

J. O. EHIGIE, S. A. OKUNUGA AND A. B. SOFOLUWE

**A CLASS OF EXPONENTIALLY FITTED  
SECOND DERIVATIVE EXTENDED BACKWARD  
DIFFERENTIATION FORMULA FOR SOLVING  
STIFF PROBLEMS**

**ABSTRACT.** An exponentially fitted second derivative extended backward differentiation formula (SDEBDF) is derived from the class of composite, multiderivative linear multistep method with a free parameter to allow for the exponential fitting. Some numerical properties such as stability of the methods are investigated as a pair of predictor-corrector (P-C) technique based on a proposed algorithm, to which the local error estimates are also obtained. The efficiency of the new method tested on some standard problems shows that the method compares favourably with existing methods and with high accuracy.

**KEY WORDS:** exponential fitted, second derivative, backward differentiation formula, stability.

*AMS Mathematics Subject Classification:* 65L04, 65L05, 65L06.

## 1. Introduction

Science and Engineering problems are often modelled as initial value problems involving systems of Ordinary Differential Equations (ODEs) and many of these problems appear to be stiff. There are different definitions given to stiffness with respect to systems of first order ODEs,

$$(1) \quad y' = Ay + \phi(x), \quad y(a) = \eta, \quad a \leq x \leq b$$

where  $y = (y_1, y_2, \dots, y_s)$  and  $\eta = (\eta_1, \eta_2, \dots, \eta_s)$ .

**Definition 1** (Lambert [16]). *The linear system (1) is said to be stiff if*

- (i)  $Re(\lambda_i) < 0, i = 1, 2, \dots, s$
- (ii)  $Max |Re(\lambda_i)| \gg Min |Re(\lambda_i)|$

where  $\lambda_i$  are the eigenvalues of  $s \times s$  matrix  $A$ , and the stiff ratio is  $\frac{Max |Re(\lambda_i)|}{Min |Re(\lambda_i)|}$ .

Backward Differentiation Formulas (BDF) were the first numerical methods to be proposed for stiff initial value problems (IVPs) as discussed in pioneering work of Curtiss and Hirschfelder [9] and Gear's [12] book. For many years, these methods have been the most prominent and most widely used for stiff computation. More recently, Cash [8] introduced a class of methods known as the Extended Backward Differentiation Formula (EBDF) which are known to be advantageous over the usual BDF.

The concept of exponential fitting was originally proposed by Liniger and Willoughby [17]. The method is derived such that a LMM allows for free parameters which are chosen to fit some given exponential function that satisfies the integration formula exactly. Exponential fitting are numerical methods which are very robust for the integration of differential equations whose Jacobian has large imaginary eigenvalues. Cash [7].

Second derivative methods have been derived by Enright [10] and Cash [8] where he introduced a multistep formula containing the second derivatives, and was later implemented as a computer code for the integration of stiff system.

Jackson and Kenue [14] developed a fourth order method with exponential fitting based on a linear 2-step method and the method is  $A(\alpha)$ -stable for  $\alpha$  very near  $\frac{\pi}{2}$ . Cash [7] used the Jackson and Kenue [14] method as a predictor to a second order derivative method which he found to be A-Stable. Okunuga [19] derived a fourth order composite LMM in the spirit of Cash, and obtained a very high accuracy compared to methods by Jackson et al. and Cash.

In this paper, a class of exponentially fitted second derivative extended backward differentiation formula shall be derived. This method is developed as an hybrid of extended backward differentiation formula of Cash [8], the second derivative multistep methods of Enright [10] and the exponential fitting of Okunuga [18]. Hence, the new method shall contain an extra superfuture point and a second order derivative term to the usual BDF. The method shall be derived using the Taylors series approach conforming to the general composite LMM.

## 2. Theoretical procedure

There are various techniques which can be used to derive this type of methods. However in this paper, the procedure for the derivation of the new method shall be the Taylors series approach.

Consider the Initial Value Problem (IVP),

$$(2) \quad y' = f(x, y), \quad y(x_0) = y_0.$$

The general form of a composite, multiderivative linear multistep method is given by,

$$(3) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^{k+1} \beta_j f_{n+j} + h^2 \sum_{j=0}^k \delta_j g_{n+j}$$

where  $y_{n+j} \approx y(x_n + jh)$ ,  $f_{n+j} \equiv f(x_n + jh, y(x_n + jh))$  and

$$g_{n+j} \equiv \left. \frac{df(x, y(x))}{dx} \right|_{x=x_{n+j}, y=y_{n+j}}$$

$x_n$  is a discrete point at node point  $n$ . where  $\alpha_j$ ,  $\beta_j$  and  $\delta_j$  are parameters to be determined and usually  $\beta_{k+1} \neq 0$  to preserve the composite nature of the formula.

