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ON LOCALLY γ-s-CLOSED SPACES

Abstract. In this paper, we aim to continue studying the properties of γ-s-closed spaces introduced and discussed in [5] and [9]. The concept of locally γ-s-closed space have been introduced. Certain important characterizations and properties of locally γ-s-closed space have also been established.

Key words: γ-closed (open), γ-closure, γ-semi-open(closed), γ-semi-closure, γ-semi-interior, γ-s-closed, locally γ-s-closed.

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1. Introduction

Topology is an important and interesting area of mathematics, the study of which will not only introduce you to new concepts and theorems but also put into context old ones like continuous functions. It is so fundamental that its influence is evident in almost every other branch of mathematics. This makes the study of topology relevant to all who aspire to be mathematicians whether their first love is algebra, analysis, category theory, chaos, continuum mechanics, dynamics, geometry, industrial mathematics, mathematical biology, mathematical economics, mathematical finance, mathematical modeling, mathematical physics, mathematics of communication, number theory, numerical mathematics, operation research or statistics.

S. Kasahara [12] introduced and discussed an operation γ of a topology τ into the power set $P(X)$ of a space X. H. Ogata [13] introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ by using operation γ.

S. Hussain and B. Ahmad [1-6] and [8-11] continued studying the properties of γ-operations on topological spaces and investigated many interesting results. Recently S. Hussain, B. Ahmad and T. Noiri [11], B. Ahmad and S. Hussain [2] defined and discussed γ-semi-open sets in topological spaces. They explored many interesting properties of γ-semi-open sets. It is interesting to mention that γ-semi-open sets generalized γ-open sets introduced by H. Ogata [13].

In 1987, G. Di. Maio and T. Noiri [7], introduced the notion of s-closed space. S. Hussain and B. Ahmad [9] introduced a class of topological spaces called γ-s-closed space by utilizing γ-semi-closure. It is shown that the concept of γ-s-closed space generalized s-closed space [9]. It is interesting to note that γ-s-closedness is the generalization of γ_0-compactness defined and investigated in [4]. In [5], they also defined and characterized sets γ-s-closed relative to a space X. They investigated γ-semi-regular set in γ-s-closed subspace relative to a space X.

In this paper, we aim to continue studying the properties of γ-s-closed spaces introduced and discussed in [5] and [9]. The concept of locally γ-s-closed space have been introduced. Certain important characterizations and properties of locally γ-s-closed space have also been established.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. Preliminaries

Definition 1 ([12]). *Let X be a space. An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation γ.*

Definition 2 ([13]). *Let $A \subseteq X$. A point $x \in A$ is said to be γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus*

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \ \text{ and } \ N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [13] iff A $= int_\gamma(A)$. A set A is called γ- closed [1] iff $X - A$ is γ-open.

Definition 3 ([13]). *A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $cl_\gamma(A)$. A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.*

Definition 4 ([13]). *An operation γ on τ is said be regular, if for any open nbds U,V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.*

Definition 5 ([13]). *An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \supseteq B$.*

Definition 6 ([9]). *A space X is γ-extremally disconnected space, if $cl_\gamma(U)$ is γ-open set, for every γ-open set U in X.*

Definition 7 ([9]). *A subset A of a space X is called γ-regular open, if $A = int_\gamma(cl_\gamma(A))$. The set of γ-regular open sets is denoted by $RO_\gamma(X)$. Note that $RO_\gamma(X) \subseteq \tau_\gamma \subseteq \tau$.*

Definition 8 ([9]). *A subset A of a space X is called γ-regular closed, denoted by $RC_\gamma(X)$, if one of the following conditions hold:*
(i) $A = cl_\gamma(int_\gamma(A))$.
(ii) $X - A \in RO_\gamma(X)$.
Clearly A is γ-regular open iff $X - A$ is γ-regular closed.

Definition 9 ([11]). *A subset A of a space X is said to be a γ-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ-semi-open sets is denoted by $SO_\gamma(X)$. A is γ-semi-closed iff $X - A$ is γ-semi-open in X. Note that A is γ-semi-closed iff $int_\gamma(cl_\gamma(A)) \subseteq A$.*

It is shown that every γ-open sets is γ-semi-open but converse is not true in general [11].

