FASCICULI MATHEMATICI

Nr 51 2013

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SOME GENERALIZED SPACES OF VECTOR VALUED DOUBLE SEQUENCES DEFINED BY A MODULUS

Abstract. In this paper we generalize the χ_f^2 by introducing the sequence space $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ and exhibit some general properties of the space.

Key words: gai sequence, analytic sequence, modulus function, double sequences, difference sequence,vector valued.

AMS Mathematics Subject Classification: 40A05,40C05,40D05.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], BaŞarir and Solancan [2], Tripathy [17], Türkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - \lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case

$t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{0p}$, $\mathcal{L}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and BaŞar [27] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently BaŞar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. This will be accomplished by presenting the following sequence spaces:

$$\chi^2_{f_{mn}}\left(\Delta^{\mu}_{(\eta\gamma)}, p, q, r\right) = \Big\{x \in w^2(X)$$
$$:= \lim_{m,n\to\infty} (mn)^{-r}\left[f_{mn}\Big(q\Big(\,(m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\Big)^{1/(m+n)} Big)\right]^{p_{mn}} = 0\Big\}$$

and

$$\Delta^2_{f_{mn}}\left(\Delta^{\mu}_{(\eta\gamma)}, p, q, r\right) = \Big\{x \in w^2(X)$$
$$:= \sup_{mn}(mn)^{-r}\left[f_{mn}\Big(q\Big(\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|^{1/(m+n)}\Big)\Big)\right]^{p_{mn}} < \infty\Big\}$$

where f is a modulus function. Other implications,general properties and variations will also be presented.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} \|x_{mn}\|^{1/(m+n)} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\|x_{mn}\|)^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{allfinitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], BektaŞ and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty t) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)(u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell ut) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t = t^p(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p. If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X;

(ii) $X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\}$;

(iii) $X^\beta = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convegent, for each } x \in X \right\}$;

(iv) $X^\gamma = \left\{ a = (a_{mn}) : \sup_{M,N \geq 1} \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, \text{for each } x \in X \right\}$;

(v) let X be an FK-space $\supset \phi$; then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;

(vi) $X^\delta = \left\{ a = (a_{mn}) : \sup_{mn} |a_{mn}x_{mn}|^{1/(m+n)} < \infty, \text{ for each } x \in X \right\}$;

X^α, X^β, X^γ are called α-(or Köthe-Toeplitz) dual of X, β-(or generalized-Köthe-Toeplitz) dual of X, γ-dual of X, δ-dual of X respectively. X^α is defined by Gupta and Kamptan [20]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\beta \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ by BaŞar and Altay in [42] and in the case $0 < p < 1$ by Altay and BaŞar in [43]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + \sup_{k\ge 1} |\Delta x_k| \quad \text{and} \quad \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, \quad (1 \le p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. Definitions and preliminaries

$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ and $\Lambda^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ denote the Pringscheims sense of vector valued difference double gai sequence space of modulus and Pringscheims sense of vector valued difference double analytic sequence space of modulus respecctively.

By $w^2(X)$, we shall denote the space of all $X-$ valued sequences , where (X, q) is a semi normed space, semi normed by q. For $X = \mathbb{C}$, the space of complex numbers, these reduce to the corresponding scalar valued sequence spaces. The zero sequence is denoted by

$$\bar{\theta} = \begin{pmatrix} \theta, & \theta, & \cdots\theta \\ \theta, & \theta, & \cdots\theta \\ . & & \\ . & & \\ . & & \\ \theta, & \theta, & \cdots\theta \\ & & \end{pmatrix}, \quad \text{where } \theta \text{ is the zero element of } X.$$

