F A S C I C U L I   M A T H E M A T I C I

Nr 50 2013

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ON e-CONTINUOUS FUNCTIONS AND RELATED RESULTS

Abstract. In this paper, characterizations and properties of e-continuous functions are given. Moreover, Urysohn's Lemma on e-normal spaces is proved.

Key words: e-open and e-closed subsets; e-continuous function; e-irresolute function; e-normal spaces; Urysohn's lemma.

AMS Mathematics Subject Classification: 54A05, 54D15.

1. Introduction

In recent years, many researchers introduced different forms of continuous functions. El-Atik et al. [1] presented γ-open sets and γ-continuity. Hatir and Noiri et al. [5] has introduced δ-β-open sets and δ-β-continuity. Raychaudhurim and Mukherjee et al. [10] investigated δ-preopen sets and δ-almost continuity. Noiri et al. [12] not only studied δ-semi-sets and δ-semi-continuity but also discussed the relationship between δ-β-continuity and δ-semi-continuity. In 2008, Ekici et al. [3] introduced the concept of e-open sets and investigated e-continuity. The purpose of this paper is to study further e-continuity. We will give characterizations and properties of e-continuity. We also discuss the relationship between e-continuity and other forms of continuity. In addition, Urysohn's Lemma on e-normal spaces is proved.

2. Preliminaries

Throughout this paper, spaces always mean topological spaces with no separation properties assumed, and maps are onto. If X is a space and $A \subset X$, then the interior and the closure of A in X are denoted by iA, cA, respectively.

Let $f_i : 2^X \longrightarrow 2^X$ be a operator $(i = 1, 2, \ldots, n)$ and $A \subset X$. We define

$$f_1 f_2 \cdots f_n A = f_1(f_2(\ldots(f_n(A))\ldots)).$$

Let X be a space, $A \subset X$ and $x \in X$. A is called regular open (resp. regular closed) if $A = icA$ (resp. $A = ciA$). x is called a δ-cluster point of A if $A \cap icU \neq \emptyset$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure [7] of A and is denoted by $c_\delta A$. A is called δ-closed if $c_\delta A = A$ and the complements are called δ-open. The union of all δ-open sets contained in A is called the δ-interior [7] of A and is denoted by $i_\delta A$. Obviously, A is δ-open if and only if $A = i_\delta A$.

Let (X, τ) be a space and $x \in X$. Then $\tau(x)$ means the family of all open neighborhoods of x. Put

$$\tau_\delta = \{A : \ A \text{ is } \delta\text{-open in } X\}.$$

It is not difficult that τ_δ forms a topology on X and $\tau_\delta \subset \tau$.

Definition 1. *Let X be a space and $A \subset X$. Then A is called*
(*a*) *e-open [3] if $A \subset ic_\delta A \cup ci_\delta A$.*
(*b*) *δ-preopen [10] if $A \subset ic_\delta A$.*
(*c*) *δ-semiopen [6] if $A \subset ci_\delta A$.*
(*d*) *δ-β-open [4] if $A \subset cic_\delta A$.*
(*e*) *b-open [2] (or γ-open [1]) if $A \subset icA \cup ciA$.*

The family of all e-open (resp. δ-preopen, δ-semiopen, δ-β-open, b-open) subsets of X is denoted by $EO(X)$ (resp. $\delta PO(X)$, $\delta SO(X)$, $\delta\beta O(X)$, $BO(X)$).

Definition 2. *The complement of a e-open (resp. δ-preopen, δ-semiopen, δ-β-open, b-open) set is called e-closed [3] (resp. δ-preclosed [10], δ-semiclosed [6], δ-β-closed [4], b-closed [2]).*

Definition 3. *The union of all e-open (resp. δ-preopen, δ-semiopen, δ-β-open, b-open) subsets of X contained in A is called the e-interior [3] (resp. δ-preinterior [10], δ-semi-interior [12], δ-β-interior [4], b-interior [2]) of A and is denoted by $i_e A$ (resp. ${}_pi_\delta A$, ${}_si_\delta A$, ${}_\beta i_\delta A$, $i_b A$).*

Definition 4. *The intersection of all e-closed (resp. δ-preclosed, δ-semiclosed, δ-β-closed, b-closed) sets of X containing A is called the e-closure [3] (resp. δ-preclosure [10], δ-semiclosure [12], δ-β-closure [4], b-closure [2]) of A and is denoted by $c_e A$ (resp. ${}_pc_\delta A$, ${}_sc_\delta A$, ${}_\beta c_\delta A$, $c_b A$).*

Lemma 1 ([4]). *Let X be a space and $A \subset X$. Then*
(*a*) ${}_pi_\delta A = A \cap ic_\delta A$; ${}_pc_\delta A = A \cup ci_\delta A$.
(*b*) ${}_si_\delta A = A \cap ci_\delta A$; ${}_sc_\delta A = A \cup ic_\delta A$.
(*c*) ${}_\beta i_\delta A = A \cap cic_\delta A$; ${}_\beta c_\delta A = A \cup ici_\delta A$.

Proposition 1 ([3]). *Let X be a space and $A \subset X$. Then A is e-open in X if and only if $A = {}_pi_\delta A \cup {}_si_\delta A$.*

Theorem 1 ([3]). *Let X be a space and $A \subset X$. Then*
(a) $i_eA = A \cap (ic_\delta A \cup ci_\delta A)$.
(b) $c_eA = A \cup (ci_\delta A \cap ic_\delta A)$.
(c) $i_e(X - A) = X - c_eA$.
(d) *$x \in i_eA$ if and only if $U \subset A$ for some $U \in EO(X)$ containing x.*
(e) *A is e-open in X if and only if $A = i_eA$.*

Theorem 2 ([3]). *Let X be a space. Then*
(a) *The union of any family of e-open subsets of X is e-open.*
(b) *The intersection of any family of e-closed subsets of X is e-closed.*

Proposition 2. *Let X be a space. Then the intersection of an open subset and a e-open subset is e-open in X.*

Proof. Suppose $A \in EO(X)$ and $B \in \tau$. By Proposition 1, then $A \cap B = ({}_pi_\delta A \cup {}_si_\delta A) \cap B = ({}_pi_\delta A \cap B) \cup ({}_si_\delta A \cap B) = ({}_pi_\delta A \cap iB) \cup ({}_si_\delta A \cap iB) \subset ({}_pi_\delta A \cap {}_pi_\delta B) \cup ({}_si_\delta A \cap {}_si_\delta B) = (A \cap ic_\delta A \cap B \cap ic_\delta B) \cup (A \cap ci_\delta A \cap B \cap ci_\delta B) \subset (ic_\delta A \cap ic_\delta B) \cup (ci_\delta A \cap ci_\delta B) = ic_\delta(A \cap B) \cup ci_\delta(A \cap B)$. Hence $A \cap B$ is e-open in X. ■