Definition 10 ([2]). *Let A be a subset of a space X. The intersection of all γ-semi-closed sets containing A is called γ-semi-closure of A and is denoted by $scl_\gamma(A)$. A is γ-semi-closed iff $scl_\gamma(A) = A$.*

Definition 11 ([2]). *Let A be a subset of a space X. The union of γ-semi-open subsets of A is called γ-semi-interior of A and is denoted by $sint_\gamma(A)$.*

Definition 12 ([9]). *A subset A of a space X is said to be γ-semi-regular, if it is both γ-semi-open and γ-semi-closed. The class of all γ-semi-regular sets of X is denoted by $SR_\gamma(A)$. If γ is regular, then the union of γ-semi-regular sets is γ-semi-regular.*

Definition 13 ([10]). *A space X is said to be γ-s-regular, if for any γ-semi-regular set A and $x \notin A$, there exist disjoint γ-open sets U and V such that $A \subseteq U$ and $x \in V$.*

Definition 14 ([2]). *A space X is said to be γ-semi-regular, if for each γ-semi-closed F and $x \notin F$, there exist disjoint γ-semi-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Definition 15 ([13]). *A space X is said to be γ-T_2 space, if for each disjoint points x, y of X, there exist open sets U and V such that $x \in U$, $y \in V$ and $U^\gamma \cap V^\gamma = \phi$.*

Definition 16 ([4]). *An operation γ on τ is said to be γ-open, if for each $U \in \tau$, U^γ is γ-open.*

Definition 17 ([5]). *A subset A of space X is called γ-pre-open, if $A \subseteq int_\gamma(cl_\gamma(A))$. Note that every γ-open set is γ-pre-open.*

Definition 18 ([2]). *A space X is said to be γ-s-closed, if for every cover $\{V_\alpha : \alpha \in I\}$ of X by γ-semi-open sets of X, there exists a finite subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.*

Definition 19 ([5]). *A subset A of a space X is said to be γ-s-closed relative to X, if for any cover $\{V_\alpha : \alpha \in I\}$ of X by γ-semi-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{\alpha \in I_0} scl_\gamma(V_\alpha)$.*

Definition 20 ([4]). *A subset A of a space X is said to be γ_0-compact relative to X, if every cover $\{V_i : i \in I\}$ of X by γ-open sets of X, there exists a finite subset I_0 of I such that $A \subseteq \bigcup_{I \in I_0} cl_\gamma(V_i)$. A space X is γ_0-compact, if $X = \bigcup_{i \in I_0} cl_\gamma(V_i)$.*

Proposition 1 ([9]). *A space X is γ-extremally disconnected iff $cl_\gamma(U) = scl_\gamma(U)$, for every $U \in SO_\gamma(X)$, where γ is a regular and an open operation.*

Lemma 1 ([9]). *A subset A of a space X is γ-semi-open iff $cl_\gamma(A) = cl_\gamma(int_\gamma(A))$, where γ is an open operation.*

Lemma 2 ([5]). *A γ-pre-open set of a space X is γ-s-closed as a subspace of X iff it is γ-s-closed relative to X.*

Corollary 1 ([5]). *Every γ-semi-regular subset of γ-s-closed space is a γ-s-closed subspace, where γ is regular.*

Lemma 3 ([9]). *If B is γ-open in a space X, then $scl_\gamma(B) = int_\gamma(cl_\gamma(B))$.*

Corollary 2 ([5]). *Let A and B be γ-open sets of a space X such that $A \subseteq B$. Then A is γ-s-closed subspace of B iff A is γ-s-closed subspace of X.*

Theorem 1 ([5]). *A γ-semi-open set A of a space X is a γ-s-closed subspace of X with regular and open operation γ, if it is γ-s-closed relative to X.*