Let γ and μ be non-negative integers and $\eta = (\eta_{mn})$ be a sequence of non-zero scalars. Also let $Z \in \left\{\Lambda^2, \chi^2\right\}$ and define the following sequence spaces

$$Z\left(\Delta^{\mu}_{\eta\gamma} x_{mn}\right) = \left\{x = (x_{mn}) \in w^2 : \left(\Delta^{\mu}_{\eta\gamma} x_{mn}\right) \in Z\right\},$$

where $\Delta^{\mu}_{\eta\gamma} x_{mn} = \Delta_{\eta\gamma}\Delta^{\mu-1}_{\eta\gamma} x_{mn} = \Delta^{\mu-1}_{\eta\gamma} x_{mn} - \Delta^{\mu-1}_{\eta\gamma} x_{mn+1} - \Delta^{\mu-1}_{\eta\gamma} x_{m+1n} + \Delta^{\mu-1}_{\eta\gamma} x_{m+1n+1}$ and $\Delta^{0}_{\eta\gamma} x_{mn} = \eta_{mn} x_{mn}$ for all $m, n \in \mathbb{N}$.

In this expansion we take $\eta_{mn} = 0$ and $x_{mn} = 0$ for non-positive values of $m, n \in \mathbb{N}$.

Definition 1. *A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0,\infty)$ to itself such that*

(a) $f(x)=0$ if and only if $x=0$

(b) $f(x+y)\le f(x)+f(y)$, for all $x\ge 0$, $y\ge 0$,

(c) f is increasing,

(d) f is continuous from the right at 0. Since $|(x)-f(y)|\le f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

Definition 2. *Let $f=(f_{mn})$ be a sequence of modulus functions, X be a semi normed space with semi norm $q, p=(p_{mn})$ be a sequence of positive real numbers and $\eta=(\eta_{mn})$ be a fixed sequence of non-zero scalars. Then for non-negative real numbers r,γ and μ, we define*

$$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)},p,q,r)=\Big\{x\in w^2(X)$$
$$:=\lim_{m,n\to\infty}(mn)^{-r}\left[f_{mn}\Big(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\Big)\right]^{p_{mn}}=0\Big\}$$

$$\Delta^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)},p,q,r)=\Big\{x\in w^2(X)$$
$$:=\sup_{mn}(mn)^{-r}\left[f_{mn}\Big(q\Big(\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|^{1/(m+n)}\Big)\Big)\right]^{p_{mn}}<\infty\Big\}.$$

Considering $X=\mathbb{C}$, $q(x)=|x|$, $p_{mn}=1$, $\eta_{mn}=1$ for all $m,n\in\mathbb{N}$; $r=0$ and $\mu=0$, we get the spaces of $\chi^2_{f_{mn}}$ and $\Lambda^2_{f_{mn}}$ respectively.

The space $\chi^2_{f_{mn}}$ is a metric space with the metric

$$d(x,y)=\inf\Big\{\sup\Big((mn)^{-r}\Big[f_{mn}\Big(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}-\Delta^{\mu}_{(\eta\gamma)}y_{mn}\right|\right)^{1/(m+n)}\Big)\Big]^{p_{mn}}\Big)\le 1\Big\}$$

and $\Lambda^2_{f_{mn}}$ is a metric space with the metric

$$d(x,y)=\inf\Big\{\sup\Big((mn)^{-r}\Big[f_{mn}\Big(q\left(\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}-\Delta^{\mu}_{(\eta\gamma)}y_{mn}\right|\right)^{1/(m+n)}\Big)\Big]^{p_{mn}}\Big)\le 1\Big\}$$

Definition 3. *Let $A=(a^{mn}_{k,\ell})$ be a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ell-$ th term of Ax is as follows:*

$$(Ax)_{k\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a^{mn}_{k\ell}x_{mn}$$

such transformation is said to be nonnegative if $a^{mn}_{k\ell}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

3. Main results

Theorem 1. *Let $p = (p_{mn})$ be a analytic double sequence of strictly positive reals, then $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ is a linear space over the filed $\mathbb{C}$.*

Proof. It is easy. Therefore omit the proof. ∎

Theorem 2. *$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ is a paranormed space (need not total paranorm) space with paranorm g, defined as follows:*

$$g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} (mn)^{-r} \times \left[f_{mn}\left(q\left((m+n)! \left| \Delta^{\mu}_{(\eta\gamma)} x_{mn} \right| \right)^{1/(m+n)} \right) \right]^{p_{mn}/M}. \tag{2}$$

if and only if $\mu > 0$, where $\mu = \lim_{n\to\infty} \inf_{mn\geq N} p_{mn}$ and $M = \max(1, \sup_{mn\geq N} p_{mn})$

