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FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACES

Abstract. This paper presents some fixed point theorems for six occasionally weakly compatible maps in fuzzy metric spaces.

Key words: Occasionally weakly compatible mappings, fuzzy metric space.

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1. Introduction

The concept of Fuzzy sets was introduced initially by Zadeh in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets. Both George and Veermani (1994) modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki (1999) proved fixed point theorems for R-weakly commutating mappings. Pant (1998) introduced the new concept reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam et al (2002) have shown that Rhoades open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha (2004) obtained some analogous results proved by Balasubramaniam et al (2002). Recently many authors have also studied the fixed point theory in fuzzy metric spaces. This paper presents some common fixed point theorems for more general commutative condition i.e. occasionally weakly compatible mappings in fuzzy metric space.

2. Preliminary Notes

Definition 1. *A fuzzy set A in X is a function with domain X and values in $[0, 1]$.*

Definition 2. *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norms if* $*$ *is satisfying conditions:*

1. $*$ *is an commutative and associative;*
2. $*$ *is continuous;*
3. $a * 1 = a$ *for all* $a \in [0,1]$;
4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Definition 3. *A 3-tuple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set of* $X^2 \times (0, \infty)$ *satisfying the following conditions, for all* $x, y, z \in X$, $s, t > 0$,

(a) $M(x, y, t) > 0$;
(b) $M(x, y, t) = 1$ *if and only if* $x = y$;
(c) $M(x, y, t) = M(y, x, t)$;
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Then M *is called a fuzzy metric on* X. *Then* $M(x, y, t)$ *denotes the degree of nearness between* x *and* y *with respect to* t.

Example 1. *Let* (X, d) *be a metric space. Define* $a * b = ab$ *or* $a * b = \min(a, b)$ *for all* $x, y \in X$ *and* $t > 0$.

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ *is a fuzzy metric space and the fuzzy metric* M *induced by the metric* d *is often referred to as the standard fuzzy metric.*

Definition 4. *Let* $(X, M, *)$ *be a fuzzy metric space. Then*

(a) *a sequence* x_n *in* X *is said to converges to* x *in* X *if for each* $\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \varepsilon$ *for all* $n \geq n_0$.
(b) *a sequence* x_n *in* X *is said to be Cauchy if for each* $\varepsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \varepsilon$ *for all* $n, m \geq n_0$.
(c) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 5. *A pair of self-mappings* (f, g) *of a fuzzy metric space* $(X, M, *)$ *is said to be*

(i) *Weakly commuting if* $M(fgx, gfx, t) \geq M(fx, gx, t)$ *for all* $x \in X$ *and* $t > 0$.
(ii) *R-weakly commuting if there exists some* $R > 0$ *such that* $M(fgx, gfx, t) \geq M(fx, gx, t/R)$ *for all* $x \in X$ *and* $t > 0$.

Definition 6. *Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim\limits_{n \to \infty} M(fgx_n, gfx_n, t) = 1$ whenever (x_n) is a sequence in X such that $\lim\limits_{n \to \infty} fx_n = \lim\limits_{n \to \infty} gx_n = x$ for some x in X.*

Definition 7. *Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called reciprocally continuous on X if $\lim\limits_{n \to \infty} fgx_n = fx$ and $\lim\limits_{n \to \infty} gfx_n = gx$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n \to \infty} fgx_n = \lim\limits_{n \to \infty} gfx_n = x$ for some x in X.*

Lemma 1. *Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that $M(x, y, qt) > M(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.*

Definition 8. *Let X be a set self maps of X. A point in X is called a coincidence point of f and g if $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g.*

Definition 9. *A pair of maps S and T is called weakly compatible pair if they commute at coincidence points. The concept occasionally weakly compatible is introduced by Thagafi and Shahzad (2008). It is stated as follows.*

Definition 10. *Two self maps f and g of a set X are occasionally weakly compatible (owc) if there is a point x in X which is a coincidence point of f and g at which f and g commute. Thagafi and Shahzad (2008) shows that occasionally weakly is weakly compatible but converse is not true.*

Example 2. Let R be the usual metric space. Define $S, T : R \to R$ by $Sx = 2x$ and $Tx = x^2$ for all $x \in R$. Then $Sx = Tx$ for $x = 0, 2$ but $ST0 = TS0$ and $ST2 \neq TS2$.

Lemma 2. *Let X be a set, f, g owc self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Main results

Theorem 1. *Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S, T, P and Q be self-mappings of X. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc. If there exists $q \in (0, 1)$ such that*

(1) $M\left(Px, Qy, qt\right)$
$$\geq \phi \left[\begin{array}{l} \min\left\{M\left(STx, ABy, t\right), M\left(STx, Px, t\right)\right\} * \\ \min\left\{M\left(Qy, ABy, t\right), M\left(Px, ABy, t\right), M\left(Qy, STx, t\right)\right\} \end{array} \right]$$

for all $x, y \in X$ and $\phi : [0, 1] \longrightarrow [0, 1]$ such that $\phi(t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S, T, P and Q.