From the general class (3), we proposed a second derivative EBDP as,

$$(4) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h\beta_k f_{n+k} + h\beta_{k+1} f_{n+k+1} + h^2 \delta_j g_{n+k}$$

that is,  $\beta_{n+j} = 0$ ,  $0 \leq j \leq k-1$  and  $\delta_{n+j} = 0$ ,  $0 \leq j \leq k-1$  in (3) and this class of method shall be associated with a predictor formula,

$$(5) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h\beta_k f_{n+k} + h^2 \delta_j g_{n+k}$$

that is,  $\beta_{n+j} = 0$ ,  $0 \leq j \leq k-1$ ,  $\beta_{n+k+1} = 0$  and  $\delta_{n+j} = 0$ ,  $0 \leq j \leq k-1$  in (3). The two formulas shall be as a predictor-corrector pair assuming that the back values are provided i.e  $y_{n+j}$ ,  $0 \leq j \leq k-1$ . We illustrate the implementation of the schemes with a proposed algorithm in the following stages:

1. Compute  $\bar{y}_{n+k}$  as a solution of the predictor (5)

$$\bar{y}_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f_{n+k} + h^2 \delta_j g_{n+k}.$$

2. Compute  $\bar{y}_{n+k+1}$  as a solution of the predictor (5) after one step,

$$\bar{y}_{n+k+1} + \bar{y}_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f_{n+k+1} + h^2 \delta_j g_{n+k+1}.$$

3. Compute  $\bar{f}_{n+k} \equiv f(x_{n+k}, \bar{y}_{n+k})$ ,  $\bar{g}_{n+k} \equiv g(x_{n+k}, \bar{y}_{n+k})$  and  $\bar{f}_{n+k+1} \equiv f(x_{n+k+1}, \bar{y}_{n+k+1})$ ,  $\bar{g}_{n+k+1} \equiv g(x_{n+k+1}, \bar{y}_{n+k+1})$ .

4. Compute  $y_{n+k}$  as a solution of the corrector (4),

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f_{n+k} + h\beta_{k+1} \bar{f}_{n+k+1} + h^2 \delta_j g_{n+k}.$$

The process above helps in the practical implementation of the exponentially fitted extended second derivative BDFs on IVPs. In deriving the new method we shall allow free parameters  $a$  and  $b$  for the predictor and corrector respectively so as to fit the method with parameter  $a$  and  $b$  to exponential functions. The predictor used shall be a method of order  $k$  while the corrector is a method of  $k + 1$ .

### 3. Derivation of the method

The predictor formula for  $k = 2$  will be of the form,

$$(6) \quad \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h\beta_2 f_{n+2} + h^2 \delta_2 g_{n+2}$$

and the corresponding corrector formula will be of the form,

$$(7) \quad \alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h\beta_2 f_{n+2} + h\beta_3 f_{n+3} + h^2 \delta_2 g_{n+2}$$

using the Taylors series approach, setting  $\alpha_k = 1$  and allowing a free parameter  $a$  and  $b$  for the predictor and corrector formula respectively, we obtain a predictor formula of as,

$$(8) \quad y_{n+2} + (a - 2) y_{n+1} + (1 - a) y_n = h a f_{n+2} + h^2 \left(1 - \frac{3}{2} a\right) g_{n+2}$$

and the corresponding corrector formula as,

$$(9) \quad y_{n+2} + \left(-\frac{8}{7} + \frac{3}{7} b\right) y_{n+1} + \left(\frac{1}{7} - \frac{3}{7} b\right) y_n \\ = h \left(\frac{6}{7} - \frac{4}{7} b\right) f_{n+2} + h b f_{n+3} + h^2 \left(-\frac{2}{7} - \frac{23}{14} b\right) g_{n+2}.$$