Definition 5. *A function $f : X \to Y$ is called δ-continuous [11] if $f^{-1}(V)$ is regular open in X for each $V \in RO(Y)$.*

Definition 6. *A function $f : X \to Y$ is called δ-β-continuous [5] (resp. γ-continuous [1], δ-almost continuous [10], δ-semi-continuous [12]) if $f^{-1}(V)$ is δ-β-open (resp. b-open, δ-preopen, δ-semiopen) in X for each open set V in Y.*

Lemma 2 ([9]). *If $f : X \to Y$ is a function, $A \subset X$ and $B \subset Y$, then $f^{-1}(B) \subset A$ if and only if $B \subset Y - f(X - A)$.*

3. e-continuous functions

Definition 7 ([3]). *A function $f : (X, \tau) \to (Y, \sigma)$ is called e-continuous if $f^{-1}(V)$ is e-open in X for each $V \in \sigma$.*

Every δ-almost continuous and δ-semi-continuous is e-continuous but the converse is not true. Every e-continuous is δ-β-continuous but the converse is also not true, as shown by the following Example 4.4 [3], Example 4.5 [3] and Example 1.

Example 1. Let $X = Y = \{x, y, z\}$, $\tau = \{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}$ and

$$\sigma = \{\emptyset, \{x, z\}, Y\}.$$

Let $f : X \to Y$ be the identity function.

Since $\tau(x) = \{\{x\}, \{x, y\}, X\}$, $\tau(y) = \{\{y\}, \{x, y\}, X\}$ and $\tau(z) = \{X\}$, then $c_\delta\{x, z\} = \{x, z\}$ and $i_\delta\{x, z\} = \emptyset$. Thus we have $cic_\delta\{x, z\} = ci\{x, z\} = c\{x\} = \{x, z\}$ and $ci_\delta\{x, z\} \cup ic_\delta\{x, z\} = \emptyset \cup \{x\} = \{x\}$. Therefore for each open subset $\{x, z\} \in \sigma$, then $f^{-1}(\{x, z\}) = \{x, z\} \subset cic_\delta f^{-1}(\{x, z\}) = \{x, z\}$ and $f^{-1}(\{x, z\})$ is δ-β-open in X. Hence f is δ-β-continuous.

But $f^{-1}(\{x, z\}) = \{x, z\} \not\subset ci_\delta f^{-1}(\{x, z\}) \cup ic_\delta f^{-1}(\{x, z\}) = \emptyset \cup \{x\} = \{x\}$ is not e-open in X. Hence f is not e-continuous.

The following Theorem 3 gives some characterizations of e-continuity.

Theorem 3. *Let $f : X \to Y$ be a function. Then the following are equivalent.*

(a) *f is e-continuous*;

(b) *For each $x \in X$ and each open neighborhood V of $f(x)$, there exists $U \in EO(X)$ containing x such that $f(U) \subset V$;*

(c) *$f^{-1}(V)$ is e-closed in X for each closed subset V of Y;*

(d) *$ci_\delta f^{-1}(B) \cap ic_\delta f^{-1}(B) \subset f^{-1}(cB)$ for each $B \subset Y$;*

(e) *$f(ci_\delta A \cap ic_\delta A) \subset cf(A)$ for each $A \subset X$;*

(f) *$f^{-1}(iB) \subset i_e f^{-1}(B)$ for each $B \subset Y$.*

Proof. $(a) \Leftrightarrow (b)$, $(a) \Leftrightarrow (c)$ are obvious.

$(c) \Rightarrow (d)$. Let $B \subset Y$. By (3), then we obtain $f^{-1}(cB)$ is e-closed subset of X. Hence $ci_\delta f^{-1}(B) \cap ic_\delta f^{-1}(B) \subset ci_\delta f^{-1}(cB) \cap ic_\delta f^{-1}(cB) \subset f^{-1}(cB)$.

$(d) \Rightarrow (c)$. For any closed subset $V \subset Y$. By (4), then we have $ci_\delta f^{-1}(V) \cap ic_\delta f^{-1}(V) \subset f^{-1}(cV) = f^{-1}(V)$. Hence $f^{-1}(V)$ is e-closed in X.

$(d) \Rightarrow (e)$. Put $B = f(A)$. By (4), then we obtain $ci_\delta f^{-1}(f(A)) \cap ic_\delta f^{-1}(f(A)) \subset f^{-1}(cf(A))$ and $ci_\delta A \cap ic_\delta A \subset f^{-1}(cf(A))$. Hence $f(ci_\delta A \cap ic_\delta A) \subset cf(A)$.

$(e) \Rightarrow (d)$ is obvious.

$(c) \Rightarrow (f)$. Let $B \subset Y$, then $Y - iB$ is closed subset in Y. By (3), then we have $f^{-1}(Y - iB) \in EC(X)$ and $ci_\delta f^{-1}(Y - iB) \cap ic_\delta f^{-1}(Y - iB) \subset f^{-1}(Y - iB)$. Thus, we obtain $(X - (ci_\delta f^{-1}(iB))) \cap (X - (ic_\delta f^{-1}(iB))) \subset X - f^{-1}(iB)$ and $X - (ci_\delta f^{-1}(iB) \cup ic_\delta f^{-1}(iB)) \subset X - f^{-1}(iB)$. Hence $f^{-1}(iB) \subset ci_\delta f^{-1}(iB) \cup ic_\delta f^{-1}(iB) \subset ci_\delta f^{-1}(B) \cup ic_\delta f^{-1}(B)$ and $f^{-1}(iB) \subset i_e f^{-1}(B)$.

