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K. Liu, P. Li and W. Zhong*

ON A SYSTEM OF RATIONAL DIFFERENCE EQUATIONS $x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}$, $y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}$, $z_{n+1} = \frac{1}{y_n z_{n-1}}$

Abstract. In this paper, we are concerned with a three- dimensional system of rational difference equations. We present solutions of the system in an explicit way and obtain the asymptotical behavior of solutions according to initial values. We also give sufficient conditions of existing four-period solutions.

Key words: system of rational difference equation, asymptotical behavior, periodicity.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

Difference equations, also referred to recursive sequence, is a hot topic. Recently there has been an increasing interest in the study of qualitative analysis of nonlinear difference equations and systems of difference equations. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economics, physics, computer sciences and so on. Especially, Gu and Ding[5] has considered the state space models described by difference equations. Although difference equations' forms are very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. The study of these equations is quite challenging and rewarding and is still in its infancy. We believe that nonlinear rational difference equations are of paramount importance in their own right, and furthermore we believe that these results about such equations offer prototypes towards the development of the basic theory of nonlinear difference equations.

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Particularly, there is a class of nonlinear difference equations, known as rational difference equation or fractional difference equation, which consists of the ratio of two polynomials in the sequence terms. A lot of work has been concentrated on it, see [12]–[13]. There is one way to study rational difference equations–giving the exact expression of solutions[1, 2]. Another way is studying the qualitative behavior such as asymptotical stability using the linearized method[4, 18], semicycle analysis and so on[12].

At the same time, more and more attention is paid to systems of rational difference equations composed by two or three rational difference equations[3]–[13]. The single equation is simple, but the coupled ways of systems are various and thus such systems have no fixed ways to follow to investigate their behavior.

In [1, 2], Çinar has obtained the solutions of the following difference equations

$$\begin{aligned} x_{n+1} &= \frac{x_{n-1}}{1 + x_n\, x_{n-1}}, \\ x_{n+1} &= \frac{a\, x_{n-1}}{1 + b\, x_n\, x_{n-1}}. \end{aligned}$$

In [4] and [18], more complicated equations have been investigated. In [3], Çinar has proved the periodicity of positive solutions of the difference equation system

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}\, y_{n-1}}.$$

In [17] and [16], higher-ordered systems have been studied. In [15], Stevic has investigated the following general system of difference equations

$$x_{n+1} = \frac{a x_{n-1}}{b\, y_n\, x_{n-1} + c}, \qquad y_{n+1} = \frac{\alpha y_{n-1}}{\beta\, x_n\, y_{n-1} + \gamma}.$$

In [11], Kurbanli, Çinar and Yalçinkaya has expressed solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n\, x_{n-1} + 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n\, y_{n-1} + 1}. \tag{1}$$

In [9], Kurbanli, Çinar and Simsek has also expressed solutions of the system

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n\, x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1} + x_n}{x_n\, y_{n-1} - 1}. \tag{2}$$

In [10], Kurbanli, Çinar and Yalçinkaya has investigated the behavior of positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n\, x_{n-1} - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{x_n\, y_{n-1} - 1}. \tag{3}$$

Based on it, three-dimensional systems have been investigated. In [6], [7] and [8], Kurbanli and his collaborates has obtained the behavior of solutions of the following systems, respectively,

$$(4)\quad x_{n+1}=\frac{x_{n-1}}{y_n\,x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n\,y_{n-1}-1},\quad z_{n+1}=\frac{z_{n-1}}{y_n\,z_{n-1}-1};$$

$$(5)\quad x_{n+1}=\frac{x_{n-1}}{y_n\,x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n\,y_{n-1}-1},\quad z_{n+1}=\frac{1}{y_n\,z_n};$$

$$(6)\quad x_{n+1}=\frac{x_{n-1}}{y_n\,x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n\,y_{n-1}-1},\quad z_{n+1}=\frac{x_n}{y_n\,z_{n-1}}.$$

In [14], we have improved the results on (4) of those in [6] and also investigated the system

$$(7)\quad x_{n+1}=\frac{x_{n-1}}{y_n\,x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n\,y_{n-1}-1},\quad z_{n+1}=\frac{z_{n-1}}{x_n\,z_{n-1}-1}.$$

And some other results would be presented in [13].