Proof. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc, so there are points $x, y \in X$ such that $Px = STx$ and $Qy = ABy$. We claim that $Px = Qy$. If not, by inequality (1)

$$
\begin{aligned}
M\left(Px, Qy, qt\right) & \\
& \geq \phi \left[\begin{array}{l} \min\left\{M\left(STx, ABy, t\right), M\left(STx, Px, t\right)\right\} * \\ \min\left\{M\left(Qy, ABy, t\right), M\left(Px, ABy, t\right), M\left(Qy, STx, t\right)\right\} \end{array} \right] \\
& = \phi \left[\begin{array}{l} \left\{\min M\left(Px, Qy, t\right), M\left(Px, Px, t\right)\right\} * \\ \min\left\{M\left(Qy, Qy, t\right), M\left(Px, Qy, t\right), M\left(Qy, Px, t\right)\right\} \end{array} \right]
\end{aligned}
$$

$$
\phi\left[M\left(Px, Qy, t\right)\right] > \left[M\left(Px, Qy, t\right)\right].
$$

Therefore $Px = Qy$, i.e. $Px = STx$ and $Qy = ABy$. Suppose that there is another point z such that $Pz = STz$ then by (1) we have $Pz = STz = Qy = ABy$, so $Px = Pz$ and $w = Px = STx$ is the unique point of coincidence of P and ST. By Lemma 2 w is the only common fixed point of P and ST. Similarly there is a unique point $z \in X$ such that $z = Qz = ABz$. Assume that $w \neq z$. We have

$$
\begin{aligned}
M\left(w, z, qt\right) & = M(Pw, Qz, qt) \\
& \geq \phi \left[\begin{array}{l} \left\{\min M\left(STw, ABz, t\right), M\left(STw, Pz, t\right)\right\} * \\ \min\left\{M\left(Qz, ABz, t\right), M\left(Pw, ABz, t\right), M\left(Qz, STw, t\right)\right\} \end{array} \right] \\
& = \phi \left[\begin{array}{l} \min\left\{M\left(w, z, t\right), M\left(w, z, t\right)\right\} * \\ \left\{M\left(z, z, t\right), M\left(w, z, t\right), M\left(z, w, t\right)\right\} \end{array} \right] \\
& = \phi\left[M(w, z, t)\right] > M(w, z, t).
\end{aligned}
$$

Therefore we have $z = w$ by Lemma 2 and z is a common fixed point of A, B, S, T, P and Q. The uniqueness of the fixed point holds from (1). ∎

Theorem 2. *Let $(X, M, *)$ be a complete fuzzy metric space A, B, S, T, P and Q let and be self-mappings of X. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc. If there exists $q \in (0, 1)$ such that*

$$
\text{(2)} \quad M\left(Px, Qy, qt\right)
$$
$$
\geq \left[\begin{array}{l} \min\left\{M\left(STx, ABy, t\right), M\left(STx, Px.t\right), M(Qy, ABy, t)\right\} * \\ \min\left\{(Px, ABy, t\right), M\left(Qy, STx, t\right)\right\} \end{array} \right]
$$

for all $x, y \in X$ and for all $t > 0$, then there exists a unique point $w \in X$ such that $Pw = STw = w$ and a unique point $z \in X$ such that $Qz = ABz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S, T, P and Q.

Proof. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc, so there are points $x, y \in X$ such that $Px = STx$ and $Qy = ABy$. We claim that $Px = Qy$. If not, by inequality (2)

$$M\left(Px, Qy, qt\right)$$
$$\geq \left[\begin{array}{l} \min\left\{M\left(STx, ABy, t\right), M\left(STx, Px.t\right), M(Qy, ABy, t)\right\} \\ *\min\left\{M\left(Px, ABy, t\right), M\left(Qy, STx, t\right)\right\} \end{array} \right]$$
$$= \left[\begin{array}{l} \min\left\{M\left(Px, Qy, t\right), M\left(Px, Px.t\right), M(Qy, Qy, t)\right\} \\ *\min\left\{M\left(Px, Qy, t\right), M\left(Qy, Px, t\right)\right\} \end{array} \right]$$
$$= M(Px, Qy, t).$$

Therefore, $Px = Qy$ i.e. $Px = STx = Qy = ABy$. Suppose that there is another point z such that $Pz = STz$ then by (2) we have $Pz = STz = Qy = ABy$, so $Px = Pz$ and $w = Px = STx$ is the unique point of coincidence of P and ST. By Lemma 2 w is the only common fixed point of P and ST. Similarly there is a unique point $z \in X$ such that $z = Qz = ABz$. Assume that

$$w \neq z.$$

We have

$$M\left(w, z, qt\right) = M(Pw, Qz, qt)$$
$$\geq \left[\begin{array}{l} \min\left\{M\left(STx, ABz, t\right), M\left(STx, Pz, t\right), M(Qz, ABz, t)\right\} * \\ \min\left\{M\left(Px, ABz, t\right), M\left(Qz, STw, t\right)\right\} \end{array} \right]$$
$$= \left[\begin{array}{l} \min\left\{M\left(w, z, t\right), M\left(w, z, t\right), M(z, z, t)\right\} * \\ \min\left\{M\left(w, z, t\right), M\left(z, w, t\right)\right\} \end{array} \right] = M(w, z, t).$$