$(f) \Rightarrow (c)$ is obvious. ■

Theorem 4. *Let $f : X \to Y$ be a function. If $if(A) \subset f(i_e A)$ for each $A \subset X$, then f is e-continuous.*

Proof. Suppose that $x \in X$ and V is an open neighborhood of $f(x)$. Since $if(A) \subset f(i_e A)$, then $V = iV = if(f^{-1}(V)) \subset f(i_e f^{-1}(V))$. Thus, we have $f^{-1}(V) \subset i_e f^{-1}(V)$. Set $U = f^{-1}(V)$, then $U \in EO(X)$ containing x and $f(U) \subset V$. By Theorem 3, then we obtain f is e-continuous. ■

4. Properties of e-continuous functions

Theorem 5. *Let X and Y be two spaces and A be an open subset of X. If $f : X \to Y$ is e-continuous, then $f|_A : A \to Y$ is also e-continuous.*

Proof. Let V be open in Y. Since f is e-continuous, then $(f|_A)^{-1}(V) = (f|_A)^{-1}(V \cap f(A)) = f^{-1}(V \cap f(A)) = f^{-1}(V) \cap A \in EO(X)$. Therefore $f|_A$ is e-continuous. ■

Definition 8. *Let X be a space. Let $\{x_\alpha, \alpha \in \bigwedge\}$ be a net in X and $x \in X$. Then $\{x_\alpha, \alpha \in \bigwedge\}$ is called e-converges to x in X, we denote $x_\alpha \to^e x$, if for every e-open set U containing x there exists a $\alpha_0 \in \bigwedge$ such that $x_\alpha \in U$ for every $\alpha \geq \alpha_0$.*

Lemma 3. *Let X be a space and $x \in X, A \subset X$. Then $x \in c_e A$ if and only if there exists a net consisting of elements of A and converging to x.*

Proof. *Necessity.* Suppose $x \in c_e A$ and we denote by $\mathcal{U}(x)$ the set of all e-open set containing x directed by the relation $\supset$, i.e., define that $U_1 \leq U_2$ if $U_1 \supset U_2$. Thus, we can easily check that $x_U \to^e x$ for each $x_U \in U \cap A$.

Sufficiency. Let $x_\alpha \to^e x$ in A. For every e-open set U containing x there exists a $\alpha_0 \in \bigwedge$ such that $x_\alpha \in U$ for every $\alpha \geq \alpha_0$. Thus, we have $U \cap A \neq \emptyset$. Hence $x \in c_e A$. ■

Theorem 6. *A function $f : X \to Y$ is e-continuous if and only if for any $x \in X$, the net $\{x_\alpha, \alpha \in \bigwedge\}$ e-converges to x in X, then the net $\{f(x_\alpha), \alpha \in \bigwedge\}$ converges to $f(x)$ in Y.*

Proof. *Necessity.* Suppose a net $\{x_\alpha, \alpha \in \bigwedge\}$ e-converges to $x \in X$ and a open subset V of Y containing $f(x)$. Then there exists a $\alpha_0 \in \bigwedge$ such that $x_\alpha \in U$ for every $\alpha \geq \alpha_0$. Since f is e-continuous, then there exists a $U \in EO(X)$ containing x such that $f(U) \subset V$ with Theorem 3. Thus, we have $f(x_\alpha) \in V$ for $\alpha \geq \alpha_0$. Hence $\{f(x_\alpha), \alpha \in \bigwedge\}$ converges to $f(x)$ in Y.

Sufficiency. By Theorem 3, we have $f(c_e A) \subset cf(A)$. By Lemma 3, then there exists a net converging to x in A for every $x \in c_e A$. By hypothesis, then there exists a net converges to $f(x)$ in $f(A)$. This implies the net e-converges to $f(x)$. Again by Lemma 3, we obtain $f(x) \in c_e f(A)$. Hence f is e-continuous. ■

Theorem 7. *Let $f, g : X \to Y$ be two functions and let $h : X \to Y \times Y$ be a function, defined by $h(x) = (f(x), g(x))$ for each $x \in X$. Then f and g are e-continuous if and only if h is e-continuous.*

Proof. *Necessity.* Let a net $\{x_\alpha, \alpha \in \bigwedge\}$ e-converges to x for every $x \in X$. For every open neighborhood W of $h(x)$ there exist open subsets U and V in Y such that $(f(x), g(x)) = h(x) \in U \times V \subset W$. Thus, we have $f(x) \in U$ and $g(x) \in V$. Since f is e-continuous, then there exists a $\alpha_1 \in \bigwedge$ such that $f(x_\alpha) \in U$ for every $\alpha \geq \alpha_1$ with Theorem 6. Similarly, there exists a $\alpha_2 \in \bigwedge$ such that $g(x_\alpha) \in V$ for every $\alpha \geq \alpha_2$. Set $\alpha_0 = \max\{\alpha_1, \alpha_2\}$, then $f(x_\alpha) \in U$ and $g(x_\alpha) \in V$ for every $\alpha \geq \alpha_0$. Thus, we obtain $h(x_\alpha) = (f(x_\alpha), g(x_\alpha)) \in U \times V \subset W$. Hence h is e-continuous.

Sufficiency. Suppose $p_Y : Y \times Y \to Y$ be the natural projections and $f = p_Y \circ h$. Let U is a open subset of Y. Then $f^{-1}(V) = h^{-1}(p_Y^{-1}(V))$. Since p_Y is continuous, then $p_Y^{-1}(V)$ is open set in $Y \times Y$. Since h is e-continuous, then $h^{-1}(p_Y^{-1}(V)$ is e-open set in X. Hence f is e-continuous. Similarly, we can prove that g is e-continuous. ■

Definition 9. *Let $\mathcal{F}$ be a filter base in a space X and $x \in X$. Then $\mathcal{F}$ is called e-converges to x, we denote $\mathcal{F} \to^e x$, if for every e-open set U containing x, there exists a $F \in \mathcal{F}$ such that $F \subset U$.*

Theorem 8. *A function $f : X \to Y$ is e-continuous if and only if the filter base $f(\mathcal{F}) = \{f(A) : A \in \mathcal{F}\}$ converges to $f(x)$ in Y for every filter base $\mathcal{F}$ e-converges to x in X.*

Proof. *Necessity.* Suppose $x \in X$ and V be an open set containing $f(x)$ in Y. Since f be e-continuous, then there exists a $U \in EO(X)$ containing x such that $f(U) \subset V$ with Theorem 3. Let $\mathcal{F} \to^e x$, then there exists a $F \in \mathcal{F}$ such that $F \subset U$ for every $U \in EO(X)$ containing x. Thus, we have $f(x) \in f(F) \subset f(U) \subset V$ in Y for every $f(F) \in f(\mathcal{F})$. Hence filter base $f(\mathcal{F})$ converges to $f(x)$.