Proof. *Necessity.* Let $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ be a paranormed space with (3.1) and suppose that $\mu = 0$.Then $\alpha = inf_{mn\geq N} p_{mn} = 0$ for all $N \in \mathbb{N}$ and hence we obtain $g(\lambda x) = \lim_{N\to\infty} \sup_{mn\geq N} |\lambda|^{p_{mn}/M} = 1$ for all $\lambda \in (0,1]$, where $x = (\alpha) \in \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ whence $\lambda \to 0$ does not imply $\lambda x \to \theta$, when x is fixed. But this contradicts (2) to be a paranorm.

Sufficiency. Let $\mu > 0$. It is trivial that $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x+y) \leq g(x) + g(y)$. Since $\mu > 0$ there exists a positive number β such that $p_{mn} > \beta$ for sufficiently large positive integer m, n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{p_{mn}} \leq \max(|\lambda|^M, |\lambda|^\beta) g(x)$ using this, one can prove that $\lambda x \to \theta$, whenever x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$. ∎

Theorem 3. *Let $f = (f_{mn})$ and $s = (s_{mn})$ be two sequences of modulus functions. Then*

$$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r) \bigcap \chi^2_{s_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r) \subseteq \chi^2_{(f+s)_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$$

Proof. The proof is easy, so omitted. ∎

Remark 1. Let $f = (f_{mn})$ be a sequence of modulus functions q_1 and q_2 be two semi norm on X, we have

(i) $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_1, r) \bigcap \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_2, r) \subseteq \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_1 + q_2, r)$.

(ii) If q_1 is stronger than q_2 then $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_1, r) \subseteq \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_2, r)$.

(iii) If q_1 is equivalent to q_2 then $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_1, r) = \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q_2, r)$.

Theorem 4. *Let $f = (f_{mn})$ and $s = (s_{mn})$ be the sequences of modulus functions. If $f_{mn} \approx s_{mn}$ for each $m, n \in \mathbb{N}$, then $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r) = \chi^2_{s_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$.*

Proof. It is obvious. ∎

Theorem 5. *Let $f = (f_{mn})$ be a sequence of modulus functions. If $\lim_{t\to 0} \frac{f_{mn}(t)}{t} > 0$ and $\lim_{t\to 0} \frac{f_{mn}(t)}{t} < \infty$ for each $m, n \in \mathbb{N}$, then*

$$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r) = \chi^2(\Delta^{\mu}_{(\eta\gamma)}, p, q_1, r).$$

Proof. If the given conditions are satisfied, we have $f_{mn} \approx t$ for each m, n.

If we take $r = 0$, the sequence space $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q, r)$ reduces to the following sequence space

$$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q) = \Big\{x \in w^2(X) := \lim_{m,n\to\infty} \Big[f_{mn}\Big(q\left((m+n)! \left|\Delta^{\mu}_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\Big)\Big]^{p_{mn}} = 0\Big\}.$$

∎

Proposition 1. *$\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$ is a BK-space.*

Definition 4. *For any sequence $f = (f_{mn})$ of modulus functions,*

$$h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q) = \Big\{x \in w^2(X) : \lim_{m,n\to\infty} \Big[f_{mn}\Big(q\left((m+n)! \left|\Delta^{\mu}_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\Big)\Big] = 0\Big\}.$$

Clearly $h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$ is a subspace of $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$. The topology of $h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$ is the one it inherits from the metric $d(x, y)$.

