq G$ where $G \in \gamma O(Y, \sigma)$. Then $V \subseteq f^{-1}(G)$ holds. Since f is ap-γ-irresolute, $\gamma Cl(V) \subseteq (f^{-1}(G))$ and hence $f(\gamma Cl(V)) \subseteq G$. Therefore, we have $\gamma Cl(f(V)) \subseteq \gamma Cl(f(\gamma Cl(V)) = f(\gamma Cl(V)) \subseteq G$. Hence $f(V)$ is $g\gamma$-closed in (Y, σ).

It should be noticed that the composition of two contra-γ-irresolute maps need not be contra-γ-irresolute. Let $X = \{a, b\}$ be the Sierpinski space and set $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, X, \{b\}\}$. The identity maps $f : (X, \tau) \to (X, \sigma)$ and $g : (X, \sigma) \to (X, \tau)$ are both contra-γ-irresolute but their composition $g \circ f : (X, \tau) \to (X, \tau)$ is not contra-γ-irresolute. ■

However the following theorem holds, the proof is easy and hence omitted.

Theorem 7. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps such that $g \circ f : (X, \tau) \to (Z, \eta)$. Then:*

(i) *$g \circ f$ is contra-γ-irresolute, if g is γ-irresolute and f is contra-γ-irresolute*;

(ii) *$g \circ f$ is contra-γ-irresolute, if g is contra-γ-irresolute and f is γ-irresolute.*

In analogous way, we have the following.

Theorem 8. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be two maps such that $g \circ f : (X, \tau) \to (Z, \eta)$. Then:*

(i) *$g \circ f$ is ap-γ-closed, if g is ap-γ-closed and f is pre-γ-closed;*

(ii) *$g \circ f$ is ap-γ-closed, if g is pre-γ-open, f is ap-γ-closed and g^{-1} preserves $g\gamma$-open sets;*

(iii) *$g \circ f$ is ap-γ-irresolute, if g is γ-irresolute and f is ap-γ-irresolute.*

Proof. (i) Suppose that A is an arbitrary γ-closed subset of (X, τ) and B is a $g\gamma$-open subset of (Z, η) for which $(g \circ f)(A) \subseteq B$. Then $f(A)$ is γ-closed in (Y, σ), because f is pre-γ-closed. Since g is ap-γ-closed, $g(f(A)) \subseteq \gamma - Int(B)$. This implies that $g \circ f$ is ap-γ-closed.

(ii) Suppose that A is an arbitrary γ-closed subset of (X, τ) and B is a $g\gamma$-open subset of (Z, η) for which $(g \circ f)(A) \subseteq B$. Hence $f(A) \subseteq g^{-1}(B)$. Then $f(A) \subseteq \gamma Int(g^{-1}(B))$ because, $g^{-1}(B)$ is $g\gamma$-open and f is ap-γ-closed. Thus $(g \circ f)(A) = g(f(A)) \subseteq g(\gamma - Int(g^{-1}(B)) \subseteq \gamma Int(gg^{-1}(B)) \subseteq \gamma Int(B)$. This implies that $g \circ f$ is ap-γ-closed.

(iii) Suppose that A is an arbitrary $g\gamma$-closed subset of (X, τ) and $G \in \gamma O(Z, \eta)$ for which $A \subseteq (g \circ f)^{-1}(G))$. Then $g^{-1}(G) \in \gamma O(Y, \sigma)$ because g is γ-irresolute. Since f is ap-γ-irresolute, $\gamma Cl(A) \subseteq f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G)$. This proves that $g \circ f$ is ap-γ-irresolute. ■

As a consequence of Theorem 8, we have.

