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OSCILLATION OF DIFFERENCE EQUATIONS WITH SEVERAL POSITIVE AND NEGATIVE COEFFICIENTS

Abstract. Our aim in this paper is to obtain sufficient conditions for the oscillation of every solution of first order difference equations

$$x_{n+1} - x_n + \sum_{i=1}^{m} (p_i x_{n-k_i} - q_i x_{n-l_i}) = 0, \quad n = 0, 1, 2, ...$$

and

$$x_{n+1} - x_n + \sum_{i=1}^{m} (p_i x_{n+k_i} - q_i x_{n+l_i}) = 0, \quad n = 0, 1, 2, ...$$

where $p_i, q_i \in \mathbb{R}^+$ and $k_i, l_i \in \mathbb{N}$ for $i = 1, 2, \ldots, m$.

Key words: difference equation, oscillation.

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1. Introduction

Consider the first order delay difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} (p_i x_{n-k_i} - q_i x_{n-l_i}) = 0, \quad n = 0, 1, 2, \ldots \tag{1}$$

and first order advanced difference equation

$$x_{n+1} - x_n + \sum_{i=1}^{m} (p_i x_{n+k_i} - q_i x_{n+l_i}) = 0, \quad n = 0, 1, 2, ... \tag{2}$$

where $p_i, q_i \in \mathbb{R}^+$ and $k_i, l_i \in \mathbb{N}$ for $i = 1, 2, ..., m$.

Recently there has been a lot of studies concerning the behavior of oscillatory of differential and difference equations, see $[1 - 15]$ and references cited therein. Ladas [9] only considered the case of which $q_i = 0$ $(i = 1, 2, \ldots, m)$

for equation (1). Furthermore, Ladas [10] (see also [4]) examined the oscillatory behavior of equation (1) in the case $m = 1$. Many authors have studied the oscillatory behavior of the some differential and difference equations with positive and negative coefficients (see, for instance [3, 11-15]. All these papers need the following condition in their main results;

$$q(k-l) \equiv 1. \tag{3}$$

In this paper, we obtain sufficient conditions for the oscillation of all solutions of the equations (1) and (2) which do not require condition (3).

Let $z = \max\{k_i, l_i\}$ for $i = 1, 2, \ldots, m$. By a solution of the equation (1) we mean a sequence $\{x_n\}$ which is defined $n \geq -z$ and which satisfies equation (1) for $n \geq 0$. A solution $\{x_n\}$ of equation (1) is said to be oscillatory if the terms x_n of the sequence $\{x_n\}$ are neither eventually positive nor eventually negative. Otherwise, the solution is called nonoscillatory.

2. Auxiliary lemmas

We need the following results, which proved in [4] (see also [5]).

Lemma 1. *We consider the following difference equation*

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} = 0\,, \quad n = 0, 1, 2, \ldots\ . \tag{4}$$

Suppose that either

$$p_i \in (0, \infty) \ \textit{and}\ k_i \in \{0, 1, 2, ...\} \ \textit{for}\ i = 1, 2, \ldots, m. \tag{5}$$

or

$$p_i \in (-\infty, 0) \ \textit{and}\ k_i \in \{\ldots, -2, -1\} \ \textit{for}\ i = 1, 2, \ldots, m. \tag{6}$$

Assume that either (5) or (6) holds and suppose that

$$\sum_{i=1}^{m} p_i \frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1.$$

Then every solution of equation (4) oscillates.

Lemma 2. *Assume that*

$$p_i \in (0, \infty) \ \textit{and}\ k_i \in \mathbb{N} \ \textit{for}\ i = 1, 2, ..., m.$$

Then the following statements are true.

a) The difference inequality

$$x_{n+1} - x_n + \sum_{i=1}^{m} p_i x_{n-k_i} \le 0\,, \quad n = 0, 1, 2, ...$$

has an eventually positive solution if and only if the difference equation

$$y_{n+1} - y_n + \sum_{i=1}^{m} p_i y_{n-k_i} = 0\,, \quad n = 0, 1, 2, ...$$

has an eventually positive solution.

b) The difference inequality

$$x_{n+1} - x_n - \sum_{i=1}^{m} p_i x_{n+k_i} \ge 0\,, \quad n = 0, 1, 2, ...$$

has an eventually positive solution if and only if the difference equation

$$y_{n+1} - y_n - \sum_{i=1}^{m} p_i y_{n+k_i} = 0\,, \quad n = 0, 1, 2, ...$$

has an eventually positive solution.