Proposition 2. *Let $f = (f_{mn})$ be a sequence of modulus functions which satisfies the Δ_2- condition. Then $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q) = h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$.*

Proof. It is enough to prove that $\chi^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q) \subseteq h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$. Let $x \in h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$. Then $\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right] \le \epsilon$ for sufficiently large m, n and every $\epsilon > 0$.

$$\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right] \le f_{mn}(\epsilon)$$

(because f_{mn} is non-decreasing)

$$\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right] \le K f_{mn}(\epsilon)$$

(by the Δ_2 - condition, for some $K > 0$)

$\le \epsilon$ (by defining $f_{mn}(\epsilon) < \epsilon K$)

$$\lim_{m,n\to\infty}\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right] = 0.$$

∎

Proposition 3. *Let (X, q) be a complete metric space, then $h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ is an AK-space.*

Proof. Let $x = (x_{mn}) \in h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ and take the $[m, n]^{th}$ sectional sequence of x. We have

$$d\left(x, x^{[r,s]}\right) = \inf\left\{\sup\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right] \le 1 : m \ge r, \ \ n \ge s\right\} \to 0$$

as $r, s \to \infty$. Therefore $x^{[r,s]} \to x$ in $h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ as $r, s \to \infty$. Thus $h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ has AK. ∎

Combining Proposition 2 and Proposition 3, we have the following Theorem.

Theorem 6. *Let $f = (f_{mn})$ be a sequence of modulus functions which satisfy Δ_2-condition, then $\chi^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ is an AK-space.*

Proposition 4. *$h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ is a closed subspace of $\chi^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$.*

Proof. Let $\left\{x^{[rs]}\right\}$ be a sequence of in $h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$ such that $d\left(x, x^{[rs]}\right) \to 0$ as $r, s \to \infty$, where $x \in h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$. To complete the proof we need to prove that $x \in h^2_{f_{mn}}(\Delta^\mu_{(\eta\gamma)}, q)$,

$$\text{(i.e)} \quad \lim_{m,n\to\infty}\left[f_{mn}\left(q\left((m+n)!\left|\Delta^\mu_{(\eta\gamma)} x_{mn}\right|\right)^{1/m+n}\right)\right] = 0.$$

There corresponds on $[i,j]$ such that $d\left(x, x^{[i,j]}\right) < \frac{1}{2}$. Then using convexity of each f_{mn},

$$\begin{aligned}
&\left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)\right] \\
&= \left[f_{mn}\left(q\left(\frac{2\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{[ij]}_{mn}\right|\right)^{1/(m+n)}}{2}\right.\right.\right. \\
&\qquad -\frac{2\left(\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{[ij]}_{mn}\right|\right)^{1/(m+n)}\right.}{2} \\
&\qquad \left.\left.\left.\left. -\frac{\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)}{2}\right)\right)\right] \\
&\le \frac{1}{2}\left[f_{mn}\left(q\left(2\,(m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{[rs]}_{mn}\right|\right)^{1/(m+n)}\right)\right] \\
&\quad + \frac{1}{2}\left[f_{mn}\left(q\left(2\,(m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}\left(x^{[rs]}_{mn}-x_{mn}\right)\right|\right)^{1/(m+n)}\right)\right] \\
&\le \frac{1}{2}\left[f_{mn}\left(q\left(2\,(m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{[rs]}_{mn}\right|\right)^{1/(m+n)}\right)\right] \\
&\quad + \frac{1}{2}\left[f_{mn}\left(q\left(\frac{\left(2\,(m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}\left(x^{[rs]}_{mn}-x_{mn}\right)\right|\right)^{1/(m+n)}}{d\left(x, x^{[i,j]}\right)}\right)\right)\right].
\end{aligned}$$

Now from Theorem 7, using the definition of metric, we have

$$\left[f_{mn}\left(q\left(\frac{\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}\left(x^{[rs]}_{mn}-x_{mn}\right)\right|\right)^{1/(m+n)}}{d\left(x, x^{[i,j]}\right)}\right)\right)\right] \le 1.$$

It follows that

$$\lim_{m,n\to\infty}\left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)\right] = 0.$$

Thus $x \in h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$. ■

Corollary 1. *$h^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, q)$ is a BK-space.*