Applying the test IVP to the predictor formula (8) that is,

$$(10) \quad y' = \lambda y, \quad q = \lambda h$$

with an exact solution as  $y(x) = e^{\lambda x}$

$$(11) \quad y_{n+2} = -(-2 + a) y_{n+1} - (1 - a) y_n + h a \lambda y_{n+2} + h^2 \left(1 - \frac{3}{2} a\right) \lambda^2 y_{n+2}$$

and substituting,

$$\frac{y_{n+1}}{y_n} = e^q, \quad \frac{y_{n+2}}{y_n} = e^{2q}$$

dividing (11) through by  $y_n$ , we obtain,

$$(12) \quad e^{2q} = -(-2 + a)e^q - (1 - a) + qae^{2q} + q^2 \left(1 - \frac{3}{2}a\right) e^{2q}$$

simplifying for  $a$  in terms of  $q$  to obtain,

$$(13) \quad a = \frac{e^{2q}(1 - q^2) - 2e^q + 1}{1 - e^q + qe^{2q} - \frac{3}{2}q^2e^{2q}}.$$

Therefore the predictor formula of the exponentially fitted extended second derivative BDF (6) shall have coefficients,

$$\alpha_0 = \frac{e^{2q}(2 - 2q + q^2) - 2e^q}{(3q^2 - 2q)e^{2q} + 2e^q - 2}$$

$$\alpha_1 = \frac{2 - 2e^{2q}(1 - 2q + 2q^2)}{(3q^2 - 2q)e^{2q} + 2e^q - 2}$$

$$\beta_2 = \frac{e^{2q}(1 - q^2) - 2e^q + 1}{1 - e^q + qe^{2q} - \frac{3}{2}q^2e^{2q}}$$

$$\delta_2 = \frac{e^{2q}(2q - 3) + 4e^q - 1}{(3q^2 - 2q)e^{2q} + 2e^q - 2}.$$

Again applying the test problem (10) on the corrector formula (9) yields,

$$\begin{aligned} y_{n+2} + \left(-\frac{8}{7} + \frac{3}{7}b\right) y_{n+1} + \left(\frac{1}{7} - \frac{3}{7}b\right) y_n \\ = h \left(\frac{6}{7} - \frac{4}{7}b\right) \lambda y_{n+2} + hb\lambda y_{n+3} + h^2 \left(-\frac{2}{7} - \frac{23}{14}b\right) \lambda^2 y_{n+2} \end{aligned}$$

substituting  $q = \lambda h$  yields,

$$(14) \quad \begin{aligned} y_{n+2} + \left(-\frac{8}{7} + \frac{3}{7}b\right) y_{n+1} + \left(\frac{1}{7} - \frac{3}{7}b\right) y_n \\ = \left(\frac{6}{7} - \frac{4}{7}b\right) qy_{n+2} + bqy_{n+3} + \left(-\frac{2}{7} - \frac{23}{14}b\right) q^2 y_{n+2} \end{aligned}$$

simplifying and obtaining  $b$  in terms of  $q$ , we obtain,

$$(15) \quad b = \frac{e^{2q}(14 - 12q + 4q^2) - 16e^q + 2}{6 - 6e^q - e^{2q}(8q + 23q^2) + 14qe^{3q}}.$$

This implies that the exponentially fitted 2-step extended second derivative BDF of the form (7) has coefficients,

$$\begin{aligned}\alpha_1 &= \frac{-2(e^{2q}(2q + 14q^2 + 3) - 8qe^{3q} - 3)}{6e^q + (8q + 23q^2)e^{2q} - 6 + 14qe^{3q}} \\ \alpha_0 &= \frac{e^{2q}(6 - 4q + 5q^2) - 2qe^{3q} - 6e^q}{6e^q + (8q + 23q^2)e^{2q} - 6 + 14qe^{3q}} \\ \beta_2 &= \frac{2(e^{2q}(11q^2 + 4) - 6qe^{3q} - 2e^q - 2)}{6e^q + (8q + 23q^2)e^{2q} - 6 + 14qe^{3q}} \\ \beta_3 &= \frac{e^{2q}(14 - 12q + 4q^2) - 16e^q + 2}{6 - 6e^q - e^{2q}(8q + 23q^2) + 14qe^{3q}} \\ \delta_2 &= -\frac{(22q - 23)e^{2q} + 28e^q - 4qe^{3q} - 5}{6 - 6e^q - e^{2q}(8q + 23q^2) + 14qe^{3q}}.\end{aligned}$$

#### 4. Stability analysis of the method

Stability of a LMM determines the manner in which the error is propagated as the numerical computation proceeds. Lambert [15, 16]. Hence it would be necessary to investigate the stability criteria of the methods (8) and (9). Since these methods involve free parameters  $a$  and  $b$  for the method to satisfy some A-stability conditions, therefore the determination of the range of values for the free parameter  $a$  and  $b$  in the open left plane  $(-\infty, 0]$  will be required.