3. Sufficient condition for oscillation of (1)

In this section, we obtain sufficient conditions for the oscillation of equations (1) and (2). The conditions will be given in terms of the p_i, q_i and the k_i, l_i for each $i = 1, 2, \dots, m$.

Theorem 1. *Assume that for $i = 1, 2, \dots, m$,*

$$p_i > q_i \ge 0\,, \quad k_i \ge l_i \ge 0\,, \quad \sum_{i=1}^{m} q_i(k_i - l_i) \le 1 \tag{7}$$

and that for $i = 1, 2, \dots, m$,

$$\left.\begin{array}{ll} \sum\limits_{i=1}^{m} (p_i - q_i)\,\frac{(k_i+1)^{k_i+1}}{k_i^{k_i}} > 1 & \text{if} \quad k_i \ge 0 \\ \sum\limits_{i=1}^{m} (p_i - q_i) \ge 1 & \text{if} \quad k_i = 0 \end{array}\right\}. \tag{8}$$

Then every solution of equation (1) oscillates.

Proof. The case $k_i = l_i$, $i = 1, 2, ..., m$ follows from Lemma 1 immediately. We now consider the case $k_i > l_i$ for $i = 1, 2, \dots, m$. For the sake of contradiction, assume that equation (1) has an eventually positive solution $\{x_n\}$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n > 0$ for $n \geq n_0$.

Let

$$c_n = x_n - \sum_{i=1}^{m} q_i \sum_{j=l_i+1}^{k_i} x_{n-j}\,, \quad n \geq n_0 + k \tag{9}$$

where $k = \max_{1 \leq i \leq m} \{k_i\}$. Then

$$\begin{aligned} c_{n+1} - c_n &= x_{n+1} - x_n - \left[\sum_{i=1}^{m} q_i \left(x_{n-l_i} - x_{n-k_i} \right) \right] \\ &= \sum_{i=1}^{m} (q_i - p_i) x_{n-k_i} \end{aligned}$$

and so

$$c_{n+1} - c_n = \sum_{i=1}^{m} (q_i - p_i) x_{n-k_i} < 0\,, \quad n \geq n_0 + k\,. \tag{10}$$

Thus c_n is a decreasing sequence for $n \geq n_0 + k$. We prove that

$$L \cong \lim_{n \to \infty} c_n \in \mathbb{R} \tag{11}$$

Otherwise, $L = -\infty$ and $\{x_n\}$ must be unbounded. Hence there exists $n_1 \geq n_0 + k$ such that $x_{n_1} = \max\{x_n : n \leq n_1\}$ and $c_{n_1} < 0$. Then

$$0 > c_{n_1} = x_{n_1} - \sum_{i=1}^{m} q_i \sum_{j=l_i+1}^{k_i} x_{n_1-j} \geq x_{n_1} \left(1 - \sum_{i=1}^{m} q_i (k_i - l_i) \right) \geq 0,$$

which is a contradiction. Thus (11) holds. By taking limits in (10), it yields that

$$\lim_{n \to \infty} x_n = 0.$$

Since the sequence $\{c_n\}$ decreases to zero, we obtain

$$c_n > 0 \quad \text{for} \quad n \geq n_0 + k. \tag{12}$$

Also, by (9) we see that $c_n < x_n$ for $n \geq n_0 + k$, so (10) yields the inequality

$$c_{n+1} - c_n + \sum_{i=1}^{m} (p_i - q_i) c_{n-k_i} < 0 \quad \text{for} \quad n \geq n_0 + 2k \tag{13}$$

But in view of Lemma 1, Lemma 2 and the hypothesis (8), the difference inequality (13) cannot have an eventually positive solution, which contradicts (12) So the proof is completed. ∎

The proof of the following result follows from Theorem 1 and using arithmetic-geometric mean inequality.