3. Some properties of γ-s-closed spaces

Theorem 2. *Let X be a γ-extermally disconnected space with a regular operation γ. Then every γ-s-closed set A relative to X in a γ-semi-regular space X is γ_0-compact relative to X.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be a γ-semi-open cover of $A \subseteq X$. Then for each $x \in A$, there exists $U_\alpha(x)$ such that $x \in U_\alpha(x)$. Since X is γ-semi-regular, by Theorem 3.4 [10], there exists a γ-open set $V_\alpha(x)$ such that $x \in V_\alpha(x) \subset cl_\gamma(V_\alpha(x)) \subset U_\alpha(x)$. The family $\{V_\alpha(x) : x \in A\}$ is a cover of A by γ-open and hence γ-semi-open sets in X. Since A is γ-s-closed relative to X, there exists a subfamily $\{V_\alpha(x_j) : j = 1, 2, ..., n\}$ such that $A \subseteq \bigcup\{scl_\gamma(V_\alpha(x_j)) : j = 1, 2, ..., n\}$. Also $scl_\gamma(V_\alpha(x_j)) \subset U_\alpha(x_j)$ implies $\bigcup\{scl_\gamma(V_\alpha(x_j)) : j = 1, 2, ..., n\} \subseteq \bigcup\{U_\alpha(x_j) : j = 1, 2, ..., n\}$ and thus $A \subseteq \bigcup\{U_\alpha(x_j) : j = 1, 2, ..., n\} \subseteq \bigcup cl_\gamma\{U_\alpha(x_j) : j = 1, 2, ..., n\}$. This proves that A is γ_0-compact relative to X. This completes the proof. ■

The following example shows that the condition of regular operation γ is necessary for the above theorem:

Example 1. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A\,. \end{cases}$$

Then the operation γ is not regular, also X is γ-extremally disconnected and γ-semi-regular space. Calculations shows that $\{a, b\}$, $\{a, c\}$, $\{b\}$, X, ϕ are γ-open sets, $\{a, c\}$, $\{b\}$, X, ϕ are γ-semi-open sets and $\{a\}$, $\{b\}$, $\{c\}$, X, ϕ are γ-s-closed sets relative to X. Clearly γ-s-closed set $A = \{a\}$ relative to X is not γ_0-compact relative to X.

Corollary 3. *Every γ-semi-regular γ-s-closed space is γ_0-compact. Where γ ia a regular operation.*

Lemma 4. *If X is γ-extremally disconnected space with regular and open operation γ and $A \in SR_\gamma(X)$, then A is γ-closed and γ-open in X.*

Proof. This follows from Proposition 1 and Lemma 1. Hence the proof. ■

Theorem 3. *A γ-extremally disconnected space X with regular and open operation γ is γ-s-closed, if every γ-semi-regular subset of X is γ-s-closed subspace of X.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a cover of X and $V_\alpha \in SO_\gamma(X)$ for each $\alpha \in I$. Suppose that $X \neq scl_{\gamma_X}(V_\beta)$ and $V_\beta \neq \phi$. Since $V_\beta \in SO_\gamma(X)$, $scl_{\gamma_X}(V_\beta) \in SR_\gamma(X)$ and hence $X - scl_{\gamma_X}(V_\beta) \in SR_\gamma(X)$. Therefore $X - scl_{\gamma_X}(V_\beta)$ is a γ-s-closed subspace and hence is γ-open and γ-closed in X by Lemma 4. By Corollary 1, $X - scl_{\gamma_X}(V_\beta)$ is γ-s-closed relative to X. Since $X - scl_{\gamma_X}(V_\beta) \subseteq \bigcup\{V_\alpha : \alpha \in I\}$, there exists a finite subfamily I_0 of I such that $X - scl_{\gamma_X}(V_\beta) \subseteq \bigcup\{scl_{\gamma_X}(V_\alpha) : \alpha \in I_0\}$. We obtain $X = \bigcup\{scl_{\gamma_X}(V_\alpha) : \alpha \in I_0 \bigcup\{\beta\}\}$. This shows that X is γ-s-closed. This completes the proof. ■

The following example shows that the condition of regular and open operation γ is necessary for the above theorem:

Example 2. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A\,. \end{cases}$$

Then the operation γ is not regular and open, also X is γ-extremally disconnected. Calculations shows that $\{a, b\}$, $\{a, c\}$, $\{b\}$, X, ϕ are γ-open sets, $\{a, c\}$, $\{b\}$, X, ϕ are γ-semi-open sets, $\{a\}$, $\{b\}$, $\{c\}$, X, ϕ are γ-s-closed sets relative to X and $\{b\}$, $\{a, c\}$, X, ϕ are γ-semi-regular subsets of X. Clearly X is not γ-s-closed, since γ-semi-regular subset $\{a, c\}$ is not γ-s-closed subspace of X.