The stability functions  $\bar{R}(q)$  and  $R(q)$  are functions derived from  $a$  and  $b$  respectively which reveal some conditions the free parameters must satisfy before the range of values for the fitting condition is obtained. This implies that, the condition where  $a$  and  $b$  satisfy such that,

$$(16) \quad |R(q)| = \left| \frac{y_{n+2}}{y_n} \right| < 1, \quad \forall q, \quad \text{with } Re(q) < 0$$

is sought for.

To obtain the stability function  $R(q)$ , applying (8) to the test problem (10), substituting  $q = \lambda h$  and dividing both sides by  $y_n$ , we have,

$$\frac{\bar{y}_{n+2}}{y_n} + (-2 + a) \frac{y_{n+1}}{y_n} + (1 - a) = aq \frac{\bar{y}_{n+2}}{y_n} + \left(1 - \frac{3}{2}a\right) q^2 \frac{\bar{y}_{n+2}}{y_n}.$$

Using the substitution,

$$\left(\frac{\bar{y}_{n+2}}{y_n}\right)^{\frac{1}{2}} = \frac{y_{n+1}}{y_n}$$

we obtain an equation of the form,

$$(17) \quad A \left( \frac{\bar{y}_{n+2}}{y_n} \right) + B \left( \frac{\bar{y}_{n+2}}{y_n} \right)^{\frac{1}{2}} + C = 0$$

with,

$$A = 1 - aq - \left( 1 - \frac{3}{2}a \right) q^2, \quad B = a - 2, \quad C = -(a - 1).$$

Solving (17) for  $\left( \frac{\bar{y}_{n+2}}{y_n} \right)$ , we obtain two solutions,

$$\left( \frac{\bar{y}_{n+2}}{y_n} \right) = -\frac{1}{2} \left( \frac{-B^2 + B\sqrt{b^2 - 4AC} + 2AC}{A^2} \right) = \bar{R}_1(q)$$

or

$$\left( \frac{\bar{y}_{n+2}}{y_n} \right) = -\frac{1}{2} \left( \frac{-B^2 - B\sqrt{b^2 - 4AC} + 2AC}{A^2} \right) = \bar{R}_2(q).$$

To actually obtain the stability function  $R(q)$  which is the stability function for the corrector, we apply the corrector formula (9) to the test problem (10) and dividing through by  $y_n$ . The substitution,

$$\left( \frac{\bar{y}_{n+3}}{y_n} \right) = \left( \frac{\bar{y}_{n+2}}{y_n} \right)^{\frac{3}{2}} = \bar{R}^{\frac{3}{2}}(q)$$

$$\left( \frac{y_{n+1}}{y_n} \right) = \left( \frac{\bar{y}_{n+2}}{y_n} \right)^{\frac{1}{2}} = \bar{R}^{\frac{1}{2}}(q)$$

reduces (9) to,

$$\frac{y_{n+2}}{y_n} = \frac{bq\bar{R}^{\frac{3}{2}}(q) - \left(-\frac{8}{7} + \frac{3}{7}b\right)\bar{R}^{\frac{1}{2}}(q) - \left(\frac{1}{7} - \frac{3}{7}b\right)}{1 - q\left(\frac{6}{7} - \frac{4}{7}b\right) + q^2\left(\frac{2}{7} + \frac{23}{14}b\right)} = R(q),$$

$R(q)$  here is known as the stability function for the SDEBDF, it unites the predictor formula (5) and the corrector formula (4).

By application of maximum modulus theorem, we establish that the necessary and sufficient conditions for the inequality (16) to hold are given by:

- (i)  $|R(q)| \leq 1$  on  $Re(q)=0$
- (ii)  $R(q)$  analytic in  $Re(q) < 0$

If condition (i) holds it means that  $R(q)$  is analytic in  $q = -\infty$ , therefore by maximum modulus theorem (i) and (ii) will guarantee A-stability. To examine this, we consider  $|R(q)| \leq 1$  for the corrector. this implies that,

$$-1 \leq R(q) \leq 1$$

$$(18) \quad \frac{bq\overline{R}^{\frac{3}{2}}(q) - \left(-\frac{8}{7} + \frac{3}{7}b\right)\overline{R}^{\frac{1}{2}}(q) + \left(\frac{1}{7} - \frac{3}{7}b\right)}{1 - q\left(\frac{6}{7} - \frac{4}{7}b\right) + q^2\left(\frac{2}{7} + \frac{23}{14}b\right)} - 1 \leq 0$$

taking limit as  $q \rightarrow -\infty$ , we have that  $b \geq -\frac{4}{23}$ . Repeating the same process for  $\overline{R}(q)$ , we obtain that  $a \geq \frac{2}{3}$ .