In this paper, motivated by the above references and the references cited therein, we consider the following system

$$(8)\quad x_{n+1}=\frac{x_{n-1}}{y_n\,x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n\,y_{n-1}-1},\quad z_{n+1}=\frac{1}{y_n\,z_{n-1}}$$

where the initial conditions are nonzero real numbers. Here, (8) is a system including three rational difference equations and the first two equations are similar to those of [10, 6, 7], the third equation is different from [6, 7]. How is the solution of (8) and is it similar to those of [7]? We try to answer these problems in the following.

In next section, we express solutions of the system (8) and try to describe the behavior of solutions.

2. Main results

Through the paper, we suppose the initial values to be

$$y_0=a,\quad x_0=c,\quad y_{-1}=b,\quad x_{-1}=d,\quad z_0=e,\quad z_{-1}=f.$$

Here, a, b, c, d, e and f are real numbers such that $(ad-1)(cb-1)\neq 0$, $abef\neq 0$. We call this to be the hypothesis H.

Theorem 1. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Then all solutions of (8) are*

$$(9)\qquad x_n=\begin{cases}\dfrac{d}{(ad-1)^k}, & n=2k-1,\\ c\,(cb-1)^k, & n=2k,\end{cases}\qquad k=1,2,\cdots$$

$$
(10) \qquad y_n = \begin{cases} \dfrac{b}{(cb-1)^k}, & n = 2k-1, \\ a(ad-1)^k, & n = 2k, \end{cases} \qquad k = 1, 2, \cdots
$$

$$
(11) \qquad z_n = \begin{cases} \dfrac{1}{a\,f\,(ad-1)^{k-1}}, & n = 4(k-1)+1, \\ \dfrac{(cb-1)^k}{be}, & n = 4(k-1)+2, \\ \dfrac{f}{(ad-1)^k}, & n = 4(k-1)+3, \\ e\,(bc-1)^k, & n = 4(k-1)+4. \end{cases} \qquad k = 1, 2, \cdots
$$

Proof. It is obvious to obtain (9) and (10) and referred to [10]. Here, we only focus on (11).

First, for $k = 1$, from (8) and (10), we easily check that

$$
\begin{aligned}
z_1 &= \frac{1}{y_0\, z_{-1}} = \frac{1}{a\, f}, \\
z_2 &= \frac{1}{y_1\, z_0} = \frac{1}{\frac{b}{(cb-1)}\, e} = \frac{cb-1}{b\, e}, \\
z_3 &= \frac{1}{y_2\, z_1} = \frac{f}{ad-1}, \\
z_4 &= \frac{1}{y_3\, z_2} = e\,(cb-1)\,.
\end{aligned}
$$

Next, we assume the conclusion is true for k, that is, (11) holds.

Then, for $k+1$, we confirm it. In fact, from (8) and (10) and (11), we have

$$
z_{4k+1} = \frac{1}{y_{4k}\ z_{4(k-1)+3}} = \frac{1}{a\,(ad-1)^{2k} \times \frac{f}{(ad-1)^k}} = \frac{1}{a\, f\,(ad-1)^k},
$$

$$
z_{4k+2} = \frac{1}{y_{4k+1}\ z_{4k}} = \frac{1}{\frac{b}{(cb-1)^{2k+1}} \times e\,(cb-1)^k} = \frac{(cb-1)^{k+1}}{b\, e},
$$

$$
z_{4k+3} = \frac{1}{y_{4k+2}\ z_{4k+1}} = \frac{1}{a\,(ad-1)^{2k+1} \times \frac{1}{a\, f\,(ad-1)^k}} = \frac{f}{(ad-1)^{k+1}},
$$

$$
z_{4k+4} = \frac{1}{y_{4k+3}\ z_{4k+2}} = \frac{1}{\frac{b}{(cb-1)^{2k+2}} \times \frac{(cb-1)^{k+1}}{be}} = e\,(cb-1)^{k+1}
$$

and complete the proof. ∎

By Theorem 1, the expressions of (9), (10) and (11) will greatly help us to investigate the asymptotical behavior of solutions of (11).