Corollary 2. *Let $f_\alpha : X \to Y_\alpha$ be a map for each $\alpha \in \Omega$ and let $f : X \to \Pi Y_\alpha$ be the product map given by $f(x) = (f_\alpha(x)$. If, f is ap-γ-irresolute, then f_α is ap-γ-irresolute for each α.*

Proof. For each γ, let $P_\gamma : \Pi Y_\alpha \to Y_\gamma$ be the projection map. Then $f_\gamma = P_\gamma \circ f$, where P_γ is γ-irresolute. By Theorem 8(iii) f_γ is ap-γ-irresolute. ∎

Lemma 2. *Let A and Y be subsets of a space X. If $A \in \gamma O(Y, \tau_Y)$ and $Y \in \gamma O(X, \tau)$, then $A \in \gamma O(X, \tau)$.*

Lemma 3. *Let X be a topological space and A, Y be subsets of X such that $A \subseteq Y \subseteq X$ and $Y \in \gamma O(X, \tau)$. Then $\gamma Cl(A) \cap Y = \gamma Cl_Y(A)$, where $\gamma Cl_Y(A)$ denotes the γ-closure of A in the subspace Y.*

Regarding the restriction f_A of a map $f : (X, \tau) \to (Y, \sigma)$ to a subset A of X, we have the following.

Theorem 9. (i) *If $f : (X, \tau) \to (Y, \sigma)$ is ap-γ-closed and A is a γ-closed set of (X, τ), then its restriction $f_A : (A, \tau_A) \to (Y, \sigma)$ is ap-γ-closed;*
(ii) *If, $f : (X, \tau) \to (Y, \sigma)$ is ap-γ-irresolute and A is an open, $g\gamma$-closed subset of (X, τ), then its restriction $f_A : (A, \tau_A) \to (Y, \sigma)$ is ap-γ-irresolute.*

Proof. (i) Suppose that B is arbitrary γ-closed subset of (A, τ_A) and G is a $g\gamma$-open subset of (Y, σ) for which $f_A(B) \subseteq G$. By Lemma 2, B is γ-closed subset of (X, τ). Since A is a γ-closed subest of (X, τ), then $f_A(B) = f(B) \subseteq G$. Using Definition 2, we have $f_A(B) \subseteq \gamma Int(G)$. Thus f_A is an ap-γ-closed map.
(ii) Assume that V is a $g\gamma$-closed subset relative to A, i.e., V is $g\gamma$-closed in (A, τ_A) and G is a γ-open subset of (Y, σ) for which $V \subseteq (f_A)^{-1}(G)$. Then $V \subseteq f^{-1}(G) \cap A$.
On the other hand, V is $g\gamma$ -closed in X. Since f is ap-γ-irresolute, then $\gamma Cl(V) \subseteq f^{-1}(G)$. This implies that $\gamma Cl(V) \cap A \subseteq f^{-1}(G) \cap A$. Using the fact that $\gamma Cl(V) \cap A = \gamma Cl_A(V)$ (Lemma 3), we have $\gamma Cl_A(V) \subseteq (f_A)^{-1}(G)$. Thus $f_A : (A, \tau_A) \to (Y, \sigma)$ is ap-γ-irresolute. ∎

3. Characterizations of $\gamma - T_{\frac{1}{2}}$-spaces

In the following result, we offer a characterization of the class of $\gamma - T_{\frac{1}{2}}$-spaces by using the concepts of ap-γ-irresolute and ap-γ-closed maps.

Definition 4. *A space (X, τ) is said to be $\gamma - T_{\frac{1}{2}}$-space, if every $g\gamma$-closed set is γ-closed.*

Theorem 10. *Let (X, τ) be a space. Then the following statements are equivalent.*

(i) *(X, τ) is a $\gamma - T_{\frac{1}{2}}$-space;*

(ii) *f is ap-γ-irresolute, for every space (Y, σ) and every map $f : (X, \tau) \to (Y, \sigma)$.*

Proof. $(i) \to (ii)$. Let V be a $g\gamma$-closed subset of (X, τ) and $V \subseteq f^{-1}(G)$, where $G \in \gamma O(Y, \sigma)$. Since (X, τ) is a $\gamma - T_{\frac{1}{2}}$-space, V is γ-closed (i.e., $V = \gamma Cl(V)$). Therefore $\gamma Cl(V) \subseteq f^{-1}(G)$ and hence f is ap-γ-irresolute.