Theorem 4. *If X is a γ-T_2 space and $A \subseteq X$ is γ-s-closed relative to X, then A is γ-closed in X.*

Proof. Let X be a γ-T_2 space and $A \subseteq X$ be a γ-s-closed relative to X. Let $x_0 \in X - A$ be a fixed point. For each $a \in A$, find disjoint nbds $U(a)$, $U_{x_0}(a)$. Since X is γ-T_2, $x_0 \notin scl_{\gamma X}(U(a))$. Then $\{scl_{\gamma_x}(U(a)) : a \in A\}$ is a covering of A by γ-semi-regular sets X. Since A is γ-s-closed relative to X, there exists a finite subcovering $scl_{\gamma X}(U(a_1))$, $scl_{\gamma X}(U(a_2))$, ...,$scl_{\gamma X}(U(a_n))$ such that $A \subseteq \bigcup\{scl_{\gamma X}(U(a_i)) : i = 1, 2, ..., n\}$ and $U(x_0) = \bigcap\{U_{x_0}(a_i) : i = 1, 2, ..., n\}$. Where $U_(x_0)$ is a nbd of x_0 disjoint from $\bigcup\{scl_{\gamma_x}(U(a_i)) : i = 1, 2, ..., n\}$. Hence $(U(x_0))^\gamma \subseteq X - A$. This proves that $X - A$ is γ-open or A is γ-closed. This completes the proof. ■

Lemma 5. *Let B be a γ-s-closed relative to X with regular operation γ and $O \in SR_\gamma(X)$ such that $O \subset B$. Then $B - O$ is γ-s-closed relative to X.*

Proof. Let $U = \{U_\alpha : \alpha \in I\}$ be a cover of $B - O$ by γ-semi-regular sets of X. Then $U \bigcup\{O\}$ is a cover of B by γ-semi-regular sets of X. Since B is γ-s-closed relative to X, there exists a finite subcollection $\{U_i : i = 1, 2, ..., n\}$ such that $B \subseteq O \bigcup scl_\gamma\{U_i : i = 1, 2, ..., n\}$. This implies that $B - O \subseteq \bigcup scl_\gamma\{U_i : i = 1, 2, ..., n\}$. Then $B - O$ is γ-s-closed relative to X. Hence the proof. ■

Example 3. In Example 2, it is easy to see that the above Lemma does not hold, since γ is not regular.

Theorem 5. *Let X be γ-T_2 space with regular and open operation γ and B a set γ-s-closed relative to X. For any $x \in B$ and any γ-semi-regular set A such that $x \in A \subseteq B$, there is a γ-open set G such that $x \in G^\gamma \subseteq cl_\gamma(G^\gamma) \subseteq A$.*

Proof. Let $x \in B$ and A any γ-semi-regular set such that $x \in A \subseteq B$. For each $y \in B - A$, there exist open nbds $G_{y(x)}$ and H_y such that $G^\gamma_{y(x)} \cap H^\gamma_y = \phi$. Furthermore, we can assume that $G^\gamma_{y(x)} \subseteq A$. The collection $\{H^\gamma_y : y \in B - A\}$ is a cover of $B - A$ by γ-semi-regular sets X. By Lemma 5, $B - A$ is γ-s-closed relative to X. There exists a finite subcollection $\{H^\gamma_{y_i} : i = 1, 2, ..., n\}$ such that $B - A \subseteq \bigcup scl_\gamma\{H^\gamma_{y_i} : i = 1, 2, ..., n\} = H^\gamma$. Let $G^\gamma = \bigcap scl_\gamma\{G^\gamma_{y_i(x)} : i = 1, 2, ..., n\}$. It follows that $G^\gamma \cap H^\gamma = \phi$. Since $H^\gamma_{y_i}$ is γ-open and hence γ-semi-open for each $i = 1, 2, ..., n$. By Lemma 3, $scl_\gamma(H^\gamma_{y_i})$ is γ-open, for each $i = 1, 2, ..., n$. Hence G^γ and H^γ are γ-open and $cl_\gamma(G^\gamma) \cap H^\gamma = \phi$. Also, by Theorem 3, B is γ-closed and $G^\gamma \subseteq A \subseteq B$. This implies that $cl_\gamma(G^\gamma) \subseteq B$. Therefore $B - A \subseteq B \cap H^\gamma \subseteq B \cap (X - cl_\gamma(G^\gamma)) = B - cl_\gamma(G^\gamma)$. This implies that $cl_\gamma(G^\gamma) \subseteq A$. Clearly, then $x \in G^\gamma \subseteq cl_\gamma(G^\gamma) \subseteq A$ as specified. This completes the proof. ■