Corollary 1. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Also, if $ad = cb = 2$, then all solutions of (8) are four periodic.*

Proof. In this case, from (9), (10) and (11), we have

$$x_n = \begin{cases} d\,, & n = 2k-1, \\ c\,, & n = 2k, \end{cases} \qquad k = 1, 2, \cdots \tag{12}$$

$$y_n = \begin{cases} b\,, & n = 2k-1, \\ a, & n = 2k, \end{cases} \qquad k = 1, 2, \cdots \tag{13}$$

$$z_n = \begin{cases} \dfrac{1}{af}\,, & n = 4(k-1)+1, \\ \dfrac{1}{be}\,, & n = 4(k-1)+2, \\ f\,, & n = 4(k-1)+3, \\ e\,, & n = 4(k-1)+4, \end{cases} \qquad k = 1, 2, \cdots \tag{14}$$

and complete the proof. ∎

Corollary 2. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Also, if $ad, cb \in (1, 2)$ and $a > 0$, then all solutions of (8) satisfy*

$$\lim_{n\to\infty} (x_{2n-1},\ y_{2n-1},\ z_{2n-1}) = (\infty,\ \infty,\ \infty)\,, \tag{15}$$

$$\lim_{n\to\infty} (x_{2n},\ y_{2n},\ z_{2n}) = (0,\ 0,\ 0)\,. \tag{16}$$

Proof. From the hypothesis and $ad, cb \in (1, 2)$ and $d > c$, we obtain that $0 < ad - 1 < 1$, $0 < cb - 1 < 1$ and thus, $(ad-1)^n$ and $(cb-1)^n$ tend to zero as n tends to ∞.

First, from (9), we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \cdot \infty = \begin{cases} -\infty, & d < 0, \\ +\infty, & d > 0. \end{cases}$$

Similarly, from (10), we have

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b\cdot\infty = \begin{cases} -\infty, & b<0, \\ +\infty, & b>0. \end{cases}$$

As far as z_{2n-1} is concerned, from (11) we could consider z_{4k+1} and z_{4k+3} for $n = k+1$, respectively,

$$\lim_{n\to\infty} z_{4k+1} = \lim_{n\to\infty} \frac{1}{a\,f\,(ad-1)^k} = \frac{1}{a\,f}\cdot\infty = \begin{cases} -\infty, & f<0,\ a>0, \\ +\infty, & f>0. \end{cases}$$

$$\lim_{n\to\infty} z_{4k+3} = \lim_{n\to\infty} = \frac{f}{(ad-1)^{k+1}} = f\cdot\infty = \begin{cases} -\infty, & f<0, \\ +\infty, & f>0. \end{cases}$$

and thus

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} -\infty, & f<0, \\ +\infty, & f>0. \end{cases}$$

Therefore,

$$\lim_{n\to\infty} (x_{2n-1},\ y_{2n-1},\ z_{2n-1}) = (\infty,\ \infty,\ \infty)\,.$$

Next, from (9) and (10), we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c\,(cb-1)^n = 0,$$
$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a\,(ad-1)^n = 0.$$

At last, for z_{2n}, we have

$$\lim_{n\to\infty} z_{4k+2} = \lim_{n\to\infty} \frac{(cb-1)^{k+1}}{be} = 0\,,$$

$$\lim_{n\to\infty} z_{4k+4} = \lim_{n\to\infty} e\,(cb-1)^{k+1} = 0$$

and thus

$$\lim_{n\to\infty} z_{2n} = 0$$

and complete the proof. ■

Corollary 3. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Also, if $a, b, c, d \in (0,1)$, then all solutions of (8) satisfy*

$$\lim_{n\to\infty} (x_{2n-1},\ y_{2n-1},\ z_{2n-1}) = (\infty,\ \infty,\ \infty)\,, \tag{17}$$

$$\lim_{n\to\infty} (x_{2n},\ y_{2n},\ z_{2n}) = (0,\ 0,\ 0)\,. \tag{18}$$

Proof. From $a, b, c, d \in (0,1)$, we have $-1 < ad-1 < 0$, $-1 < cb-1 < 0$. The remainder is similar to that of Corollary 2 and we omit here. ■

And we also have the following and omit the proof.