Sufficiency. Suppose $x \in X$ and V be an open set containing $f(x)$ in Y. Let filter base $\mathcal{U}(x)$ be the set of all e-open set U containing x in X, then $\mathcal{U}(x) \to^e x$. By hypothesis, then $f(\mathcal{U}(x))$ converges to $f(x)$. Thus, we have $F \subset V$ for some a $F \in f(\mathcal{U}(x))$ and there exists a $U \in \mathcal{U}(x)$ such that $f(U) \subset V$. Hence f is e-continuous. ■

Theorem 9. *If $f : X \to Y$ is e-continuous and $g : Y \to Z$ is continuous, then the composition $g \circ f : X \to Z$ is e-continuous.*

Proof. Suppose $x \in X$ and V be an open neighborhood of $g(f(x))$. Since g is continuous, then there exists a $g^{-1}(V)$ open in Y containing $f(x)$. Since f is e-continuous, then there exists a $U \in EO(X)$ containing x such that $f(U) \subset g^{-1}(V)$. Thus, we have $(g \circ f)(U) \subset (g \circ g^{-1})(V) \subset V$. Hence $g \circ f$ is e-continuous. ■

Definition 10. *A function $f : X \to Y$ is called e-irresolute if $f^{-1}(V) \in EO(X)$ for each $V \in EO(Y)$.*

Definition 11. *A function $f : X \to Y$ is called e-open if the image of every e-open subset is e-open.*

Every e-irresolute function is e-continuous but the converse is not true, and e-irresolute and openness are not relate to each other, as shown by the following Example 2 and Example 3.

Example 2. Let $X = Y = \{x, y, z\}, \tau = \{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}$ and

$$\sigma = \{\emptyset, \{x, y\}, Y\}.$$

Let $f : X \to Y$ be the identity function.

Since $\tau(x) = \{\{x\}, \{x, y\}, X\}, \tau(y) = \{\{y\}, \{x, y\}, X\}$ and $\tau(z) = \{X\}$, then $c_\delta\{x, y\} = \{X\}$ and $i_\delta\{x, y\} = \emptyset$. Thus we have $ci_\delta\{x, y\} \cup ic_\delta\{x, y\} = \{X\} \cup \emptyset = \{X\}$. Therefore for each open set $\{x, y\} \in \sigma$, then $f^{-1}(\{x, y\}) = \{x, y\} \subset i_\delta f^{-1}(\{x, y\}) \cup ic_\delta f^{-1}(\{x, y\}) = \{X\}$ and $f^{-1}(\{x, y\})$ is e-open in X. Hence f is e-continuous.

Since $\sigma(x) = \sigma(y) = \{\{x, y\}, Y\}$ and $\sigma(z) = \{Y\}$, then $c_\delta\{x, z\} = \{Y\}$ and $i_\delta\{x, z\} = \emptyset$. Therefore $\{x, z\} \subset ic_\delta\{x, z\} \cup ci_\delta\{x, z\} = \{Y\}$ and $\{x, z\}$ is e-open set in Y. But $f^{-1}(\{x, z\}) = \{x, z\} \not\subset ci_\delta f^{-1}(\{x, z\}) \cup ic_\delta f^{-1}(\{x, z\}) = \emptyset \cup \{x\} = \{x\}$ is not e-open in X. Hence f is not e-irresolute.

Example 3. Let $X = Y = \{x, y, z\}, \tau = \{\emptyset, \{x\}, \{x, z\}, X\}$ and

$$\sigma = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z\}, Y\}.$$

Let $f : X \to Y$ be the identity function.

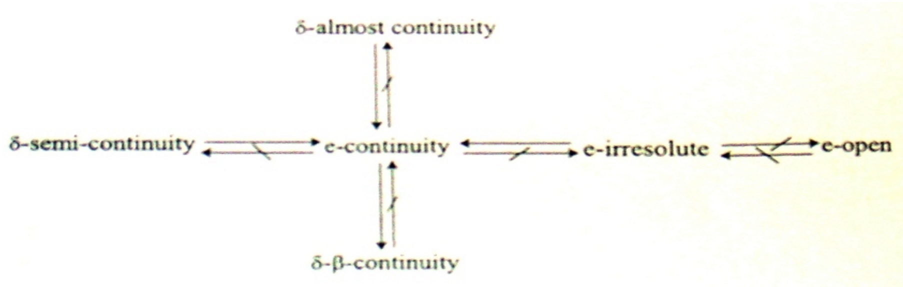
Since $\tau(x) = \{\{x\}, \{x, z\}, X\}, \tau(y) = \{Y\}$ and $\tau(z) = \{\{x, z\}, X\}$, then $c_\delta\{x, y\} = c_\delta\{y, z\} = c_\delta\{z\} = c_\delta\{y\} = \{X\}$ and $i_\delta\{x, y\} = i_\delta\{y, z\} = i_\delta\{z\} = i_\delta\{y\} = \emptyset$. Thus we have $ci_\delta\{x, y\} \cup ic_\delta\{x, y\} = \{X\} \cup \emptyset = \{X\}$, $ci_\delta\{y, z\} \cup ic_\delta\{y, z\} = \{X\} \cup \emptyset = \{X\}$, $ci_\delta\{z\} \cup ic_\delta\{z\} = \{X\} \cup \emptyset = \{X\}$ and $ci_\delta\{y\} \cup ic_\delta\{y\} = \{X\} \cup \emptyset = \{X\}$. Hence $EO(X) = \tau \cup \{\{x, y\}, \{y, z\}, \{y\}, \{z\}\}$.

Since $\sigma(x) = \{\{x\}, \{x, y\}, Y\}, \sigma(y) = \{\{y\}, \{x, y\}, \{y, z\}, Y\}$ and $\sigma(z) = \{\{y, z\}, Y\}$ then $c_\delta\{x, z\} = \{Y\}$, $c_\delta\{z\} = \{y, z\}$ and $i_\delta\{x, z\} = i_\delta\{z\} = \emptyset$. Thus we have $ci_\delta\{x, z\} \cup ic_\delta\{x, z\} = \{Y\} \cup \emptyset = \{Y\}$ and $ci_\delta\{z\} \cup ic_\delta\{z\} = \{y, z\} \cup \emptyset = \{y, z\}$. Hence $\{x, z\}, \{z\} \in EO(Y)$.

Because $f(\{x\}) = \{x\} \in \sigma$, $f(\{y\}) = \{y\} \in \sigma$, $f(\{z\}) = \{z\} \in EO(Y)$, $f(\{x,y\}) = \{x,y\} \in \sigma$, $f(\{y,z\}) = \{y,z\} \in \sigma$ and $f(\{x,z\}) = \{x,z\} \in EO(Y)$. Thus f is e-irresolute.

Let $\{x,z\} \in \tau$, then $f(\{x,z\}) = \{x,z\} \notin \sigma$. Hence f is not open.