$(ii) \to (i)$. Let B be a $g\gamma$-closed subset of (X, τ) and Y be the set X with the topology $\sigma = \{\phi, Y, B\}$. Finally let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. By the assumption f is ap-γ-irresolute. Since B is $g\gamma$-closed in (X, τ) and γ-open in (Y, σ) and $B \subseteq f^{-1}(B)$, it follows that $\gamma Cl(B) \subseteq f^{-1}(B) = B$. Hence B is γ-closed in (X, τ). Therefore (X, τ) is a $\gamma - T_{\frac{1}{2}}$-space. ■

Theorem 11. *Let (Y, σ) be a space. Then the following statements are equivalent.*

(i) *(Y, σ) is a $\gamma - T_{\frac{1}{2}}$-space;*

(ii) *f is ap-γ-closed, for every space (X, τ) and every map $f : (X, \tau) \to (Y, \sigma)$.*

Proof. This is analogous to the proof of Theorem 10. ■

References

[1] ANDRIJEVIC D., On b-open sets, *Mat. Vesnik,* 48(1)(1996), 59-64.

[2] ARYA S.P., NOUR T.M., Characterizations of s-normal spaces, *Indian J. Pure Appl. Math.,* 21(1990), 717-719.

[3] BHATTACHARYYA P., LAHIRI B.K., Semi-generalized closed sets in topology, *Indian J. Math.,* 29(1987), 375-382.

[4] CALDAS M., Weak and strong forms of irresolute maps, *Internat. J. Math. Math. Sci.,* 24(2000), 253-259.

[5] DONTCHEV J., PRZEMKI M., On the various decompositions of continuous and some weakly continuous functions, *Acta Math. Hungar.,* 71(1-2)(1996), 109-120.

[6] EL-ATIK A.A., A Study of Some types of Mappings on Topological Spaces, *M. Sc. Thesis,* Tanta Univ. Egypt, 1997.

[7] EL-MAGHRABI A.I., Some properties of contra-γ-continuous mappings, *Internat. J. of General Topology,* 3(1-2)(2010), 55-64.

[8] EL-MAGHRABI A.I., NASEF A.A., More on γ-closed sets in topology, (*preparing*).

[9] KESKIN A., NOIRI T., On bD-sets and associated separation axioms, *Bull. of the Iranian Math. Soc.,* 35(1)(2009), 179-198.

[10] LEVINE N., Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly,* 70(1963), 36-41.

[11] Levine N., Generalized closed sets in topology, *Rend. Circ. Mat. Palermo,* 19(2)(1970), 89-96.

[12] Maki H., Devi R., Balachandran K., Generalized α-closed sets in topology, *Bull. Fukuoka Univ. Ed.,* Part III(42)(1993), 13-21.

[13] Masheshwari S.N., Thakur S.S., On α-irresolute mappings, *Tamkang J. Math.,* 11(1980), 209-214.

[14] Nasef A.A., Some properties of contra-γ-continuous functions, *Chaos Solitons and Fractals,* 24(2005), 471-477.

[15] Njastad O., On some classes of nearly open sets, *Pacific J. Math.,* 15(1965), 961-970.

[16] Noiri T., A unified theory of generalized closed sets, The 6^{th} Meeting on topological spaces theory and its applications, *Yatsushiro College of Technology,* August, 18-19(2001), 27-34.

A. A. Nasef
Department of Mathematics and Physics
Faculty of Engineering
Kafr El-Sheikh University
Kafr El-Sheikh, Egypt
e-mail: nasef50@yahoo.com

A. I. El-Maghrabi
The permenent address:
Department of Mathematics
Faculty of Science
Kafr El-Sheikh University
Kafr El-Sheikh, Egypt
e-mail: aelmaghrabi@yahoo.com
The current address:
Department of Mathematics
Faculty of Science
Taibah University
AL-Madinah AL-Munawarah
P. O. BOX, 30002, K.S.A.
e-mail: amaghrabi@taibahu.edu.sa

Received on 08.02.2011 and, in revised form, on 07.07.2012.