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Bipan Hazarika

ON ACCELERATION CONVERGENCE OF MULTIPLE SEQUENCES

Abstract. In this article the notion of acceleration convergence of double sequences in Pringsheim's sense has been introduced and some theorems related to that concept have been established using the four dimensional matrix transformations.

Key words: converging faster, converging at the same rate, acceleration field, P-convergent, P-lim-inf, P-lim-sup, double natural density.

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1. Introduction

In 1968, D.F. Dawson [3] had characterized the summability field of matrix $A = (a_{kn})$ by showing A is convergence preserving over the set of all sequences which converge faster than some fixed sequence x or A only preserves the limit of a set of constant sequences. In 1979, Smith and Ford [13] introduced the concept of acceleration of linear and logarithmic convergence. Subsequently, Keagy and Ford [6] had established the results analogues to the results of Dawson [3] dealing with the acceleration field of subsequence transformation. Later, Brezinski, Delahaye and Gesmain-Bonne [2], Brezinski [1] and many other authors have worked in this areas (see [11], [12], [14], [15]).

The sequence $x = (x_n)$ converges to σ faster than the sequence $y = (y_n)$ converges to λ, usually written as $x < y$, if

$$\lim_{n\to\infty} \frac{|x_n - \sigma|}{|y_n - \lambda|} = 0, \text{ provided } y_n - \lambda \neq 0 \text{ for all } n \in \mathbb{N}.$$

The matrix $A = (a_{kn})$ is said to accelerate the convergence of the sequence $x = (x_n)$ if $Ax < x$. The acceleration field of A is defined to be the class of sequences $\{x = (x_n) \in w : Ax < x\}$, where w is the space of all sequences.

The sequence $x = (x_n)$ converges to σ at the same rate as the sequence $y = (y_n)$ converges to λ, written as $x \approx y$, if

$$0 < \lim-\inf\left|\frac{x_n - \sigma}{y_n - \lambda}\right| \leq \lim-\sup\left|\frac{x_n - \sigma}{y_n - \lambda}\right| < \infty.$$

Let $A = (a_{kn})$ be an infinite matrix. For a sequence $x = (x_n)$, the A-transform of x is defined as

$$Ax = \sum_{k=1}^{\infty} a_{kn} x_n, \text{ for all } n \in \mathbb{N}.$$

The subsequence (x_{k_n}) of the sequence $x = (x_n)$ can be represented by a matrix transformation Ax, where the matrix $A = (a_{kn})$ is defined by

$$a_{i,k_i} = \begin{cases} 1, & \text{for all } i \in \mathbb{N}; \\ 0, & \text{otherwise} \end{cases}$$

For a matrix summabilty transformation A, we define the domain of A, usually denoted by $d(A)$, as

$$d(A) = \left\{ x = (x_n) \in w : \lim_k \sum_{n=1}^{\infty} a_{kn} x_n \text{ exists} \right\}.$$

We denote

$$l^1 = \left\{ x = (x_n) \in w : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

$S_\delta = \{x = (x_n) \in w : x_n \geq \delta > 0, \text{ for all } n\}$, $S_0 =$ the set of all nonnegative sequences which have at most a finite number of zero entries.

2. Definitions and preliminaries

In 1900, A. Pringsheim [9] introduced the notion of double sequence and presented the definition of convergence of a double sequence. Robinson [10] studied the divergent of double sequences and series. Later on Hamilton ([4], [5]) introduced the transformation of multiple sequences. Subsequently Patterson [8] introduced the concept of rate of convergence of double sequences.

In this article various notions and definitions on double sequence and double sequence spaces have been presented. Some interesting results on sequence spaces of convergent double sequences and bounded double sequences have been presented. Further, summability field of a four dimensional matrix $A = (a_{k,l,m,n})$ and the acceleration field of subsequence transformation of double sequences have been characterized.