Also showing analytically that  $a$  and  $b$  have finite limits for both  $q = 0$  and  $q = -\infty$ , and at  $q = 0$  we have  $a = \frac{6}{7}$ ,  $b = 0$  while at  $q = -\infty$  we have  $a = 1$ ,  $b = \frac{1}{3}$ . This means that the SDEBDF will be A-stable in the interval  $b \in \left[\frac{8}{73}, \frac{1}{3}\right]$  and  $a \in \left[\frac{6}{7}, 1\right]$ .

The values of some sample of  $q$  is given in the Table 1, this is to show the relationship between the numerical stability of the the method within the range  $a$  and  $b$ . It is however observed that as  $q$  is decreasing  $a$  and  $b$  are monotonically increasing for the samples  $q$ . This suggest that all values  $a$  and  $b$  within this range are convergent and bounded in the range as obtained by the maximum modulus theorem as  $q \rightarrow 0$  and  $q \rightarrow -\infty$ .

**Table 1.** Parameter values  $a$  and  $b$  as a function of  $q$  associated with the SDEBDF

$q$	$a$	$b$
-1.0	0.89945	0.16327
-2.0	0.93907	0.22103
-5.0	0.99405	0.32014
-10.0	0.99995	0.33323
-20.0	0.99999	0.33333
-50.0	1.00000	0.33333
-100.0	1.00000	0.33333

**Lemma 1** (Cash [8]). *If equation (4) is of order  $k + 1$  and the BDF for predictor are of order  $k$ , then the whole Predictor-Corrector method (5) - (4) is of order  $k + 1$ .*

**Theorem 1.** *The 2-step SDEBDF is of order 3.*

**Proof.** The theorem is proved based on the lemma of Cash [8]. The proof of this theorem shall involve the Predictor-Corrector Pair (8) and (9). Suppose the exact solution of  $y_n$  and  $y_{n+1}$  are given, then by Taylors expansion of  $y(x_{n+2}) - \overline{y}_{n+2}$ , it is easily derived that,

$$(19) \quad y(x_{n+2}) - \overline{y}_{n+2} = h^3 \left( \frac{5}{3} + 2a \right) y'''(x_n) + O(h^4)$$

this implies that the predictor is of order 2 whenever  $a \neq -\frac{5}{6}$ . Following the proposed algorithm, step 2 implies,

$$(20) \quad \overline{y}_{n+3} = (a - 1)y_{n+1} + (2 - a)y_{n+2} + haf_{n+3} + h^2 \left( 1 - \frac{3}{2}a \right) g_{n+3}.$$



Also expanding by Taylor series and simplifying, we obtain,

$$(21) \quad y_{n+3} - \bar{y}_{n+3} = h^3 \left( -1 + \frac{7}{6}a \right) y'''(x_n) + O(h^4)$$

which shows that it also of order 2 which is dependent on  $(y(x_{n+2}) - \bar{y}_{n+2})$ . On approaching stage four(4) of the algorithm which replaces  $\bar{y}_{n+3}$  in  $\bar{f}_{n+3}$  as  $f(x_{n+3}, \bar{y}_{n+3})$ . The corrector (9) becomes,

$$(22) \quad \begin{aligned} y_{n+2} + \left( -\frac{8}{7} + \frac{3}{7}b \right) y_{n+1} + \left( \frac{1}{7} - \frac{3}{7}b \right) y_n \\ = h \left( \left( \frac{6}{7} - \frac{4}{7}b \right) f_{n+2} + b f(x_{n+3}, \bar{y}_{n+3}) \right) \\ + h^2 \left( \left( -\frac{2}{7} - \frac{23}{14}b \right) \right) g_{n+2}. \end{aligned}$$