The proof of the following theorem is straightforward:

Theorem 6. *If A_i, $i = 1, 2, \ldots, n$ are γ-s-closed relative to X with regular operation γ, then $\bigcup_{i=1}^n A_i$ is γ-s-closed relative to X.*

4. Locally γ-s-closed spaces

Definition 21. *A space X is said to be locally γ-s-closed, if each point of X has a γ-regular-open nbd which is γ-s-closed subspace.*

Note that every γ-s-closed space is locally γ-s-closed. However the converse is not true in general, because every uncountable set with discrete γ-topology is locally γ-s-closed but not γ-s-closed.

Lemma 6. *If A is a γ-regular open set in (X, τ) with regular operation γ, then every γ-regular open set in the subspace (A, τ_A) is also γ-regular open in (X, τ).*

Proof. Let $V \subseteq A$ be a γ-regular open in the subspace (A, τ_A). Then $V = int_{\gamma_A}(cl_{\gamma_A}(V)) = int_{\gamma_A}(A \cap cl_{\gamma_X}(V)) = int_{\gamma_X}(A \cap cl_{\gamma_X}(V) = int_{\gamma_X}(A) \cap int_{\gamma_X}(cl_{\gamma_X}(V)) = A \cap int_{\gamma_X}(cl_{\gamma_X}(V)) = int_{\gamma_X}(cl_{\gamma_X}(V))$, since $V \subseteq A$ implies $int_{\gamma_X}(cl_{\gamma_X}(V)) \subseteq int_{\gamma_X}(cl_{\gamma_X}(A)) = A$. Therefore V is γ-regular open in X. This completes the proof. ■

Theorem 7. *A space X is locally γ-s-closed with regular operation γ iff for each point $x \in X$, there exists a γ-regular open set U containing x such that U is locally γ-s-closed.*

Proof. *Necessity.* The proof is straightforward.

Sufficiency. Let $x \in X$. Then by supposition, there exists a γ-regular open set U of X containing x such that U is locally γ-s-closed. By Lemma 6, there exists a γ-regular open set V in U such that $x \in V$ and V is a γ-s-closed subspace of U. Therefore V is a γ-regular open set in X and by Corollary 2, V is γ-s-closed subspace of X. Hence X is locally γ-s-closed. ■

Theorem 8. *A space X is locally γ-s-closed iff each point of $x \in X$ has a γ-open nbd which is γ-s-closed relative to X.*

Proof. The proof follows from Lemma 3 and the fact that every γ-open set is γ-pre-open [5]. Hence the proof. ■

Remark 1 ([2])**.** Let γ be a regular operation on τ. If A and B are γ-semi-open sets, then $A \cap B$ is γ-semi-open.

Theorem 9. *If X is a locally γ-s-closed space with regular and open operation γ and $A \in SO_\gamma(X)$, then A is locally γ-s-closed.*

Proof. Let $x \in A$ and X is locally γ-s-closed, by Theorem 8, there exists a γ-open nbd U of x such that U is γ-s-closed relative to X. As A is γ-semi-open , by Theorem 1, $A \cap U$ is γ-s-closed relative to A. Since $A \cap U$ is γ-open nbd of x in the subspace A, by Theorem 8, A is locally γ-s-closed. This completes the proof. ■

Corollary 4. *If γ is a regular operation, then locally γ-s-closedness is a γ-open hereditary property.*

Theorem 10. *A space X is locally γ-s-closed iff for each point $x \in X$, there exists a γ-open set A of X such that $x \in A$ and A is locally γ-s-closed.*