Definition 1 ([9]). *A double sequence $x = (x_{m,n})$ is said to converge to a number L in Pringsheim's sense, symbolically we write $P-\lim_{m,n\to\infty} x_{m,n} = L$, if for every $\varepsilon > 0$, there exists a positive integer n_0 depending upon ε, such that $|x_{m,n} - L| < \varepsilon$, whenever $m, n \geq n_0$. The number L is called the Pringsheim's limit of the sequence x.*

Definition 2 ([9]). *A double sequence $x = (x_{m,n})$ is said to be bounded if there exists a real number $M > 0$ such that $|x_{m,n}| < M$, for all m and n.*

Definition 3 ([7]). *The double sequence y is a double subsequence of a sequence $x = (x_{m,n})$, if there exists two increasing index sequences $\{m_j\}$ and $\{n_j\}$ such that $y = (x_{m_j,n_j})$.*

Definition 4 ([7]). *A number β is said to be a Pringsheim limit point of the double sequence $x = (x_{m,n})$ if there exists a subsequence y of x such that $P - \lim y = \beta$.*

Definition 5 ([7]). *Let $x = (x_{m,n})$ be a double sequence of real numbers and for each k, let $\alpha_k = \sup_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit superior of x is defined as follows:*

(i) If $\alpha_k = +\infty$, for each k, then $P - \lim - \sup x = +\infty$;

(ii) If $\alpha_k < +\infty$, for each k, then $P - \lim - \sup x = \inf_k\{\alpha_k\}$.

Similarly, let $\beta_k = \inf_k\{x_{m,n} : m, n \geq k\}$. Then the Pringsheim limit inferior of x is defined as follows:

(iii) If $\beta_k = -\infty$, for each k, then $P - \lim - \sup x = -\infty$;

(iv) If $\beta_k > -\infty$, for each k, then $P - \lim - \sup x = \sup_k\{\beta_k\}$.

Definition 6. *A four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

Definition 7. *Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers. The cardinality of A, usually denoted by $|A(m,n)|$, is defined to be the number of (i,j) in A such that $i \leq m$ and $j \leq n$.*

Definition 8. *A two-dimensional set of positive integers A is said to have a double natural density, if the sequence $\left(\frac{|A(m,n)|}{mn}\right)$ has a limit in Pringsheim's sense. If this exists, it is denoted by $\delta_2(A)$. Thus we have*

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|A(m,n)|}{mn}.$$

Clearly we have $\delta_2(A^c) = \delta_2(\mathbb{N}\times\mathbb{N} - A) = 1 - \delta_2(A)$. Further, it is also clear that all finite subsets of $\mathbb{N} \times \mathbb{N}$ have zero double natural density. Moreover, some infinite subsets also have zero density. For example, the set

$$A = \{(i,j) : i \in [2^k, 2^k + k) \text{ and } j \in [2^l, 2^l + l), k, l = 1, 2, 3 \ldots\}$$

has double natural density zero.

Definition 9. *A double sequence $x = (x_{m,n})$ is said to satisfy a property P for "almost all (m,n)" if it satisfies the property P for all (m,n) except a set of double natural density zero. We abbreviate this by "a.a.(m,n)".*

Throughout this paper we use the following notations:
${}_2w$ = the space of all double sequences of real numbers.
${}_2l_\infty$ = the space of all bounded double sequences of real numbers.
${}_2c$ = the space of all convergent in Pringsheim's sense double sequences of real numbers.
${}_2c_0$ = the space of all null in Pringsheim's sense double sequences of real numbers.
${}_2c_0^B = {}_2c_0 \cap {}_2l_\infty$.
${}_2S_0^B$ = the subset of the space ${}_2c_0^B$.
${}_2S_\delta$ = the set of all real double sequences $x = (x_{m,n})$ such that $x_{m,n} \geq \delta > 0$, for all m and n.
${}_2S_0$ = the set of all non-negative sequences which have at most finite number of columns and / or rows with zero entries.

$$ {}_2l = \left\{ x = (x_{m,n}) : \sum_{m,n=1,1}^{\infty,\infty} |x_{m,n}| < \infty \right\}, $$

$$ {}_2d(A) = \left\{ x = (x_{m,n}) : P - \lim_{k,l} \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} x_{m,n} \text{ exists} \right\}. $$

$$ \mu_{k,l}(x) = \sup_{m,n \geq k,l} |x_{m,n}|, \text{ for } x \in {}_2l. $$

Definition 10. *Let $A = (a_{k,l,m,n})$ be a four-dimensional matrix. For any $x = (x_{m,n}) \in {}_2w$, the A-transform of x is defined as*

$$ Ax = \sum_{m,n=1,1}^{\infty,\infty} a_{k,l,m,n} x_{m,n}, \text{ for all } m, n \in \mathbb{N}. \tag{1} $$

The subsequence (x_{m_k,n_l}) of the sequence $x = (x_{m,n})$ can be represented by a matrix transformation represented by (1), where

$$ a_{k,l,m,n} = \begin{cases} 1, & \text{if } (m,n) = (m_k, n_l); \\ 0, & \text{if } (m,n) \neq (m_k, n_l) \end{cases} \tag{2} $$

It can be easily verified that the matrix as defined in (2) is a RH-regular matrix.