But,

$$(23) \quad \begin{aligned} \bar{y}_{n+3} = y(x_n) + 3hy'(x_n) + \frac{(3h)^2}{2!}y''(x_n) \\ + \left( -\frac{7}{6}a + \frac{11}{2} \right) h^3y'''(x_n) + o(h^4) \end{aligned}$$

simplifying using some elementary algebra, we have,

$$(24) \quad \begin{aligned} y_{n+2} = y(x_n) + 2hy'(x_n) + \frac{(2h)^2}{2!}y''(x_n) + \frac{(2h)^3}{3!}y'''(x_n) \\ + \left( \frac{13}{21} + \frac{241}{168}b - \frac{7}{6}ab \right) h^4y^{(iv)}(x_n) \end{aligned}$$

hence the method (8)-(9) P-C pair is of order 3 depending on the choice of  $a$  and  $b$ . ■

**Theorem 2.** *The 2-step SDEBDF has a Local Truncation Error (L.T.E) of  $(\frac{1}{21} - \frac{241}{168}b + \frac{7}{6}ab) h^4y^{(iv)}(x_n)$*

**Proof.** The L.T.E is defined as,

$$L.T.E = y(x_{n+2}) - y_{n+2}$$

hence, expanding  $y(x_{n+2})$  with Taylors series and subtracting (24) from it, the result follows. ■

### 5. Numerical experiment

In this paper we apply the 2-step SDEBDF to some standard problem in the literatures [1, 4, 7, 11, 14]. The implementation of the schemes were carried out on a digital computer.

**Problem 1.** We consider the linear problem considered by Jackson et al. [14] and Cash [7].

$$y' = -y + 95z, \quad y(0) = 1, \quad z' = -y - 97z, \quad z(0) = 1, \quad x \in [0, 1]$$

The eigenvalues of the Jacobian matrix at  $x = 0$  are  $\lambda_1 = -2$  and  $\lambda_2 = -96$ . The analytical solution of Problem 1 is given as:

$$y = \frac{1}{47} (95e^{-2x} - 48e^{-96x}), \quad z = \frac{1}{47} (48e^{-96x} - e^{-2x}).$$

**Problem 2.** The Second problem considered is a problem considered from Enright and Pryce [11],

$$\begin{aligned} y_1' &= -10^4 y_1 + 100y_2 - 10y_3 + y_4; & y_1(0) &= 1 \\ y_2' &= -1000y_2 + 10y_3 - 10y_4; & y_2(0) &= 1 \\ y_3' &= -y_3 + 10y_4; & y_3(0) &= 1 \\ y_4' &= -0.1y_3; & y_4(0) &= 1. \end{aligned}$$

This problem is solved within the range  $0 \leq x \leq 20$ . The eigenvalues of the Jacobian matrix are  $\lambda_1 = -0.1$ ,  $\lambda_2 = -1.0$ ,  $\lambda_3 = -1000.0$  and  $\lambda_4 = -10000.0$ . The analytical solution of Problem 2 is given as:

$$\begin{aligned} y_1(x) &= -\frac{89990090}{8999010009}e^{-0.1x} + \frac{818090}{89901009}e^{-x} \\ &\quad + \frac{9989911}{899010090}e^{-1000x} + \frac{89071119179}{89990100090}e^{-10000x} \\ y_2(x) &= \frac{9100}{89991}e^{-0.1x} - \frac{910}{8991}e^{-x} + \frac{9989911}{9989001}e^{-1000x} \\ y_3(x) &= \frac{100}{9}e^{-0.1x} - \frac{91}{9}e^{-x} \\ y_4(x) &= e^{-0.1x}. \end{aligned}$$

**Problem 3.** A mildly stiff linear problem considered by Akinfenwa, Jator and Yao [4] given by,

$$(25) \quad \begin{aligned} y_1' &= 998y_1 + 1998y_2, & y_1(0) &= 1 \\ y_2' &= -999y_1 - 1999y_2, & y_2(0) &= 1 \end{aligned}$$

is also considered on the interval  $0 < x < 10$  having an analytical solution given by the sum of two decaying components.

$$\begin{aligned} y_1(x) &= 4e^{-x} - 3e^{-1000x} \\ y_2(x) &= -2e^{-x} + 3e^{-1000x}. \end{aligned}$$

With an eigenvalue of 1 and 1000, the problem has a stiffness ratio of 1 : 1000. Solving this problem with SDEBDF, we compare the absolute errors on the interval at  $x = 10$  with the Block Backward Differentiation Formula of Order 8 of Akinfenwa et al. [4] (BBDF8) using a step length  $h = 0.1$ .