Theorem 7. *Let*

$$N_1 = \min\left\{n_0 : \sup_{m,n\geq n_0} \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}} < \infty\right\}$$

$$N_2 = \min\left\{n_0 : \sup_{m,n\geq n_0} p_{mn} < \infty\right\}$$

and $N = \max(N_1, N_2)$. *Let* $p = (p_{mn})$ *be double analytic sequence of positive reals and* (X, q) *be a complete seminormed space, then* $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q)$ *is a complete paranormed*

$$(3) \quad g(x) = \lim_{N\to\infty} \sup_{m,n\geq N} \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M}.$$

Proof. Let $(\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn})$ be a Cauchy sequence in $\chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q)$, where $(\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}) = (\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn})_{m,n\in\mathbb{N}}$. Then for every $\epsilon > 0$ $(0 < \epsilon < 1)$ there exists positive integer s_0 such that

$$(4) \quad g(x^{k,\ell} - x^{rt}) = \lim_{N\to\infty} \sup_{m,n\geq N} \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} < \epsilon/2$$

for all $k, \ell, r, t > s_0$.

By (4) there exists a positive integer n_0 such that

$$\sup_{m,n\geq N} \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)} x^{rt}_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} < \epsilon/2$$

for all $k, \ell, r, t > s_0$, and for $n > n_0$. Hence we obtain

$$(5) \quad \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)} x^{rt}_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} < \epsilon/2 < 1$$

so that

$$\begin{aligned} &\left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)} x^{rt}_{mn}\right|\right)^{1/m+n}\right)\right] \\ &\quad < \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)} x^{rt}_{mn}\right|\right)^{1/m+n}\right)\right]^{p_{mn}/M} \\ &\quad < \epsilon/2 \ \text{ for all } \ k, \ell, r, t > s_0. \end{aligned}$$

This implies that $(\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn})_{k\ell\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m, n > n_0$. Hence the sequence $(\Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn})_{k\ell\in\mathbb{N}}$ is convergent to (Δx_{mn}) say,

$$(6) \qquad \lim_{k\ell\to\infty} \Delta^{\mu}_{(\eta\gamma)} x^{k\ell}_{mn} = (\Delta^{\mu}_{(\eta\gamma)} x_{mn}) \ \text{ for each fixed } \ m, n > n_0.$$

Getting (Δx_{mn}), we define $x = (\Delta x_{mn})$. From (4) we obtain

$$gt\left(x^{k,\ell} - x\right) = \lim_{N\to\infty} \sup_{m,n\geq N} \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} < \epsilon/2. \tag{7}$$

as $r, t \to \infty$ for all $k, \ell > s_0$, by (6). This implies that $\lim_{k,\ell\to\infty} \Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn} = \Delta^{\mu}_{(\eta\gamma)}x_{mn}$.

Now we show that $x = (x_{mn}) \in \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q)$. Since $\Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn} \in \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q)$ for each $(k, 1) \in \mathbb{N}\times\mathbb{N}$ for every $\epsilon > 0$ $(0 < \epsilon < 1)$ there exists a positive integer $n_1 \in \mathbb{N}$ such that

$$\left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} < \epsilon/2 \tag{8}$$

for every $m, n > n_1$.

From (6) and (7) in (1) we obtain

$$\begin{aligned}
&\left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} \\
&\quad\leq \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn}\right|\right)^{1/m+n}\right)\right]^{p_{mn}/M} \\
&\qquad + \left[f_{mn}\left(q\left((m+n)!\left|\Delta^{\mu}_{(\eta\gamma)}x^{k\ell}_{mn} - \Delta^{\mu}_{(\eta\gamma)}x_{mn}\right|\right)^{1/(m+n)}\right)\right]^{p_{mn}/M} \\
&\quad< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

for all $k\ell > \max(s_0, s_1)$ and $m, n > \max(n_0, n_1)$. This implies that $x \in \chi^2_{f_{mn}}(\Delta^{\mu}_{(\eta\gamma)}, p, q)$. ■

Acknowledgements. The author wishes to express his sincere thanks and gratitude to the referee for his valuable suggestions towards the improvement of the paper.

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Received on 25.07.2012 and, in revised form, on 06.09.2012.