Definition 11. *A matrix transformation associated with the four-dimensional matrix A is said to be an ${}_2c_0 - {}_2c_0$ if Ax is in the set ${}_2c_0$, whenever x is in ${}_2c_0$ and is bounded (for details see [7]).*

The following is an important result on the characterization of ${}_2c_0 - {}_2c_0$ matrices:

Lemma 1 ([4]). *A four-dimensional matrix $A = (a_{k,l,m,n})$ is an ${}_2c_0 - {}_2c_0$, if and only if,*

(*a*) $\sum_{p,q=1,1}^{\infty,\infty} |a_{k,l,p,q}| < \infty$ *for all* k, l;

(*b*) *for* $q = q_0$, *there exists* $C_q(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $q > C_q(k,l)$ *for all* k, l, p;

(*c*) *for* $p = p_0$, *there exists* $C_p(k,l)$ *such that* $a_{k,l,p,q} = 0$, *whenever* $p > C_p(k,l)$ *for all* k, l, q;

(*d*) $P - \lim_{k,l} a_{k,l,p,q} = 0$, *for all* p *and* q.

3. Acceleration convergence of multiple sequences

In this section some more definitions related to double sequence have been defined and some interesting theorems regarding acceleration convergence of double sequences have been discussed.

Definition 12. *Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two double sequences of real numbers. Then the sequence x is said to converge P-faster than the sequence y, written as $x <^P y$, if*

$$P - \lim_{m,n} \left| \frac{x_{m,n}}{y_{m,n}} \right| = 0.$$

Definition 13. *The sequence $x = (x_{m,n})$ is said to converge at the same rate in Pringsheim's sense as the sequence $y = (y_{m,n})$, written as $x \approx^P y$, if*

$$0 < P - \lim - \inf \left| \frac{x_{m,n}}{y_{m,n}} \right| \leq P - \lim - \sup \left| \frac{x_{m,n}}{y_{m,n}} \right| < \infty.$$

Definition 14. *The four-dimensional matrix $A = (a_{k,l,m,n})$ is said to P-accelerate the convergence of the sequence $x = (x_{m,n})$ if $Ax <^P x$.*

We define the P-acceleration field of A as the set

$$\{x = (x_{m,n}) \in {}_2w : Ax <^P x\}.$$

Theorem 1. *Let $x = (x_{m,n})$ and $y = (y_{m,n})$ be two elements of ${}_2S_0^B$ such that $x <^P y$, then there exists an element $z = (z_{m,n})$ in ${}_2S_0^B$ such that $x <^P z <^P y$.*

Proof. Let $x, y \in {}_2S_0^B$ be such that $x <^P y$. Define the sequence $z = (z_{m,n})$ as follows:

$$z = x_{m,n}^{\frac{4}{5}} \, y_{m,n}^{\frac{1}{5}}.$$

This implies that $x <^P z <^P y$. ■

Theorem 2. *Let* $x <^P y$ *and* $y \approx^P z$*, then* $x <^P z$*.*

Proof. The proof is omitted as it is straight forward. ∎

Theorem 3. *Let* A *be a nonnegative* ${}_2c_0 - {}_2c_0$ *summability matrix and let* x *and* y *be two elements in* ${}_2l$ *such that* $x <^P y$ *with* $x \in {}_2S_0^B$ *and* $y \in {}_2S_\delta$ *for some* $\delta > 0$*, then* $\mu(Ax) <^P \mu(Ay)$*.*