**Problem 4.** A second order ordinary differential equation given by,

$$(26) \quad y'' + 1001y' + 1000y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

transformed to a system of first order equation as,

$$(27) \quad \begin{aligned} y' &= z, & y(0) &= 1 \\ z' &= -1000y - 1001z, & z(0) &= 0 \end{aligned}$$

is also considered. The stiff system has eigenvalues  $\lambda_1 = -1$  and  $\lambda_2 = 1000$ . For the purpose of comparison, we solve the problem on the interval  $0 < x < 1$ . Numerical results is compared with methods of Abhulimen and Okunuga [2], Okunuga [18] and Abhulimen [1] denoted as AB-OK, OK, and AB5 respectively and presented in Table 5.

The result obtained for Problem 1 using stepsizes of 0.0625 and 0.03125 are given in Table 1. The method is implemented with these stepsizes to be able to compare the newly developed method with the existing methods. In this paper, we denote J-K, Cash4, Cash5 and ABOT due to the method of Jackson and Kenue, Cash of Order 4 and Cash of Order 5 and the method of Abhulimen et al. [3] respectively. While SDEBDF denotes the newly derived method.

**Table 2.** Result of the Problem 1.

h	Method	$y(1)( error )$	$z(1) \times 10^2( error  \times 10^2)$
0.0625	J-K	0.2735503( $3 \times 10^{-7}$ )	-0.2879477( $4 \times 10^{-7}$ )
	Cash4	0.2735498( $3 \times 10^{-7}$ )	-0.2879471( $3 \times 10^{-7}$ )
	Cash5	0.27355005( $1 \times 10^{-8}$ )	-0.28794742( $1 \times 10^{-8}$ )
	ABOT	0.27354656( $3.5 \times 10^{-6}$ )	-0.2879474( $1.1 \times 10^{-8}$ )
	SDEBDF	0.27355004( $3.4 \times 10^{-9}$ )	-0.28794741( $3.6 \times 10^{-9}$ )
Analytical Solution		0.2735500405	-0.2879474111 $\times 10^2$
0.03125	J-K	0.2735505( $1 \times 10^{-8}$ )	-0.28794742( $1 \times 10^{-8}$ )
	Cash4	0.27354004( $1 \times 10^{-8}$ )	-0.28794740( $1 \times 10^{-8}$ )
	ABOT	0.27355046( $4.2 \times 10^{-7}$ )	-0.28796700( $1.9 \times 10^{-5}$ )
	SDEBDF	0.2735500371( $3.4 \times 10^{-9}$ )	-0.2879474081( $3.5 \times 10^{-9}$ )

The result in the Table 2 above shows the superiority of the newly derived method, SDEBDF with the existing methods and it shows the method is efficient for the integration of stiff problems.

The results obtained from the solution of Problem 2 is solved with the new SDEBDF using stepsizes 0.05 and 0.1 and numerical results are also compared with . The result is shown in the Table 2 below.

**Table 3.** Result of the Problem 2.

h	Method	$y_1(20)( error )$	$y_2(20)( error )$
0.05	SDEBDF	$(-0.00135335)5.31 \times 10^{-12}$	$(0.01368527)7.27 \times 10^{-11}$
0.1	SDEBDF	$(-0.00135335)2.25 \times 10^{-10}$	$(0.01368527)2.29 \times 10^{-9}$
Analytical Solution		-0.00135335	0.01368527

  

h	Method	$y_3(20)( error )$	$y_4(20)( error )$
0.05	SDEBDF	$(1.50372535)5.90 \times 10^{-9}$	$(0.13533528)1.34 \times 10^{-9}$
0.1	SDEBDF	$(1.50372560)2.50 \times 10^{-7}$	$(0.13533530)2.06 \times 10^{-8}$
Analytical Solution		1.50372535	0.13533528

The numerical results presented in Table 3 shows that the error tolerance could be raise to  $10^{-7}$  as against  $10^{-4}$  prescribed in Enright and Pryce [11]. It is observed that the results for the numerical solution for  $y_1(20)$  for both stepsizes was the most accurate of all other dependent variables.

Numerical results for problem 3 was compared with the Block Backward Differentiation formula of order 8 as derived in Akinfenwa et al. [4] denoted by BBDF8 in Table 4.

**Table 4.** Result of the Problem 3.

Method	$x$	$ y_{1,100} - y_1(1) $
BBDF8	10	$4.18 \times 10^{-13}$
SDEBDF	10	$8.92 \times 10^{-18}$

Our results also generates a more accurate result than the block methods of Akinfenwa [4].