Proof. *Sufficiency.* Let $x \in X$. By Theorem 8, there exists a γ-open set A such that $x \in A$ and A is locally γ-s-closed. Therefore, there exists a γ-open nbd U of $x \in A$ such that U is a γ-s-closed subspace of A. Since A is γ-open in X, U is γ-open in X. By Corollary 2, U is γ-s-closed subspace of X. This proves that X is locally γ-s-closed.
Necessity. This is obvious. Hence the proof. ■

Theorem 11. *Let X be a space. The following are equivalent:*
(a) *X is locally γ-s-closed.*
(b) *Every point has a γ-regular-open set which is γ-s-closed relative to X.*
(c) *Every point $x \in X$ has a γ-open nbd U such that $int_{\gamma X}(cl_{\gamma X}(U))$ is γ-s-closed relative to X.*
(d) *Every point $x \in X$ has a γ-open nbd U such that $scl_\gamma(U)$ is γ-s-closed relative to X.*

(*e*) *For every point* $x \in X$*, there exists a* γ*-open set* V *containing* x *such that* $scl_\gamma(V)$ *is* γ*-s-closed relative to* X*.*

(*f*) *For every point* $x \in X$*, there exists a* γ*-open set* V *containing* x *such that* $int_{\gamma_X}(cl_{\gamma_X}(U))$ *is* γ*-s-closed relative to* X*.*

(*g*) *For each* $x \in X$*, there exists a* γ*-pre-open set* V *containing* x *such that* $scl_\gamma(V)$ *is* γ*-s-closed relative to* X*.*

(*h*) *For every* $x \in X$*, there exists a* γ*-pre-open set* V *containing* x *such that* $int_{\gamma_X}(cl_{\gamma_X}(U))$ *is* γ*-s-closed relative to* X*.*

(*i*) *For every* $x \in X$*, there exists a* γ*-pre-open set* V *containing* x *such that* $int_{\gamma_X}(cl_{\gamma_X}(U))$ *is a* γ*-s-closed subspace of* X*.*

Proof. $(a) \Rightarrow (b)$. Follows from Lemma 3 and the fact that every γ-regular-open set is γ-pre-open.

$(b) \Rightarrow (c)$. This is obvious.

$(c) \Rightarrow (d)$. This follows from Lemma 2.

$(d) \Rightarrow (e)$. This is evident, since every γ-open set is open.

$(e) \Rightarrow (f)$, $(f) \Rightarrow (g)$, $(g) \Rightarrow (h)$, $(h) \Rightarrow (i)$ are straightforward because of the facts that every γ-open set is γ-pre-open and by Lemma 2.

$(i) \Rightarrow (a)$. This follows from Lemma 3. This completes the proof. ■

Theorem 12. *If* X *is a* γ*-*T_2 *space with a regular and open operation* γ*, then the following conditions are equivalent:*

(*a*) X *is locally* γ*-s-closed.*

(*b*) *For each* $x \in X$ *and each* γ*-open nbd* U *of* x*, there is a* γ*-open set* V *such that* $cl_\gamma(V^\gamma)$ *is* γ*-s-closed relative to* X *and* $x \in V^\gamma \subseteq cl_\gamma(V^\gamma) \subseteq scl_\gamma(U^\gamma)$*.*

(*c*) *For each* $x \in X$ *and each* γ*-regular open nbd* U *of* x*, there is a* γ*-open set* V *in* X *such that* $cl_\gamma(V^\gamma)$ *is* γ*-s-closed relative to* X *and* $x \in V^\gamma \subseteq cl_\gamma(V^\gamma) \subseteq U^\gamma$*.*

(*d*) *For each set* C*,* γ*-s-closed relative to* X *and* γ*-regularly open set* U *containing* C*, there is a* γ*-open set* V *such that* V *is* γ*-s-closed relative to* X *and* $C \subseteq V^\gamma \subseteq scl_\gamma(V^\gamma) \subseteq U^\gamma$*.*