∎

Theorem 3. *Let $(X, M, *)$ be a complete fuzzy metric space A, B, S, T, P and Q let and be self-mappings of X. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc. If there exists $q \in (0, 1)$ such that*

$$(3) \qquad M\left(Px, Qy, qt\right) \geq \phi\left\{ \begin{array}{l} M(STx, ABy, t), M(STx, Qy, t), \\ M(Qy, ABy, t), M(Px, ABy, t) \end{array} \right\}$$

for all $x, y \in X$ and $\phi : [0, 1]^4 \longrightarrow [0, 1]$ such that $\phi(t) >)t, t, 1, t)$ for all $0 < t < 1$,then there exists a unique common fixed point of A, B, S, T, P and Q.

Proof. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ are owc, there are points point $x, y \in X$ such that $Px = STx$ and $Qy = ABy$. We claim that $Px = Qy$. By inequality (3) we have

$$M\left(Px, Qy, qt\right)$$
$$\geq \phi\left\{ \begin{array}{l} M(STx, ABy, t), M(STx, Qy, t), \\ M(Qy, ABy, t), M(Px, ABy, t) \end{array} \right\}$$

$$= \phi \left\{ M(Px, Qy, t), M(Px, Qy, t), M(Qy, Qy, t), M(Px, Qy, t) \right\}$$
$$= \phi \left\{ M(Px, Qy, t), M(Px, Qy, t), 1, M(Px, Qy, t) \right\}$$
$$> M(Px, Qy, t),$$

a contradiction, therefore $Px = Qy$, i.e. $Px = STx = Qy = ABy$. Suppose that there is $*$ another point z such that $Pz = STz$ then by (3) we have $Pz = STz = Qy = ABy$, so $Px = Pz$ and $w = Px = ABx$ is the unique point of coincidence of P and AB. By Lemma 2 w is a unique common fixed point of P and ST. Similarly there is a unique point $z \in X$ such that $z = Qz = ABz$. Thus z is a common fixed point of A, B, S, T, P and Q. The uniqueness of the fixed point holds from (3).

$$M(w, z, qt) = M(Pw, Qw, qt)$$
$$\geq \phi \left\{ M(STw, ABz, t), M(STw, Qz, t)M(Qz, ABz, t), \right.$$
$$\left. M(Pw, ABz, t) \right\}$$
$$= \phi \left\{ M(w, z, t), M(w, z, t)M(z, z, t), M(w, z, t) \right\} > M(w, z, t).$$

∎

Theorem 4. *Let $(X, M, *)$ be a complete fuzzy metric space A, B, S, T, P and Q let and be self-mappings of X. Let the pairs $\{P, ST\}$ and $\{Q, AB\}$ be owc. If there exists $q \in (0, 1)$ such that*

$$(4) \qquad M(Px, Qy, qt)$$
$$\geq \phi \left\{ \begin{array}{l} M(STx, ABy, t), M(STx, Qy, t), M(Qy, ABy, t), \\ M(Px, Qy, t), M(STx, Px, t), M(Px, ABy, t) \end{array} \right\}$$

for all $x, y \in X$ and $\phi : [0, 1]^6 \longrightarrow [0, 1]$ such that $\phi(t) > (t, t, 1, t, 1, t)$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S, T, P and Q.

Proof. Let the pairs P, ST and Q, AB are owc, there are points point $x, y \in X$ such that $Px = STx$ and $Qy = ABy$. We claim that $Px = Qy$. By inequality (4) we have

$$M(Px, Qy, qt)$$
$$\geq \phi \left\{ \begin{array}{l} M(STx, ABy, t), M(STx, Qy, t), M(Qy, ABy, t), \\ M(Px, Qy, t), M(STx, Px, t), M(Px, ABy, t) \end{array} \right\}$$
$$= \phi \left\{ \begin{array}{l} M(Px, Qy, t), M(Px, Qy, t), M(Qy, Qy, t), \\ M(Px, Qy, t), M(Px, Px, t), M(Px, Qy, t) \end{array} \right\}$$
$$= \phi \left\{ M(Px, Qy, t), M(Px, Qy, t), 1, M(Px, Qy, t), \right.$$
$$\left. 1, M(Px, Qy, t) \right\} > M(Px, Qy, t).$$

a contradiction, therefore $Px = Qy$, i.e. $Px = STx = Qy = ABy$. Suppose that there is $*$ another point z such that $Pz = STz$ then by (4) we have $Pz = STz = Qy = ABy$, so $Px = Pz$ and $w = Px = ABx$ is the unique point of coincidence of P and AB. By Lemma 2 w is a unique common fixed point of P and ST. Similarly there is a unique point $z \in X$ such that $z = Qz = ABz$.

Thus z is a common fixed point of A, B, S, T, P and Q. The uniqueness of the fixed holds from (4). ∎

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