Proof. Since $x <^P y$, then there exists a bounded double sequence $z = (z_{m,n})$ with Pringsheim's limit zero such that $x_{m,n} = y_{m,n}\, z_{m,n}$. For each k and l, we have

$$\begin{aligned}
\frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} &= \frac{\sup_{r,s\geq k,l}(Ax)_{r,s}}{\sup_{r,s\geq k,l}(Ay)_{r,s}} \\
&= \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, x_{m,n}}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}} \\
&= \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}\, z_{m,n}}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\left|\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}\, z_{m,n}\right|}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}\, |z_{m,n}|}{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}} \\
&\leq \frac{\sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}\, |z_{m,n}|}{\delta \sup_{r,s\geq k,l}\sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}}.
\end{aligned}$$

Since y and z are bounded real double sequences with z is in ${}_2c_0$ and A is a nonnegative ${}_2c_0 - {}_2c_0$ matrix , then

$$P - \lim_{k,l} \sup_{r,s\geq k,l} \sum_{m,n=1,1}^{\infty,\infty} a_{r,s,m,n}\, y_{m,n}\, |z_{m,n}| = 0.$$

Hence

$$P - \lim_{k,l} \frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} \leq 0. \tag{3}$$

In a similar manner we can establish

$$P - \lim_{k,l} \frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} \geq 0. \tag{4}$$

Hence from (3) and (4), we have

$$P - \lim_{k,l} \frac{\mu_{k,l}(Ax)}{\mu_{k,l}(Ay)} = 0$$

which implies $\mu(Ax) <^P \mu(Ay)$. This establishes the result. ∎

Theorem 4. *Let* $x = (x_{m,n}) \in {}_2S_0^B$ *and* A *be a subsequence transformation such that* $Ax <^P x$. *Then there exists* $y = (y_{m,n}) \in {}_2S_0^B$ *such that* $x_{m,n} = y_{m,n}$ *a.a.(m,n) and* $Ay <^P y$.

Proof. Let $x = (x_{m,n}) \in {}_2S_0^B$. Then there exists a subset $B_1 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_1) = 1$ such that

$$P - \lim\ x_{m,n} = 0, \text{ over } \ B_1.$$

Let $(x_{m_k,n_l}) \in {}_2S_0^B$. Then there exists a subset $B_2 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_2) = 1$ such that

$$P - \lim\ x_{m_k,n_l} = 0, \text{ over } \ B_2.$$

Since $Ax <^P x$, we have

$$P - \lim \left| \frac{x_{m_k,n_l}}{x_{m,n}} \right| = 0.$$

Then there exists a subset $B_3 \subset \mathbb{N} \times \mathbb{N}$ with $\delta_2(B_3) = 1$ such that

$$P - \lim \left| \frac{x_{m_k,n_l}}{x_{m,n}} \right| = 0, \text{ over } \ B_3.$$

Let $D = B_1 \cap B_2 \cap B_3$. Then clearly $\delta_2(D) = 1$. For $r \neq m_k, s \neq n_l, (k,l) \in \mathbb{N} \times \mathbb{N}$, let us define the sequence $y = (y_{m,n})$ as follows:

$$y_{r,s} = \begin{cases} x_{r,s} & \text{if } (r,s) \in D; \\ (rs)^{-3} & \text{otherwise} \end{cases}$$

and

$$y_{m_k,n_l} = \begin{cases} x_{m_k,n_l} & \text{if } (k,l) \in D; \\ y_{m,n}\ (mn)^{-3} & \text{otherwise} \end{cases}$$

Then we have $y = (y_{m,n}) \in {}_2S_0^B$ such that $x_{m,n} = y_{m,n}$ *a.a.(m,n)* and this implies $Ay <^P y$. ■

Theorem 5. *Let* $x = (x_{m,n}) \in {}_2S_0^B$ *and* A *be a subsequence transformation such that* $Ax <^P x$. *Then there exists* $y = (y_{m,n}) \in {}_2S_0^B$ *such that* $x <^P y$ *and* $Ay <^P y$.

Proof. Consider the sequence

$$y_{m,n} = |x_{m,n}|^{\frac{1}{2}} \text{ for all } m,n \in \mathbb{N}.$$

Then clearly $y = (y_{m,n}) \in {}_2S_0^B$ such that $x <^P y$ and $Ay <^P y$. This establishes the theorem. ■

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Bipan Hazarika
Department of Mathematics
Rajiv Gandhi University
Rono Hills, Doimukh-791 112
Arunachal Pradesh, India
e-mail: bh_rgu@yahoo.co.in

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