From Example 1, Example 2, Example 3, Example 4.4 [3] and Example 4.5 [3], we have the following relationships:



Theorem 10. *Let $f : X \to Y$ be e-open and $g : Y \to Z$ be a function. If $g \circ f : X \to Z$ is e-continuous, then g is e-continuous.*

Proof. Suppose B is open in Z. Since $g \circ f$ is e-continuous, then $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ is e-open. Since f is e-open, then $f(f^{-1}(g^{-1}(B))) = g^{-1}(B)$ is e-open. Hence g is e-continuous. ■

Theorem 11. *Let $f : X \to Y$ be e-open and $g : Y \to Z$ be a function. If $g \circ f : X \to Z$ is e-continuous, then g is e-continuous.*

Proof. Suppose $y \in Y$ and V is an open neighborhood of $g(y)$. Then there exists a $x \in X$ such that $f(x) = y$. Since $g \circ f$ is e-continuous, then there exists a $U \in EO(X)$ containing x such that $g(f(U)) = (g \circ f)(U) \subset V$. Since f is e-open, then $f(U) \in EO(Y)$. Hence g is e-continuous. ■

Let $\{(X_\alpha, \tau_\alpha) : \alpha \in \bigwedge\}$ and $\{(Y_\alpha, \sigma_\alpha) : \alpha \in \bigwedge\}$ be two families of pairwise disjoint spaces, i.e., $X_\alpha \cap X_{\alpha'} = Y_\alpha \cap Y_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$ and let $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha)$ be a function for each $\alpha \in \bigwedge$.

Denote the product space $\prod\limits_{\alpha \in \bigwedge} \{(X_\alpha, \tau) : \alpha \in \Lambda\}$ of $\prod\limits_{\alpha \in \bigwedge} \{(X_\alpha, \tau_\alpha) : \alpha \in \Lambda\}$ by $\prod\limits_{\alpha \in \bigwedge} X_\alpha$ and $\prod\limits_{\alpha \in \bigwedge} f_\alpha : \prod\limits_{\alpha \in \bigwedge} X_\alpha \to \prod\limits_{\alpha \in \bigwedge} Y_\alpha$ denote the product function defined by $f(\{x_\alpha\}) = \{f(x_\alpha)\}$ for each $\{x_\alpha\} \in \prod\limits_{\alpha \in \bigwedge} X_\alpha$. Let $P_\alpha : \prod\limits_{\alpha \in \bigwedge} X_\alpha \to X_\alpha$ and $Q_\alpha : \prod\limits_{\alpha \in \bigwedge} Y_\alpha \to Y_\alpha$ be the natural projections.

Theorem 12. *The product function $\prod\limits_{\alpha \in \bigwedge} f_\alpha : \prod\limits_{\alpha \in \bigwedge} X_\alpha \to \prod\limits_{\alpha \in \bigwedge} Y_\alpha$ is e-continuous if and only if $f_\alpha : X_\alpha \to Y_\alpha$ is e-continuous for every $\alpha \in \bigwedge$.*

Proof. Denote $X = \prod_{\alpha\in\bigwedge} X_\alpha, Y = \prod_{\alpha\in\bigwedge} Y_\alpha$ and $f = \prod_{\alpha\in\bigwedge} f_\alpha$.

Necessity. Suppose f is e-continuous and Q_α is continuous for any $\alpha \in \bigwedge$. By Theorem 10, then $f_\alpha \circ P_\alpha = Q_\alpha \circ f$ is e-continuous. Since P_α is continuous surjection, then f_α is e-continuous with Theorem 11.

Sufficiency. Let $x = \{x_\alpha\} \in X$ and V be an open subset of Y containing $f(x)$, then there exists a basic open set $\prod_{\alpha\in\bigwedge} W_\alpha$ such that $f(x) \in \prod_{\alpha\in\bigwedge} W_\alpha \subset V$ and $\prod_{\alpha\in\bigwedge} W_\alpha = \prod_{i=1}^{n} W_{\alpha i} \times \prod_{\alpha\neq\alpha i} Y_\alpha$ where W_α be an open subset of Y for each $\alpha \in \{\alpha_i : 1 < i < n\}$. Since f_α is e-continuous, then there exists a e-open set $U_{\alpha i}$ such that $f_\alpha(U_\alpha) \in W_\alpha$ for each $x_{\alpha i} \in X_{\alpha i}$ and for each $W_{\alpha i}$ be an open subset of Y_α containing $f(x_{\alpha i})$. Put $U = \prod_{i\in n} U_{\alpha i} \times \prod_{\alpha\neq\alpha i} X_\alpha$, then U is e-open in X and $f(x) \in f_\alpha(\{x_\alpha\}) \in f(U) \subset \prod_{i\in n} f_{\alpha i}(U_{\alpha i}) \times \prod_{\alpha\neq\alpha i} Y_\alpha$. Let $\{y_\alpha\} = y \in \prod_{i\in n} f_{\alpha i}(U_{\alpha i}) \times \prod_{\alpha\neq\alpha i} Y_\alpha$, then there exists a $x^*_{\alpha i} \in U_{\alpha i}$ such that $y_{\alpha i} = f_\alpha(x^*_{\alpha i})$ for every $y_{\alpha i} \in \prod_{i\in n} f_{\alpha i}(U_{\alpha i})$. Set $x^* = \{x^*_\alpha\}$, then $x^* \in \prod_{i\in n} U_{\alpha i} \times \prod_{\alpha\neq\alpha i} X_\alpha$. If $\alpha \neq \alpha i$, then there exists $y_\alpha \in Y_\alpha = f(X_\alpha)$ and $x^*_\alpha \in X_\alpha$ such that $y_\alpha = f_\alpha(x^*_\alpha)$. Thus, we have $\{y_\alpha\} = y \in \prod_{i=1}^{n} W_{\alpha i} \times \prod_{\alpha\neq\alpha i} Y_\alpha \subset f(U) \times Y \subset f(U) \subset V$.