Proof. $(a) \Rightarrow (b)$. Let X be locally γ-s-closed space and $x \in X$. There exists a γ-open set W with $x \in W^\gamma \subseteq scl_\gamma(W^\gamma)$. By Theorem 11(*d*), $scl_\gamma(W^\gamma)$ is γ-s-closed relative to X. The set $scl_\gamma(U^\gamma \cap W^\gamma)$ is γ-semi regular and is contained in $scl_\gamma(W^\gamma)$. By Theorem 5, there exists a γ-open set V such that $x \in V^\gamma \subseteq cl_\gamma(V^\gamma) \subseteq scl_\gamma(U^\gamma \cap W^\gamma) \subseteq scl_\gamma(U^\gamma)$. The set $cl_\gamma(V^\gamma)$ is γ-semi regular and contained in $scl_\gamma(W^\gamma)$. Since γ is regular, by Proposition 4.9(2) [5], $cl_\gamma(V^\gamma)$ is γ-s-closed relative to X. This proves (*b*).

$(b) \Rightarrow (c)$. This is obvious.

$(c) \Rightarrow (d)$. For each $c \in C$, find a γ-open set V_1 such that by Theorem 11(*d*), $scl_\gamma(V_1^\gamma)$ is γ-s-closed relative to X and $scl_\gamma(V_1^\gamma) \subseteq U^\gamma$. Since

C is γ-s-closed relative to X, there exists a finite subcollection $\{V_{1_i}^{\gamma} : i = 1, 2, \dots, n\}$ such that $C \subseteq \bigcup\{V_{1_i}^{\gamma} : i = 1, 2, \dots, n\} = V^{\gamma}$. Since V is the finite union of the sets γ-open as well as γ-s-closed relative to X. Thus V is γ-open and by Theorem 6, V is γ-s-closed relative to X. Clearly, $C \subseteq V^{\gamma} \subseteq scl_{\gamma}(V^{\gamma}) \subseteq U^{\gamma}$. This proves (d).

$(d) \Rightarrow (a)$. A point is certainly γ-s-closed relative to X. Let X be U in (d). The by (d), there exists a γ-open set V containing x such that V is γ-s-closed relative to X. hance by Theorem 4, X is locally γ-s-closed. This proves (a). Hence the proof. ■

References

[1] AHMAD B., HUSSAIN S., Properties of γ-operations on topological spaces, *Aligarh Bull. Math.,* 22(1)(2003), 45-51.

[2] AHMAD B., HUSSAIN S., γ-semi-open sets in topological spaces II, *Southeast Asian Bull. Maths.,* 34(6)(2010), 997-1008.

[3] AHMAD B., HUSSAIN S., γ-convergence in topological space, *Southeast Asian Bull. Maths.,* 29(5)(2005), 832-842.

[4] AHMAD B., HUSSAIN S., γ^*-regular and γ-normal spaces, *Math. Today.,* 22(1)(2006), 37-44.

[5] AHMAD B., HUSSAIN S., On γ-s-closed subspaces, *Far East Jr. Math. Sci.,* 31(2)(2008), 279-291.

[6] AHMAD B., HUSSAIN S., NOIRI T., On some mappings in topological space, *Eur. J. Pure Appl. Math.,* 1(2008), 22-29.

[7] MAIO G.DI., NOIRI T., On s-closed spaces, *Indian J. Pure Appl. Math.,* 18(1987), 226-233.

[8] HUSSAIN S., AHMAD B., On Minimal γ-open sets, *Eur. J. Pure Appl. Maths.,* 2(3)(2009), 338-351.

[9] HUSSAIN S., AHMAD B., On γ-s-closed spaces, *Sci. Magna Jr.,* 3(4)(2007), 89-93.

[10] HUSSAIN S., AHMAD B., On γ-s-regular spaces and almost γ-s-continuous functions, *Lobackevskii. J. Math.,* 30(4)(2009), 263-268. DOI:10.1134/51995080209040039.

[11] HUSSAIN S., AHMAD B., NOIRI T., γ-semi-open sets in topological spaces, *Asian Eur. J. Math.,* 3(3)(2010), 427-433. DOI:10.1142/S1793557110000337.

[12] KASAHARA S., Operation-compact spaces, *Math. Japon.,* 24(1979), 97-105.

[13] OGATA H., Operations on topological spaces and associated topology, *Math. Japon.,* 36(1)(1991), 175-184.

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