**Table 5.** Result of the Problem 4.

Method	$x$	$ y_{10} - y(1) $
AB-OK	1	$5.29 \times 10^{-9}$
OK	1	$5.26 \times 10^{-8}$
AB5	1	$1.8 \times 10^{-7}$
SEDBDF	1	$1.83 \times 10^{-15}$

Clearly from Table 4, it is easily observed that our method yields a more accurate result than the methods derived in Abhulimen and Okunuga [2], Okunuga [18], and Abhulimen [1].

## 6. Conclusion

An A-stable exponentially fitted second derivative extended backward differentiation method has been derived for the integration of Stiff problems, the method is found to be A-stable within some range of free parameters. The method is found to be highly efficient on some standard problems which shows its superiority over some existing methods.

## References

- [1] ABHULIMEN C.E., An exponentially fitted Predictor-Corrector formula for stiff system of Ordinary Differential Equations, *Intenational Journal of Computational and Applied Mathematics*, 4(2)(2009), 115-126.

- [2] ABHULIMEN C.E., OKUNUGA, Exponentially fitted second derivative multi-step methods for stiff initial value problems in ordinary differential equations, *International Journal of Computational and Applied Mathematics*, 4(2)(2008), 115-126.
- [3] ABHULIMEN C.E., OTUNTA F.O., A new class of exponential fitting for initial value problems in ordinary differential equations, *J. of Nig. Math. Soc.*, 28(2009), 13-28, 982-983.
- [4] AKINFENWA O., JATOR S., YOA N., Eight order backward differentiation formula with continuous coefficients for stiff ordinary differential equations, *World Academy of Science, Engineering and Technology*, 74(2011), 848-853.
- [5] CASH J.R., On the integration of stiff systems of ordinary differential equations using extended backward differentiation formula, *Numer. Math.*, 3(1980), 235-246.
- [6] CASH J.R., On the exponential fitting of composite, multiderivate linear multistep methods, *SIAM J. Numer. Anal.*, 18(5)(1981), 808-821.
- [7] CASH J.R., Second Derivative Extended Backward Differentiation Formulas for the Numerical Solution of stiff Systems, *SIAM J. Numer. Anal.*, 18(1981), 21 - 36.
- [8] CURTISS C.F., HIRSCHFELDER J.O., Integration of stiff equations, *Proc. Nat. Acad. Sci.*, 38(1952), 235 - 243.
- [9] ENRIGHT W.H., Second derivative multistep methods for stiff ordinary differential equations, *SIAM Journal of Numerical Analysis*, 11(2)(1974), 321-331.
- [10] ENRIGHT W.H., PRYCE J.D., *Two Fortran Packages for Assessing IV methods*, Technical Report 83(16), Department of Computer Science, University of Toronto, Canada, 1983.
- [11] GEAR C.W., *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, New Jersey, 1971.
- [12] HAIRER E., WANNER G., *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems*, Second Revised Edition, Springer Verlag, Germany, 1996.
- [13] JACKSON L.W., KENUE S.K., A fourth-order exponentially fitted method, *SIAM J. Numer. Anal.*, 11(1974), 965-978.
- [14] LAMBERT J.D., *Computational methods in Ordinary Differential Equations*, John Wiley and sons, New York, 1973.
- [15] LAMBERT J.D., *Numerical Methods for Ordinary Differential Systems*, John Wiley and sons, New York, 1991.
- [16] LINIGER W., WILLOUGHBY R.A., Efficient integration methods for stiff systems of ordinary differential equations, *SIAM J. Numer. Anal.*, 7(1970), 47-65.
- [17] OKUNUGA S.A., *Composite Multiderivative method for Solving Stiff IVPs*, Ph.D Thesis, University of Lagos, Nigeria, 1992.
- [18] OKUNUGA S.A., A fourth order composite two-step method for Stiff problems, *Int. J. of Computer Mathematics*, 72(1)(1999), 39-47.

J. O. EHIGIE

DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF LAGOS, NIGERIA  
*e-mail*: jehigie@unilag.edu.ng

S. A. OKUNUGA  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF LAGOS, NIGERIA  
*e-mail*: sokunuga@unilag.edu.ng

A. B. SOFOLUWE  
DEPARTMENT OF COMPUTER SCIENCES  
UNIVERSITY OF LAGOS, NIGERIA  
*e-mail*: absofoluwe@yahoo.com

*Received on 19.10.2011 and, in revised form, on 30.10.2012.*