Hence f is e-continuous. ■

Denote the topological sum $(\bigcup_{\alpha\in\bigwedge} X_\alpha, \tau)$ of $\{(X_\alpha, \tau_\alpha) : \alpha \in \bigwedge\}$ by $\bigoplus_{\alpha\in\bigwedge} X_\alpha$ and the topological sum $(\bigcup_{\alpha\in\bigwedge} Y_\alpha, \sigma)$ of $\{(Y_\alpha, \sigma_\alpha) : \alpha \in \bigwedge\}$ by $\bigoplus_{\alpha\in\bigwedge} Y_\alpha$, where

$$\tau = \{A \subset X : A \cap X_\alpha \in \tau_\alpha \text{ for every } \alpha \in \bigwedge\},$$

and

$$\sigma = \{B \subset Y : B \cap Y_\alpha \in \sigma_\alpha \text{ for every } \alpha \in \bigwedge\},$$

A function $\bigoplus_{\alpha\in\bigwedge} f_\alpha : \bigoplus_{\alpha\in\bigwedge} X_\alpha \to \bigoplus_{\alpha\in\bigwedge} Y_\alpha$, called a sum function of $\{f_\alpha : \alpha \in \Lambda\}$, is defined as follows: for every $x \in \bigcup_{\alpha\in\bigwedge} X_\alpha$,

$$(\bigoplus_{\alpha\in\bigwedge} f_\alpha)(x) = f_\beta(x) \text{ if there exists unique } \beta \in \bigwedge \text{ such that } x \in X_\beta.$$

Theorem 13. *The sum function* $\bigoplus_{\alpha\in\bigwedge} f_\alpha : \bigoplus_{\alpha\in\bigwedge} X_\alpha \to \bigoplus_{\alpha\in\bigwedge} Y_\alpha$ *is* e*-continuous if and only if* $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \sigma_\alpha)$ *is* e*-continuous for every* $\alpha \in \bigwedge$.

Proof. Denote $f = \bigoplus_{\alpha\in\bigwedge} f_\alpha$, $X = \bigoplus_{\alpha\in\bigwedge} X_\alpha$, $Y = \bigoplus_{\alpha\in\bigwedge} Y_\alpha$.

Necessity. Suppose f is e-continuous. Then $f|_{X_\alpha} = f_\alpha$ is e-continuous with Theorem 5.

Sufficiency. Let V be an open subset of Y. Then $V \cap Y_\alpha \in \sigma_\alpha$ for every $\alpha \in \bigwedge$. Let $x \in f^{-1}(V) \cap X_\alpha$, then $f(x) \in V$ and $f(x) \in Y_\alpha$. This implies that $f(x) \in f_\alpha(x)$. Thus, we have $f_\alpha(x) \in V$ and $f_\alpha(x) \in V \cap Y_\alpha$. Hence $x \in f_\alpha^{-1}(V \cap Y_\alpha)$. Conversely, $f_\alpha^{-1}(V \cap Y_\alpha) \subset f^{-1}(V) \cap X_\alpha$. Thus, we obtain $f^{-1}(V) \cap X_\alpha = f_\alpha^{-1}(V \cap Y_\alpha)$ for every $\alpha \in \bigwedge$. Since f_α is e-continuous, then $f^{-1}(V) \cap X_\alpha$ is e-open in X_α. Thus, we have $f^{-1}(V)$ is e-open in X. Hence f is e-continuous. ■

5. Separation axioms and graph properties

Definition 12. *A space X is called*

(a) *Urysohn [8] if for each pair of distinct points x and y in X, there exist open subsets U and V such that $x \in U$, $y \in V$ and $cU \cap cV = \emptyset$.*

(b) *e-T_1 if for each pair of distinct points x and y in X, there exist e-open subsets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.*

(c) *e-T_2 if for each pair of distinct points x and y in X, there exist e-open subsets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

Theorem 14. *Let $f : X \to Y$ be a e-continuous injection. Then the following hold.*

(a) *If Y is a T_1-space, then X is e-T_1.*

(b) *If Y is a T_2-space, then X is e-T_2.*

(c) *If Y is Urysohn, then X is e-T_2.*

Proof. (a) Let x and y be any distinct points in X. Since Y is a T_1-space, then there exist open subsets U and V of Y such that $f(x) \in U, f(y) \notin U$ and $f(x) \in V, f(y) \notin V$. Since f is e-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are e-open in X such that $x \in f^{-1}(U), y \notin f^{-1}(U)$ and $x \notin f^{-1}(V), y \in f^{-1}(V)$. Hence X is e-T_1.

(b) Let x and y be any distinct points in X. Since Y is a T_2-space, then there exist open subsets U and V containing $f(x)$ and $f(y)$ in Y, respectively, such that $U \cap V = \emptyset$. Since f is e-continuous, then there exist e-open subsets A and B containing x and y, respectively, such that $f(A) \subset U$ and $f(B) \subset V$. This implies that $A \cap B = \emptyset$. Hence X is e-T_2.

(c) Let x and y be any distinct points in X. Since Y is Urysohn, then there exist open subsets U and V in Y such that $f(x) \in U$, $f(y) \in V$ and $cU \cap cV = \emptyset$. Since f is e-continuous, then there exist e-open subsets A and B containing x and y, respectively, such that $f(A) \subset U \subset cU$ and $f(B) \subset V \subset cV$. This implies that $A \cap B = \emptyset$. Hence X is e-T_2. ■

Theorem 15. *Let $f, g : X \to Y$ be two functions. If f is continuous, g is e-continuous and Y is e-T_2, then $\{x \in X : f(x) = g(x)\}$ is e-closed in X.*

Proof. Denote $A = \{x \in X : f(x) = g(x)\}$. Let $x \in X - A$. Then $f(x) \neq g(x)$. Since Y is an e-T_2 space, then there exist e-open subsets U and V containing $f(x)$ and $g(x)$ in Y, respectively, such that $U \cap V = \emptyset$. Since f is continuous and g is e-continuous, then $f^{-1}(U)$ is open and $g^{-1}(V)$ is e-open in X. This implies that $x \in f^{-1}(U)$ and $x \in g^{-1}(V)$. Put $W = f^{-1}(U) \cap g^{-1}(V)$, then W is e-open in X with Proposition 2. Thus, we have $f(W) \cap g(W) \subset U \cap V = \emptyset$. This implies that $W \cap A = \emptyset$ and $x \in W \subset X - A$. Hence $X - A$ is e-open and A is e-closed in X. ■

Definition 13. *A space X is called e-regular if for each e-closed subset F and each point $x \notin F$, there exist disjoint open subsets U and V such that $x \in U$ and $F \subset V$.*

Theorem 16. *Let a function $f : X \to Y$ be a e-irresolute surjection. If X is e-regular, then Y is e-regular.*

Proof. Suppose $y \in Y$ and F is e-closed in Y such that $y \notin F$. Since f is e-irresolute surjection, then there exists a $x \in X$ such that $y = f(x)$ and $f^{-1}(F)$ is e-closed in X such that $x \notin f^{-1}(F)$. Since X is e-regular, then there exist disjoint open subsets U and V such that $x \in U$ and $f^{-1}(F) \subset V$. This implies $y = f(x) \in f(U) \subset Y - f(X - U)$. By Lemma 2, $F \subset Y - f(X - V)$. Note that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open subsets of Y. Hence Y is e-regular. ■