Corollary 4. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Also, if $ad, cb \in (2, +\infty)$ and $b > 0$, then all solutions of (8) satisfy*

$$\lim_{n\to\infty} (x_{2n-1},\ y_{2n-1},\ z_{2n-1}) = (0,\ 0,\ 0)\,, \tag{19}$$

$$\lim_{n\to\infty} (x_{2n},\ y_{2n},\ z_{2n}) = (\infty,\ \infty,\ \infty)\,. \tag{20}$$

Corollary 5. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). Also, if $ad, cb \in (-\infty, 0)$ and $b > 0$, then all solutions of (8) satisfy*

$$\lim_{n\to\infty} (x_{2n-1},\ y_{2n-1},\ z_{2n-1}) = (0,\ 0,\ 0)\,, \tag{21}$$

$$\lim_{n\to\infty} (x_{2n},\ y_{2n},\ z_{2n}) = (\infty,\ \infty,\ \infty)\,. \tag{22}$$

The above theorems describe the asymptotical behavior of solutions in case of the initial values existing in different intervals. At last, we describe the behavior in another way.

Corollary 6. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). If one of the following holds,*

(a) $1 < cb < ad$;
(b) $ad < cb < 1$;
(c) $ad < 1 < cb$ *and* $ad + cb < 2$;
(d) $cb < 1 < ad$ *and* $ad + cb > 2$

then all solutions of (8) *satisfy*

$$\lim_{n\to\infty} x_{2n}\, y_{2n-1} = cb\,, \tag{23}$$

$$\lim_{n\to\infty} x_{2n-1}\, y_{2n} = ad\,, \tag{24}$$

$$\lim_{n\to\infty} z_{2n-1}\, z_{2n} = 0. \tag{25}$$

Proof. In view of (9), (10) and (11), we have

$$\lim_{n\to\infty} x_{2n}\, y_{2n-1} = \lim_{n\to\infty} \left(c\,(cb-1)^n \times \frac{b}{(cb-1)^n} \right) = cb,$$

$$\lim_{n\to\infty} x_{2n-1}\, y_{2n} = \lim_{n\to\infty} \left(\frac{d}{(ad-1)^n} \times a(ad-1)^n \right) = ad.$$

As far as z_{2n-1} and z_{2n} are concerned, from (11) we could consider z_{4k+1} and z_{4k+2}, z_{4k+3} and z_{4k+4} for $n = k+1$, respectively. In fact, we have

$$z_{4k+1}\, z_{4k+2} = \frac{1}{a\,f\,(ad-1)^k} \times \frac{(cb-1)^{k+1}}{b\,e} = \frac{cb-1}{abef}\left(\frac{cb-1}{ad-1}\right)^k,$$

$$z_{4k+3}\, z_{4k+4} = \frac{f}{(ad-1)^{k+1}} \times e\,(cb-1)^{k+1} = ef\left(\frac{cb-1}{ad-1}\right)^{k+1}.$$

If one of the four conditions holds, we obtain $|(cb-1)/(ad-1)| < 1$ and the conclusion is apparent. ■

Corollary 7. *Suppose that the hypothesis H holds and let $\{x_n, y_n, z_n\}$ be a solution of the system (8). If one of the following holds,*

(a) $1 < ad < cb$;
(b) $cb < ad < 1$;
(c) $ad < 1 < cb$ *and* $ad + cb > 2$;
(d) $cb < 1 < ad$ *and* $ad + cb < 2$,

and $(cb-1)/ab > 0$, then all solutions of (8) satisfy

$$\lim_{n\to\infty} x_{2n}\, y_{2n-1} = cb, \tag{26}$$

$$\lim_{n\to\infty} x_{2n-1}\, y_{2n} = ad, \tag{27}$$

$$\lim_{n\to\infty} z_{2n-1}\, z_{2n} = \infty. \tag{28}$$

The proof is omitted here. In fact, we could obtain $|(ad-1)/(cb-1)| > 1$ if one of the four conditions holds and the condition of $(ad-1)/cd > 0$ is to keep the sign.

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KEYING LIU
SCHOOL OF ECONOMICS AND FINANCE
XI'AN JIAOTONG UNIVERSITY
XI'AN 710061, CHINA
AND

School of Mathematics
North China University
of Water Resources and Electric Power
Zhengzhou 450045, China
e-mail: liukeying@ncwu.edu.cn

Peng Li
School of Mathematics
North China University of Water Resources
and Electric Power
Zhengzhou 450045, China
e-mail: lipeng@ncwu.edu.cn

Weizhou Zhong
School of Economics and Finance
Xi'an Jiaotong University
Xi'an 710061, China
and
School of Business Administration
Huaqiao University
Quanzhou 362021, China
e-mail: weizhou@mail.xjtu.edu.cn

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