Corollary 1. *Assume that (7) holds and suppose that for $i = 1, 2 \ldots, m$*

$$\left.\begin{array}{ll} m\left[\prod_{i=1}^{m}(p_i - q_i)\right]^{\frac{1}{m}} \frac{(k+1)^{k+1}}{k^k} > 1 & \text{if } \ k_i \geq 0 \\ m\left[\prod_{i=1}^{m}(p_i - q_i)\right]^{\frac{1}{m}} \geq 1 & \text{if } \ k_i = 0 \end{array}\right\} \tag{14}$$

where $k = \frac{1}{m}\sum_{i=1}^{m} k_i$.

Then every solution of equation (1) oscillates.

Theorem 2. *Assume that for $i = 1, 2, \ldots, m$*

$$0 \leq p_i < q_i\,, \quad 0 \leq k_i \leq l_i\,, \quad \sum_{i=1}^{m} p_i(l_i - k_i) \leq 1 \tag{15}$$

and that for $i = 1, 2, \ldots, m$

$$\left.\begin{array}{ll} \sum_{i=1}^{m}(q_i - p_i)\frac{(l_i-1)^{l_i-1}}{l_i^{l_i}} > 1 & \text{if } \ l_i \geq 0 \\ \sum_{i=1}^{m}(q_i - p_i) \geq 1 & \text{if } \ l_i = 0 \end{array}\right\}. \tag{16}$$

Then every solution of equation (2) oscillates.

Proof. The case $k_i = l_i$ for $i = 1, 2, \ldots, m$ follows from Lemma 1 immediately. So suppose $k_i < l_i$ for $i = 1, 2, \ldots, m$. For the sake of contradiction, assume that equation (2) has an eventually positive solution $\{x_n\}$. Let

$$c_n = x_n - \sum_{i=1}^{m} p_i \sum_{j=n+k_i}^{n+l_i-1} x_n. \tag{17}$$

Then

$$\begin{aligned} c_{n+1} - c_n &= x_{n+1} - x_n - \left[\sum_{i=1}^{m} p_i\left(x_{n+l_i} - x_{n+k_i}\right)\right] \\ &= \sum_{i=1}^{m}(q_i - p_i)\, x_{n+l_i} > 0. \end{aligned} \tag{18}$$

Thus $\{c_n\}$ is eventually increasing and either

$$\lim_{n\to\infty} c_n = \infty \tag{19}$$

or

$$\lim_{n\to\infty} c_n = c \in \mathbb{R}\,. \tag{20}$$

Assume that (20) holds. Then by (18) and (17) we see that

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} c_n.$$

Hence there eixsts an index n such that

$$c_{n_1} < 0 \quad \text{and} \quad x_{n_1} \geq x_n > 0 \ \text{ for } n \geq n_1.$$

Then (17) yields

$$\begin{aligned} 0 \, > \, c_{n_1} \, &= \, x_{n_1} - \sum_{i=1}^{m} p_i \sum_{j=n_1+k_i}^{n_1+l_i-1} x_{n_1} \\ &\geq \, x_{n_1} \left[1 - \sum_{i=1}^{m} p_i \left(l_i - k_i \right) \right] \geq 0, \end{aligned}$$

which is a contradiction. Therefore (19) holds. By (18) and (17) we find

$$c_{n+1} - c_n - \sum_{i=1}^{m} \left(p_i - q_i \right) c_{n+l_i} \geq 0.$$

Also $c_n > 0$. In view of Lemma 2 this implies that the difference equation

$$y_{n+1} - y_n - \sum_{i=1}^{m} \left(p_i - q_i \right) y_{n+l_i} = 0$$

has an eventually positive solution. This contradicts (16) and completes the proof. ■

The proof of the following result follows from Theorem 2 and using arithmetic-geometric mean inequality.

Corollary 2. *Assume that (16) holds and suppose that for* $i = 1, 2, \ldots, m$

$$\left.\begin{aligned} m \left[\prod_{i=1}^{m} \left(q_i - p_i \right) \right]^{\frac{1}{m}} \frac{(l_i-1)^{l_i-1}}{l_i^{l_i}} > 1 \quad &\textit{if} \quad l_i \geq 0 \\ m \left[\prod_{i=1}^{m} \left(q_i - p_i \right) \right]^{\frac{1}{m}} \geq 1 \qquad\qquad &\textit{if} \quad l_i = 0 \end{aligned}\right\}.$$

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