Definition 14. *A space X is called e-normal if for every pair of disjoint e-closed subsets A and B, there exist disjoint open subsets U and V such that $A \subset U$ and $B \subset V$.*

Theorem 17. *Let a function $f : X \to Y$ be e-irresolute. If X is e-normal, then Y is also e-normal.*

Proof. Let A and B be disjoint e-closed subsets of Y. Since f is e-irresolute, then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint e-closed subsets of X. Since X is e-normal, then there exist disjoint open subsets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By Lemma 2, $A \subset Y - f(X - U)$ and $B \subset Y - f(X - V)$. Note that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open subsets of Y. Hence Y is e-normal. ■

Lemma 4. *A space X is e-normal if and only if for each e-closed subset F and e-open subset U containing F, there exists an open set V such that $F \subset V \subset c_e V \subset U$.*

Proof. *Necessity.* Let F be a e-closed set and U be a e-open set containing F. Then we have $X-U$ is e-closed and $F\cap(X-U)=\emptyset$. Since X is an e-normal space, then there exist disjoint open subsets U_1, V_1 such that $F\subset U_1$ and $X-U\subset V_1$. This implies that $X-V_1\subset U$. Since $U_1\cap V_1=\emptyset$, then we obtain $c_eU_1\subset X-V_1$. Set $V=U_1$, then $c_eU_1\subset X-V_1\subset U$. Therefore, $F\subset V\subset c_eV\subset X-V_1\subset U$.

Sufficiency. The proof is obvious. ■

Below we give Urysohn's Lemma on e-normal spaces.

Theorem 18. *A space X is e-normal if and only if for each pair of disjoint e-closed subsets A and B of X, there exists a continuous map $f: X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$.*

Proof. *Sufficiency.* Suppose that for each pair of disjoint e-closed subsets A and B of X, there exists a continuous map $f: X\to[0,1]$ such that $f(A)=\{0\}$ and $f(B)=\{1\}$. Put $U=f^{-1}([0,1/2))$, $V=f^{-1}((1/2,1])$, then U and V are disjoint open subsets of X such that $A\subset U$ and $B\subset V$. Hence X is e-normal.

Necessity. Suppose X is e-normal. For each pair of disjoint e-closed subsets A and B of X, $A\subset X-B$, where A is e-closed in X and $X-B$ is e-open in X, by Lemma 4, there exists an open subset $U_{1/2}$ of X such that

$$A\subset U_{1/2}\subset c_eU_{1/2}\subset X-B.$$

Since $A\subset U_{1/2}$, A is e-closed in X and $U_{1/2}$ is e-open in X, then there exists an open subset $U_{1/4}$ of X such that $A\subset U_{1/4}\subset c_eU_{1/4}\subset U_{1/2}$ by Lemma 4. Since $c_eU_{1/2}\subset X-B$, $c_eU_{1/2}$ is e-closed in X and $X-B$ is e-open in X, then there exists an open subset $U_{3/4}$ of X such that $c_eU_{1/2}\subset U_{3/4}\subset c_eU_{3/4}\subset X-B$ by Lemma 4. Thus, there exist two open subsets $U_{1/2}$ and $U_{3/4}$ of X such that

$$A\subset U_{1/4}\subset c_eU_{1/4}\subset U_{1/2}\subset c_eU_{1/2}\subset U_{3/4}\subset c_eU_{3/4}\subset X-B.$$

We get a family $\{U_{m/2^n}: 1\le m<2^n, n\in N\}$ of open subsets of X, denotes $\{U_{m/2^n}: 1\le m<2^n, n\in N\}$ by $\{U_\alpha:\alpha\in\Gamma\}$. $\{U_\alpha:\alpha\in\Gamma\}$ satisfies the following condition:

(a) $A\subset U_\alpha\subset c_eU_\alpha\subset X-B$,

(b) if $\alpha<\alpha'$, then $c_eU_\alpha\subset U_{\alpha'}$.

We define $f: X\to[0,1]$ as follows:

$$f(x)=\begin{cases}\inf\{\alpha\in\Gamma: x\in U_\alpha\}, & \text{if } x\in U_\alpha \text{ for some } \alpha\in\Gamma,\\ 1, & \text{if } x\notin U_\alpha \text{ for any } \alpha\in\Gamma.\end{cases}$$

For each $x\in A$, $x\in U_\alpha$ for any $\alpha\in\Gamma$ by (1), so $f(x)=\inf\{\alpha\in\Gamma: x\in U_\alpha\}=\inf\Gamma=0$. Thus, $f(A)=\{0\}$.

For each $x \in B$, $x \notin X - B$ implies $x \notin U_\alpha$ for any $\alpha \in \Gamma$ by (1), so $f(x) = 1$. Thus, $f(B) = \{1\}$.

We have to show f is continuous.

For $x \in X$ and $\alpha \in \Gamma$, we have the following Claim:

Claim 1: if $f(x) < \alpha$, then $x \in U_\alpha$.

Suppose $f(x) < \alpha$, then $\inf\{\alpha \in \Gamma : x \in U_\alpha\} < \alpha$, so there exists $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha_1 < \alpha$. By (2), $c_eU_{\alpha_1} \subset U_\alpha$. Notice that $x \in U_{\alpha_1}$. Hence $x \in U_\alpha$.

Claim 2: if $f(x) > \alpha$, then $x \notin c_eU_\alpha$.

Suppose $f(x) > \alpha$, then there exists $\alpha_1 \in \Gamma$ such that $\alpha < \alpha_1 < f(x)$. Notice that $\alpha_1 \in \{\alpha \in \Gamma : x \in U_\alpha\}$ implies $\alpha_1 \geq \inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$. Thus, $\alpha_1 \notin \{\alpha \in \Gamma : x \in U_\alpha\}$. So $x \notin U_{\alpha_1}$. By (2), $c_eU_\alpha \subset U_{\alpha_1}$. Hence $x \notin c_eU_\alpha$.

Claim 3: if $x \notin c_eU_\alpha$, then $f(x) \geq \alpha$.

Suppose $x \notin c_eU_\alpha$, we claim that $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Otherwise, there exists $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$ such that $\alpha \geq \beta$. $x \notin c_eU_\alpha$ implies $\alpha \notin \{\alpha \in \Gamma : x \in U_\alpha\}$. So $\alpha \neq \beta$. Thus $\alpha > \beta$. By (2), $c_eU_\beta \subset U_\alpha$. So $x \notin \beta$, contridiction. Therefore $\alpha < \beta$ for any $\beta \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Hence $\alpha \leq \inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$.

For $x_0 \in X$, if $f(x_0) \in (0,1)$, suppose V is an open neighborhood of $f(x_0)$ in $[0,1]$, then there exists $\varepsilon > 0$ such that $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset V \bigcap (0,1)$. Pick $\alpha', \alpha" \in \Gamma$ such that

$$0 < f(x_0) - \epsilon < \alpha' < f(x_0) < \alpha" < f(x_0) + \epsilon < 1.$$

By Claim 1 and Claim 2, $x_0 \in U_\alpha"$, $x_0 \notin c_eU'_\alpha$. Put $U = U_\alpha" - c_eU'_\alpha$, then U is an open neighborhood of x_0 in X.

We will prove that $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$. if $y \in f(U)$, then $y = f(x)$ for some $x \in U$. $x \in U$ implies that $x \in U_\alpha"$ and $x \notin c_eU'_\alpha$. Since $x \in U_\alpha"$, then $\alpha" \in \{\alpha \in \Gamma : x \in U_\alpha\}$. Thus, $\alpha" \geq inf\{\alpha \in \Gamma : x \in U_\alpha\} = f(x)$. Notice that $\alpha" < f(x_0) + \epsilon$. Therefore $f(x) < f(x_0) + \epsilon$. Since $x \notin c_eU'_\alpha$, then $f(x) \geq \alpha'$ by Claim 3. Notice that $f(x_0) - \epsilon < \alpha'$. Therefore $f(x) > f(x_0) - \epsilon$. Hence, $f(U) \subset (f(x_0) - \epsilon, f(x_0) + \epsilon)$.

Therefore, $f(U) \subset V$. This implies f is continuous at x_0. If $f(x_0) = 0$, or 1, the proof that f is continuous at x_0 is similar. ■

Theorem 19. *Let $f : X \to Y$ be a function and $G : X \to X \times Y$ be the graph function of f, defined by $G(x) = (x, f(x))$ for each $x \in X$. Then f is e-continuous if and only if G is e-continuous.*

Proof. *Necessity.* Let $x \in X$ and V be an open subset in $X \times Y$ containing $G(x)$. Then there exist open subsets $U_1 \subset X$ and $W \subset Y$

such that $G(x) = (x, f(x)) \subset U_1 \times W \subset V$. Since f is e-continuous, then there exists a $U_2 \in EO(X)$ such that $f(U_2) \subset W$. Set $U = U_1 \cap U_2$, then $U \in EO(X)$ with Proposition 2. Thus, we have $G(U) \subset V$. Hence G is e-continuous.

Sufficiency. Let $x \in X$ and V be an open subset of Y containing $f(x)$. Then $X \times V$ is an open subset containing $G(x)$. Since G is e-continuous, then there exists $U \in EO(X)$ such that $G(U) \subset X \times V$. Thus, we have $f(U) \subset V$. Hence f is e-continuous. ■

Definition 15. *A graph $G(f)$ of a function $f : X \to Y$ is called strongly e-closed if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exists a $U \in EO(X)$ containing x and an open subset V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Theorem 20. *Let $f : X \to Y$ be e-continuous and Y be e-T_2. Then $G(f)$ is e-strongly closed.*

Proof. Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $f(x) \neq y$. Since Y is e-T_2, then there exist disjoint e-open subsets V and W of Y such that $f(x) \in V$ and $y \in W$. Since f is e-continuous, then there exists a $U \in EO(X)$ such that $f(U) \subset V$. Thus, we have $f(U) \cap (W) = \emptyset$. Hence $(U \times W) \cap G(f) = \emptyset$ and $G(f)$ is strongly e-closed. ■

Theorem 21. *Let $f : X \to Y$ be a e-continuous and injective. If $G(f)$ is strongly e-closed, then X is e-T_2.*

Proof. Let x, $y \in X$ such that $x \neq y$. Since f is injective, then $f(x) \neq f(y)$ and $(x, f(y)) \notin G(f)$. Since $G(f)$ is strongly e-closed, there exists a $U \in EO(X)$ and an open subset W of Y such that $(x, f(y)) \in U \times W$ and $(U \times W) \cap G(f) = \emptyset$. Thus, we have $f(U) \cap W = \emptyset$. Since f is e-continuous, then there exists a $y \in V \in EO(X)$ such that $f(V) \subset W$. This implies that $f(U) \cap f(V) = \emptyset$. Hence $U \cap V = \emptyset$ and X is e-T_2. ■

6. e-connectedness and covering properties

Definition 16. *A space X is called e-connected if X is not the union of two disjoint nonempty e-open subsets.*

Theorem 22. *Let $f : X \to Y$ be e-continuous. If X is e-connected, then Y is connected.*

Proof. Suppose Y is not a connected space. Then there exist nonempty disjoint open subsets A and B such that $Y = A \cup B$. Since f is e-continuous, then $f^{-1}(A)$ and $f^{-1}(B)$ are e-open subsets of X. Thus, we obtain $f^{-1}(A)$

and $f^{-1}(B)$ are nonempty disjoint subsets and $X = f^{-1}(A) \cup f^{-1}(B)$. This is contrary to the hypothesis that X is a e-connected space. Hence Y is connected. ■

Corollary 1. *Let $f : X \to Y$ be e-irresolute. If X is e-connected, then Y is e-connected.*

Definition 17. *A space X is called e-Lindelöf (resp. e-compact) if every e-open cover of X has a countable (resp. finite) subcover.*

Theorem 23. *Let $f : X \to Y$ be e-continuous. If X is e-Lindelöf, then Y is Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \bigwedge\}$ is an open cover of Y. Since f is an e-continuous function, then $f^{-1}(\{U_\alpha : \alpha \in \bigwedge\})$ is an e-open cover of X. Since X is e-Lindelöf, then there exists a countable subcover $f^{-1}(\{U_{\alpha i} : U_{\alpha i} \in \{U_\alpha\}, 1 < i < \infty, \alpha \in \bigwedge\})$ in X. Thus, we have $\{U_{\alpha i} : U_{\alpha i} \in \{U_\alpha\}, 1 < i < \infty, \alpha \in \bigwedge\}$ is a countable subcover of Y. Hence Y is Lindelöf. ■

Similarly, we can prove the following Theorem 24.

Theorem 24. *Let $f : X \to Y$ be e-continuous. If X is e-compact, then Y is compact.*

Acknowledgement. This paper is supported by the Innovation Project of Guangxi University for Nationalities (No. gxun-chx2011081).

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Received on 07.11.2011 and, in revised form, on 23.03.2012.